Pressure contribution of bound states and resonances at nonzero T

Francesco Giacosa

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In collaboration with Subhasis Samanta (UJK Kielce)

21st ZIMÁNYI SCHOOL
WINTER WORKSHOP ON HEAVY ION PHYSICS

(Hybrid: Budapest and zoom)
6/12/2021-10/12/2021

Outline

1. Thermal gas: recall + resonances and phase-shift formula

2. Lee Hamiltonian: a QM model for QFT
   (using a simple example: the rho meson)

3. Two resonances: $f_0(500)$ at nonzero T

4. Bound state at nonzero temperaturę

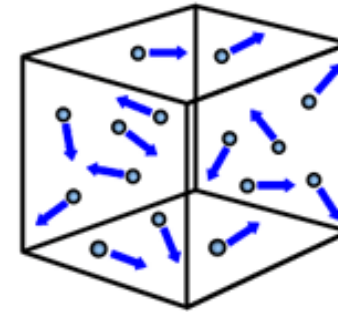5. Conclusions

Part 1: thermal gas + resonances

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Heavy-ion collisions

made by Chun Shen
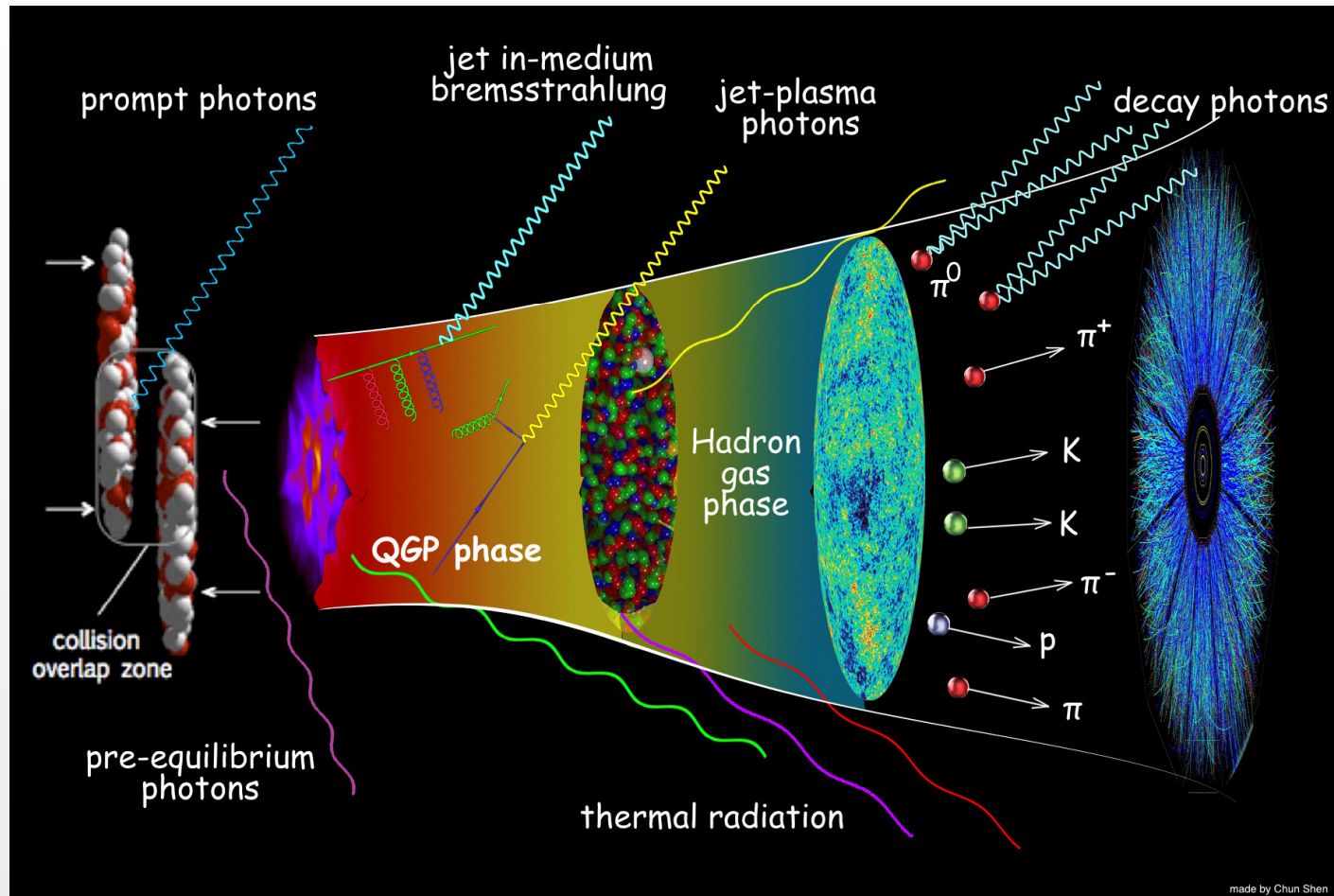
At the freeze-out, the emission of hadrons is well described by thermal models.
Question: how to include resonances, such as the rho meson?
Does the $f_0(500)$ and his brother, the light k, play a role?

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Theoretical description of a thermal gas

$$\ln Z = \sum_k \ln Z_k^{\text{stable}} + \sum_k \ln Z_k^{\text{res}}$$
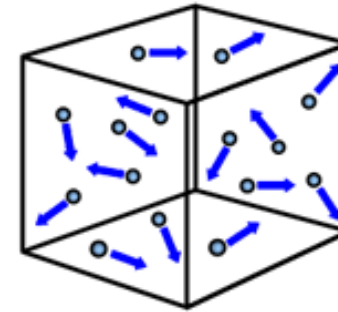

The stable part is "easy":

$$\ln Z_k^{\text{stable},} = f_k V \int \frac{d^3 p}{(2\pi)^3} \ln \left[1 \pm e^{-E_p/T} \right]^{\pm 1}$$

$$E_p = \sqrt{\vec{p}^2 + M_k^2}$$

$$P = \frac{T}{V} \ln Z$$

Theoretical description of a thermal gas

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$$E_p = \sqrt{\vec{p}^2 + M_k^2}$$

Yet, how to threat the unstable states (the resonances) ?

(Simplified) Thermal gas in QCD

It is well known that the 'dominant' term is given by pions
(since these are the lightest mesons).
For simplicity, let us first consider only the pions
and the rho meson.
The question is: how to treat the rho meson?

(Simplified) Thermal gas in QCD – the pions

$$\ln Z = \ln Z_\pi + \ln Z_\rho$$

$$\ln Z_\pi = 3V \int \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta E_\pi}} \right]$$

$$E_\pi = \sqrt{\mathbf{p}^2 + M_\pi^2}$$

In the pion gas one performs the sum over all occupation number

(Simplified) Thermal gas in QCD –the rho /1

In the easiest possible approximation, just neglect the fact that the rho is a resonance… treat it as stable
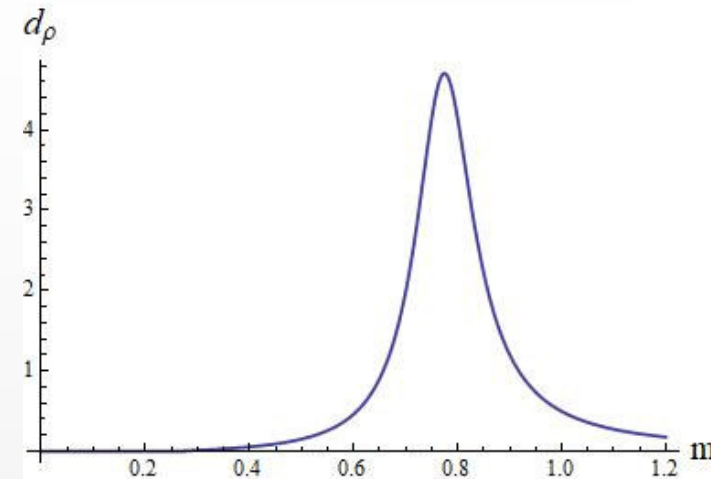
$$\ln Z_\rho = 3 \cdot 3V \int \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta E_\rho}} \right]$$

$$E_\rho = \sqrt{\mathbf{p}^2 + M_\rho^2}$$

Use, for instance, the PDG mass for the rho meson

(Simplified) Thermal gas in QCD –the rho /2

But one can do better. We may take into account that the rho meson is described by a mass distribution

$$\ln Z_\rho = 3 \cdot 3V \int_0^\infty dM d_\rho(M) \int \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta\sqrt{\mathbf{p}^2 + M^2}}} \right]$$

As a parameterization of $d_\rho(M)$ one may use BW, relativistic BW, or even more advanced function. (The best is as the imaginary part of the propagator).

$$\int_0^\infty dM d_\rho(M) = 1$$

$$d_\rho(M) \xrightarrow{\text{zero-width limit}} \delta(M - M_\rho)$$

One can go even further

R. Dashen, S.-K. Ma, and H. J. Bernstein, Phys.Rev. **187**, 345 (1969).
R. Dashen and R. Rajaraman, Phys.Rev. **D10**, 694 (1974).

W. Weinhold, B. Friman, and W. Noerenberg, Acta Phys.Polon. **B27**, 3249 (1996).
W. Weinhold, B. Friman, and W. Norenberg, Phys.Lett. **B433**, 236 (1998), arXiv:nucl-th/9710014 [nucl-th].

Result in agreement with the scattering theory of QM and by TD potential in QFT in the low density approximation

Instead of the spectral function or energy distribution, use the (derivative of the) phase-shit.
In the example that we study, consider the pion-pion scattering data in the rho-channel

Thermal gas: connection to scattering data

The density function can be directly extracted from two-body scattering data (phase shifts).

$$\ln Z_\rho = \ln Z_{(I=1,J=1)} = 3 \cdot 3V \int_0^\infty \mathrm{dM} \left[\frac{1}{\pi} \frac{d\delta_{\pi\pi,(I=1,J=1)}(M)}{dM} \right] \int \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta\sqrt{\mathbf{p}^2 + M^2}}} \right]$$

This is a model-independent way of taking the resonances into account.

Indeed, it is a justification of the validity of thermal gas models (**but**…).

It is actually even more general, since it allows also to include any other reaction and **even repulsions** in some channels.

Important point

$$\frac{1}{\pi}\frac{d\delta_{\pi\pi,(I=1,J=1)}(M)}{dM} \neq d_\rho(M)$$

These two quantities are not equal. They are actually very often similar,
Sometimes very similar…but they are not the same. The equal sign only
holds in the BW limit.

$$\frac{1}{\pi}\frac{d\delta(M)}{dM} \overset{\text{BW}}{=} \frac{\Gamma}{2\pi}\frac{1}{(M-M_\rho)^2+\Gamma_\rho^2/4} = d_\rho^{\text{BW}}(M)$$

The reason is that the two-pion states are also modified.

$$\ln Z_{tot} = \ln Z_\pi + \ln Z_{(I=1,J=1)}$$

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Part 2: the Lee Hamiltonian
and
its application to thermodynamics

P. M. Lo and F. Giacosa, *Thermal contribution of un-stable states* Eur. Phys. J. C **79** (2019) no.4, 336 [arXiv:1902.03203 [hep-ph]].

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Lee Hamiltonian

$$H = H_0 + H_1$$

$$H_0 = M_0 |S\rangle\langle S| + \int_{-\infty}^{+\infty} dk\, \omega(k) |k\rangle\langle k|$$

$$H_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, (g \cdot f(k))(|S\rangle\langle k| + |k\rangle\langle S|)$$

|S> is the initial unstable state, coupled to an infinity of final states |k>. (Poincare-time is infinite. Irreversible decay). General approach, similar Hamiltonians used in many areas of Physics. (Jaynes-Cummings approach, Friedrichs model.)

In our example: rho-meson decay. |S> represents a rho meson, |k> represents a two-photon state (flying back-to-back). The quantity k is the three-momentum of one of the outgoing pion.
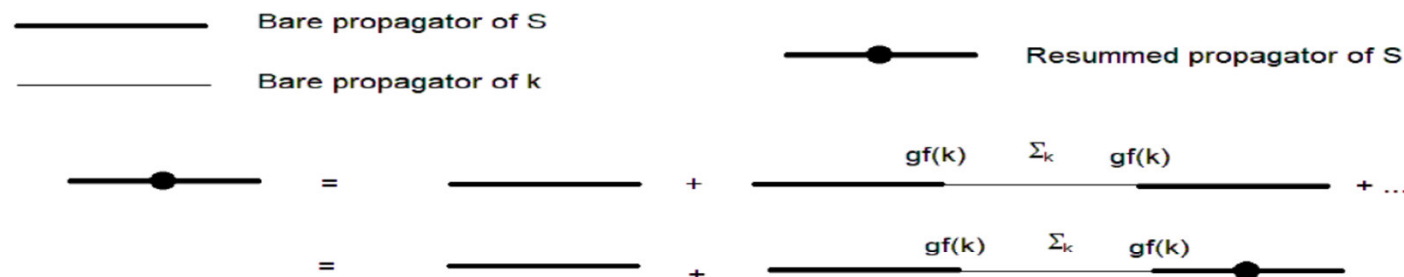
F.G J.Phys.Conf.Ser. 1612 (2020) 1, 012012, 2001.07781

Figure 1. Schematic presentation of the sum leading to the dressed propagator. In the last part the Bethe-Salpeter resummation is depicted.

Lee Hamiltonian

Used in many areas of physics…also called Jaynes-Cummings Hamiltonian or Friedrichs model

T. D. Lee, *Some Special Examples in Renormalizable Field Theory*, Phys. Rev. **95** (1954) 1329-1334. C. B. Chiu, E. C. G. Sudarshan and G. Bhamathi, *The Cascade model: A Solvable field theory*, Phys. Rev. D **46** (1992) 3508.

Z. W. Liu, W. Kamleh, D. B. Leinweber, F. M. Stokes, A. W. Thomas and J. J. Wu, *Hamiltonian effective field theory study of the N(1535) resonance in lattice QCD* Phys. Rev. Lett. **116** (2016) no.8, 082004 [arXiv:1512.00140 [hep-lat]].

Z. Xiao and Z. Y. Zhou, *On Friedrichs Model with Two Continuum States*, J. Math. Phys. **58** (2017) no.6, 062110 [arXiv:1608.06833 [hep-ph]]. Z. Y. Zhou and Z. Xiao, *Understanding X(3862), X(3872), and X(3930) in a Friedrichs-model-like scheme* Phys. Rev. D **96** (2017) no.5, 054031 Erratum: [Phys. Rev. D **96** (2017) no.9, 099905]

F. Giacosa, *Non-exponential decay in quantum field theory and in quantum mechanics: the case of two (or more) decay channels*, Found. Phys. **42**, 1262 (2012).

Application to TD

We consider that also the state |k> (the two-pion state) is modified by the presence of the interaction. Not only the rho is modified (quite "expected"), but the pions are not any longer 'free'.

$$P_{\text{KFL}} \approx P_\pi^{(0)} + P_\rho^{(0)} + \Delta P_{\mathcal{K}}$$
$$= P_\pi^{(0)} + P_\rho^{(0)} + \Delta P_\rho + \Delta P_{2\pi}.$$

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Rho spectra function

$$\ln Z_\rho = 3 \cdot 3V \int_0^\infty \mathrm{d}M d_\rho(M) \int \frac{d^3 p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta\sqrt{\mathbf{p}^2 + M^2}}} \right]$$

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Phase-shift's derivative formula

$$\ln Z_\rho = \ln Z_{(I=1,J=1)} = 3{\cdot}3V \int_0^\infty \mathrm{dM} \left[\frac{1}{\pi} \frac{d\delta_{\pi\pi,(I=1,J=1)}(M)}{dM} \right] \int \frac{d^3p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta\sqrt{\mathbf{p}^2 + M^2}}} \right]$$

Numerical example
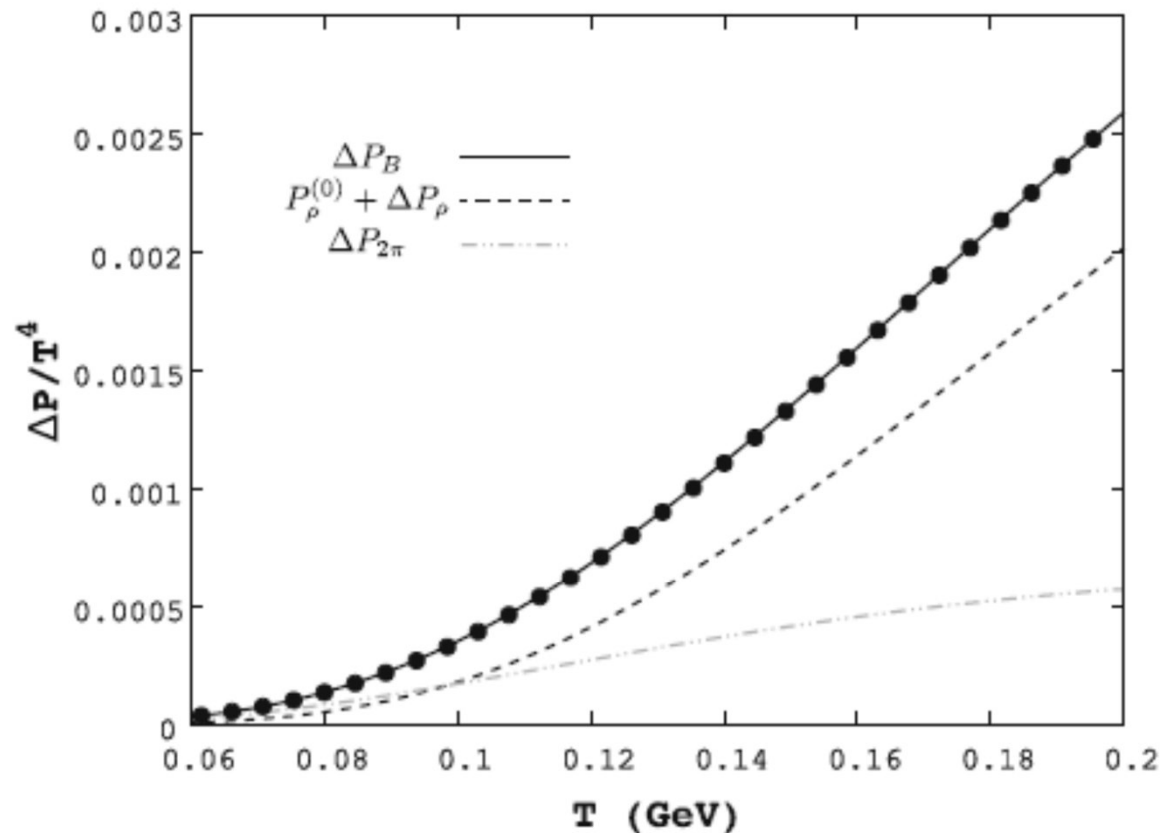
Fig. 2 Interacting contributions to the thermodynamics pressure (normalized to T^4) from different effective spectral functions. The top line (B) is equal to the sum of the contributions from the lower two (see Eq. (69)). The points are the results obtained by directly constructing the partition function from the eigenvalues of the Hamiltonian. See text

Contributions/T^4

$$P_\rho^{(0)} + \Delta P_\rho + \Delta P_{2\pi}$$

$$\cdot \; P_\rho^{(0)} + \Delta P_\rho$$

$$\Delta P_{2\pi}$$

Uniwersytet
Jana Kochanowskiego w Kielcach

Part 3: Resonances $f_0(500)$ and $K_0^*(700)$

based on W. Broniowski, F.G., V. Begun,

Cancellation of the sigma meson in thermal models
Phys. Rev. C 92 (2015) no.3, 03490
[arXiv:1506.01260 [nucl-th]].

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Inclusion fo two additional channels

$$\ln Z = \ln Z_\pi + \ln Z_K + \boxed{\ln Z_{(1,1--)}} + \boxed{\ln Z_{(0,0++)} + \ln Z_{(2,0++)}} + \boxed{\ln Z_{(1/2,0++)} + \ln Z_{(3/2,0++)}} + ...$$

rho-meson   f0(500)   K0*(700)

$$\ln Z_K = 4V \int \frac{d^3p}{(2\pi)^3} \ln \left[\frac{1}{1 - e^{-\beta E_K}} \right]$$

$$E_K = \sqrt{\mathbf{p}^2 + M_K^2}$$

$$P = \frac{T}{V} \ln Z$$

For the various channels:

Simple and model-independent procedure: **just use scattering data!**

In the scalar channel

$f_0(500)$ $\quad I^G(J^{PC}) = 0^+(0^{++})$

also known as σ; was $f_0(600)$
See the related review(s):
Scalar Mesons below 2 GeV

$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\, \mathrm{Im}(\sqrt{s_{pole}})$.

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
$(400{-}550){-}i(200{-}350)$ OUR ESTIMATE			

M. Soltysiak, T. Wolkanowski and F. G.,
K0*(700) as a companion pole of
K0*(1430),'
Nucl. Phys. B 909 (2016) 418
[arXiv:1512.01071 [hep-ph]].

$K_0^*(700)$ $\quad I(J^P) = \frac{1}{2}(0^+)$

also known as κ; was $K_0^*(800)$
Needs confirmation. See the mini-review on scalar mesons under
$f_0(500)$ (see the index for the page number).

$K_0^*(700)$ T-Matrix Pole $\sqrt{s}$

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
$(630{-}730) = i\,(260{-}340)$ OUR EVALUATION			

What to do?
Simple…phase shifts

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The f0(500) spectral function **and** the isotensor repulsion/1

arxiv: 1506.01260
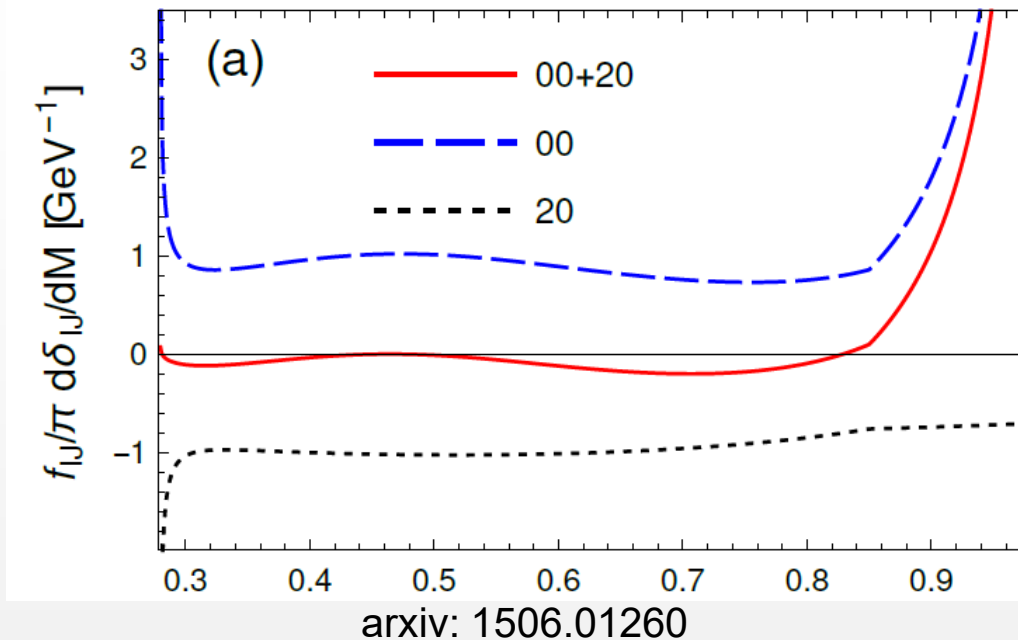
The total contribution from J=0

is the red curve: $\ln Z_{(0,0)} + \ln Z_{(2,0)}$

M [GeV]

$\ln Z_{(0,0)}$ is the contribution of f0(500). It is indeed nonzero and even non-negligible, but it is almost exactly cancelled by the isotensor repulsion. Thermal models however usually neglect repulsions.

Either you take into account both I=0 and I =2, or –simply- you neglect both of them

$$\ln Z_{(0,0++)} + \ln Z_{(2,0++)} = \int_0^{\Lambda_0} dM \left[\frac{d\delta_{(0,0)}}{\pi dM} + 5 \frac{d\delta_{(2,0)}}{\pi dM} \right] \int_p \ln \left[1 - e^{-\frac{\sqrt{p^2 + M^2}}{T}} \right]^{-1}$$

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Part 4: how to deal with bound states

based on S. Samanta, F.G,

QFT treatment of a bound state in a thermal
Phys.Rev.D 102 (2020) 116023
[arXiv:2009.13547 [nucl-ph]]

and

Thermal role of bound states and resonances in scalar QFT,
[arXiv:2110.14752 [hep-ph]].

Various states could be hadronic bound states: X(3872), a0(980), …

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QFT with quartic interaction term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

+...
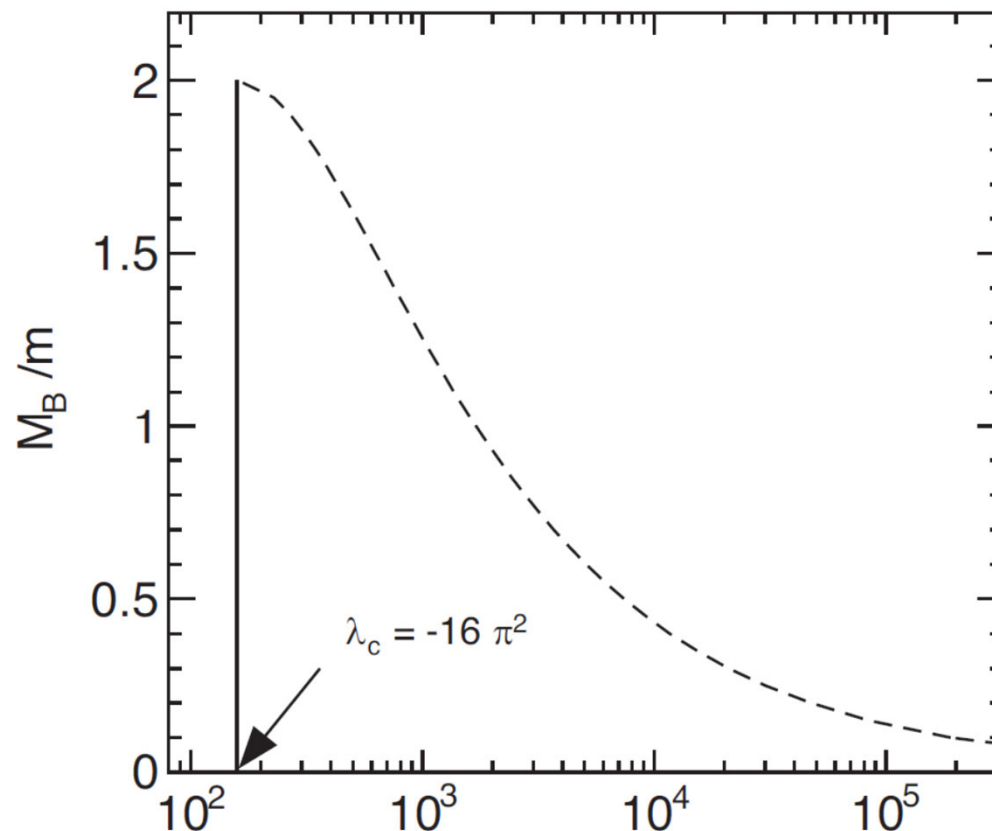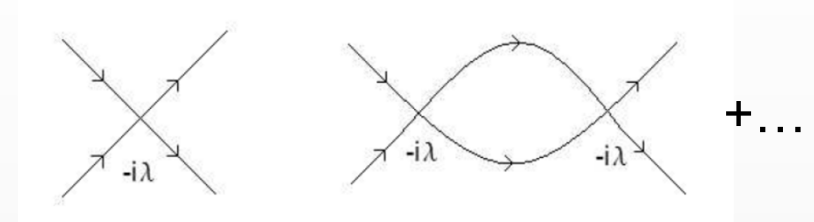
Recall:
λ > 0 repulsion
λ < 0 attraction

One-loop resummed unitarization scheme, one subtraction with

$$\Sigma(s \to 0) = 0$$

$\lambda_c = -16\,\pi^2$

arxiv: 2009.13547

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Pressure of the system

$$P_{\varphi,\text{free}} = -T \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+m^2}}\right]$$

$$P_{\varphi\varphi\text{-int}} = -T \int_{2m}^{\infty} dx \frac{1}{\pi} \frac{d\delta_0(s=x^2)}{dx} \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+x^2}}\right]$$

Interaction contribution

$$P_B = -\theta(\lambda_c - \lambda)T \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+M_B^2}}\right]$$

Bound state contribution

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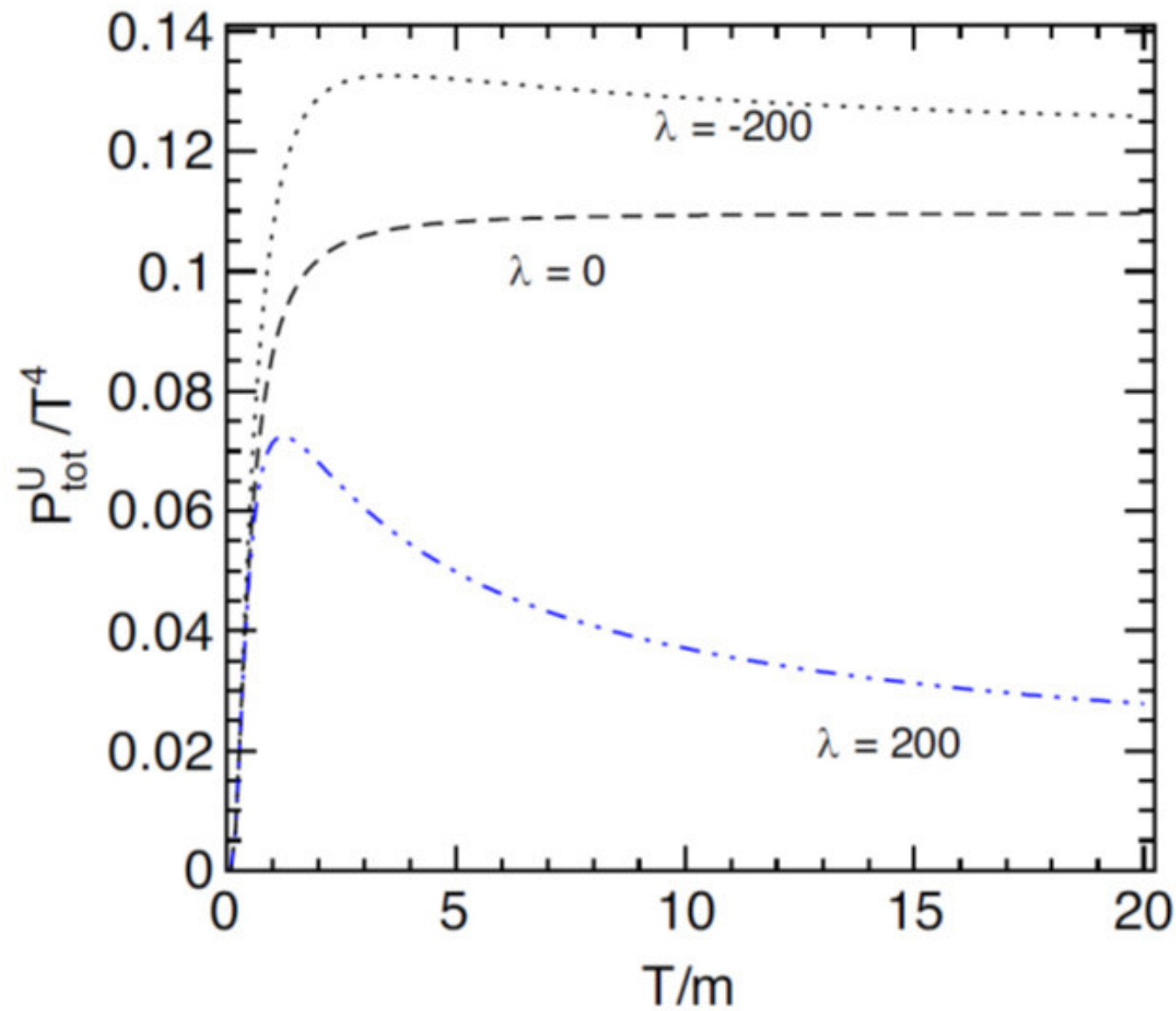
Pressure vs T

Interacting part of the pressure (without BS) as function of λ >0

arxiv: 2009.13547
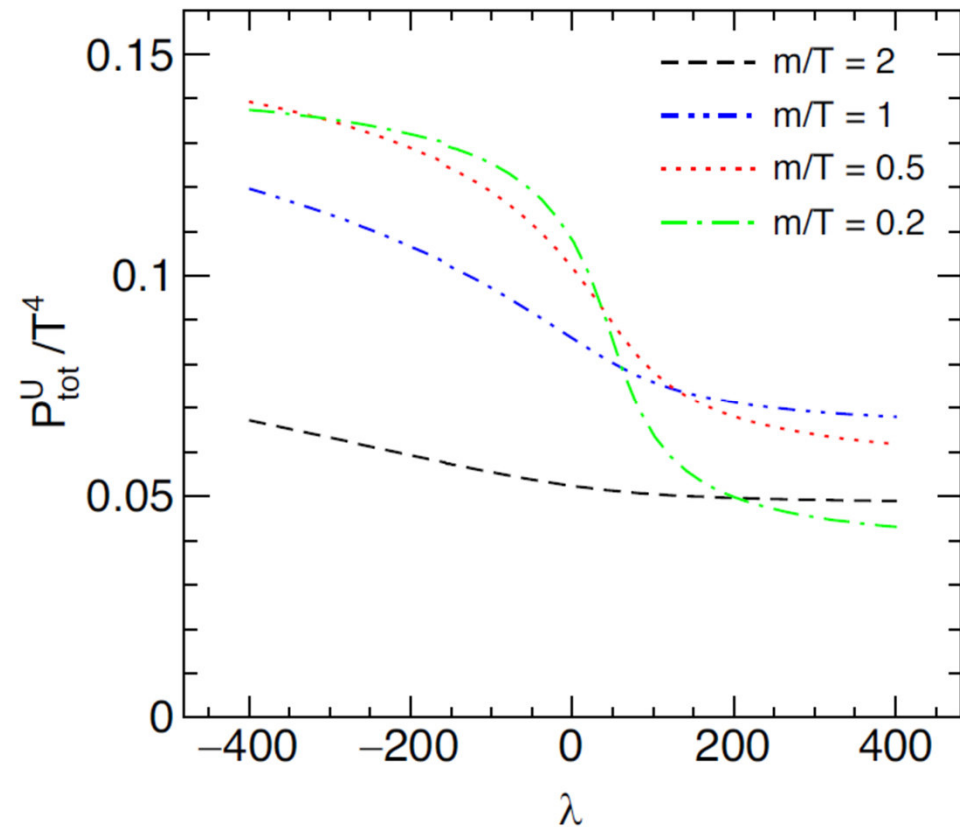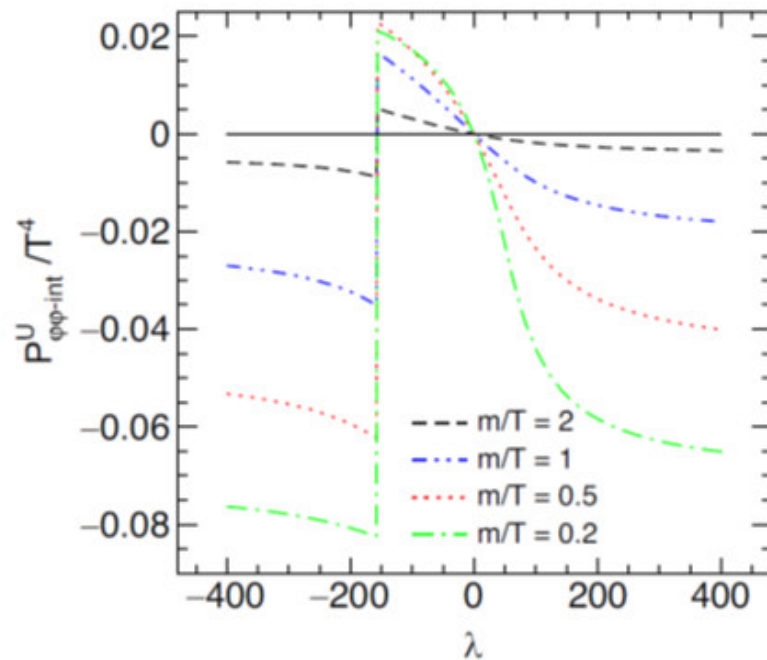
Role of the bound state

The results suggest that for a bound state created close to threshold (thus λ smaller but close to λ_c), the bound state is indeed important

2009.13547

$$P_B + P_{\varphi\varphi\text{-int}} = \xi P_B$$

$$P_B = -\theta(\lambda_c - \lambda)T \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+M_B^2}}\right]$$

See also:
Ortega et al.,
Counting states and the Hadron Resonance Gas:
Does X(3872) count?
 PLB 781 (2018) arxiv: 1707.01915

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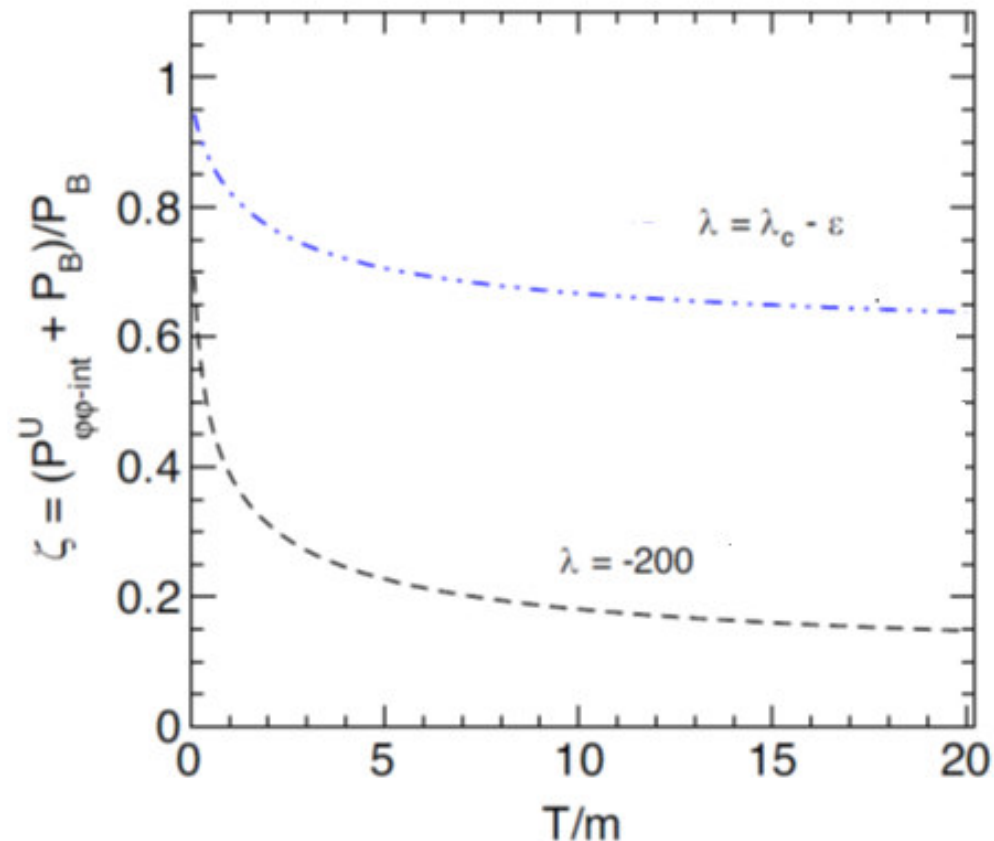
φ^3 interaction

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3,$$

$$A(s,t,u) = -\frac{g^2}{s - m^2 + i\epsilon} - \frac{g^2}{t - m^2 + i\epsilon} - \frac{g^2}{u - m^2 + i\epsilon}.$$
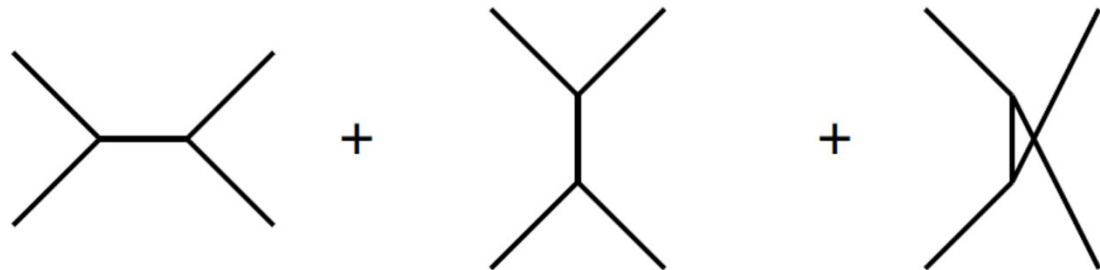
At tree-level the s-wave amplitude:

$$A_0(s) = \frac{1}{2}\int_{-1}^{+1} d\xi A(s,\theta) = -\frac{g^2}{s - m^2} + 2g^2\frac{\log\left[1 + \frac{s - 4m^2}{m^2}\right]}{s - 4m^2}$$
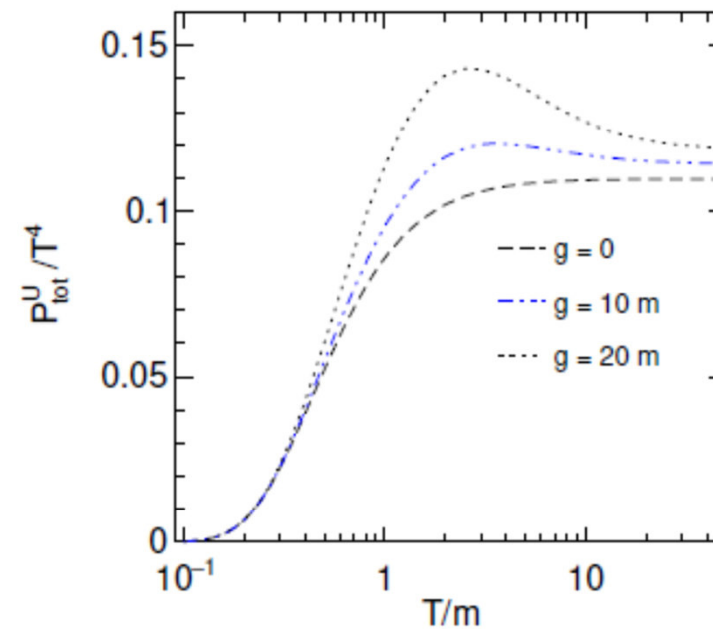
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Bound state

The bound state equation (s-channel and s-wave, for s below the threshold) reads:

$$\left[A_0^{-1}(s) - \Sigma(s)\right] = 0 .$$
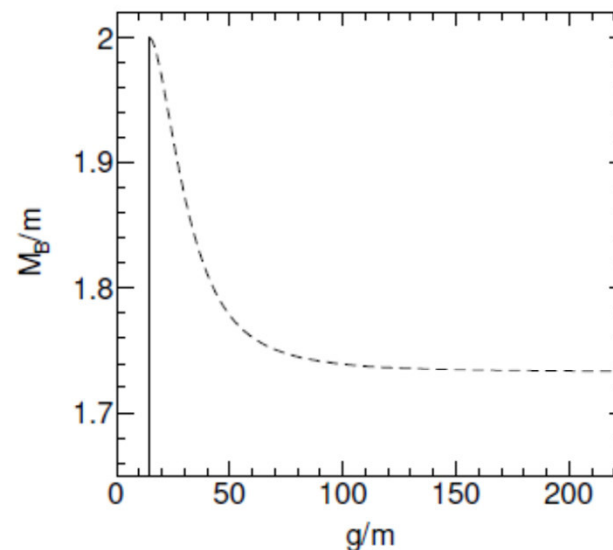
★ $A_0(s = 4m^2) \neq 0 \Rightarrow$ bound state is formed in s-wave at the threshold
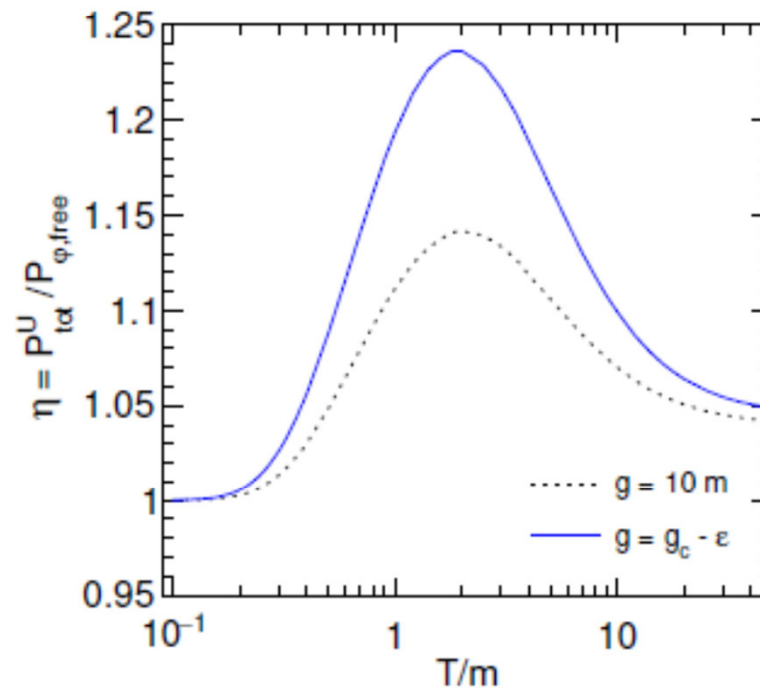
Pressure vs T

★ For both $g = 10m$ and $20m$ total pressure is greater than that of free particle

When no bound state is formed ($g < g_c$), let us write P^U_{tot} as:

$$P^U_{tot} = \eta P_{\varphi,free}$$

$$\eta = \frac{P^U_{tot}}{P_{\varphi,free}} = 1 + \frac{P^U_{\varphi\varphi-int}}{P_{\varphi,free}} \; .$$
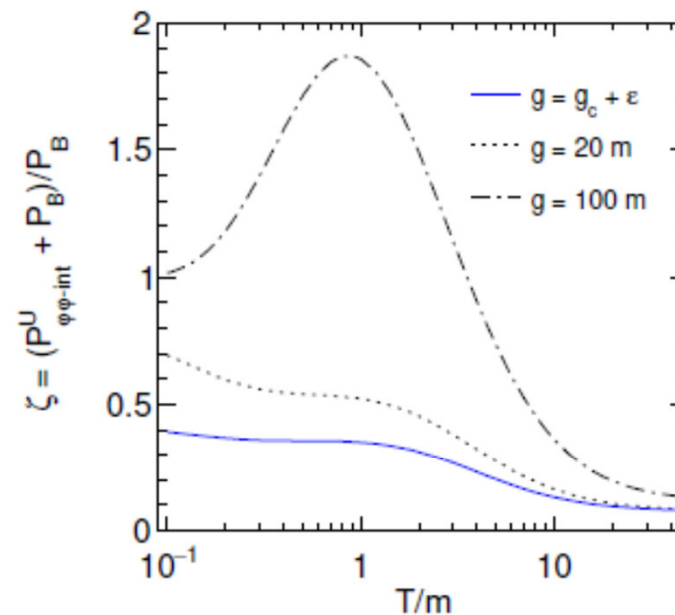

* At low T, $\eta \approx 1 \Rightarrow P^U_{\varphi\varphi-int} \approx 0$
* Effect of interaction is large at some intermediate temperature

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When $g > g_c$, let us write P_{tot}^U as:

$$P_{tot}^U = P_{\varphi,free} + \zeta P_B$$

$$\zeta = \frac{P_{\varphi\varphi-int}^U + P_B}{P_B},$$

* When g is not very large, $\zeta < 1$ ($P_{\varphi\varphi-int}^U < 0$)
* for large g the role of the d- and g-waves is sizable and the additional pressure is due to the higher waves

Conclusions

For resonances: mass distribution is important, but sometimes not sufficient.

If you have scattering data, easy to use them!

Bound states are relevant, especially if the mass is just below the threshold of constitutents. Yet, the answer is: it is complicated!
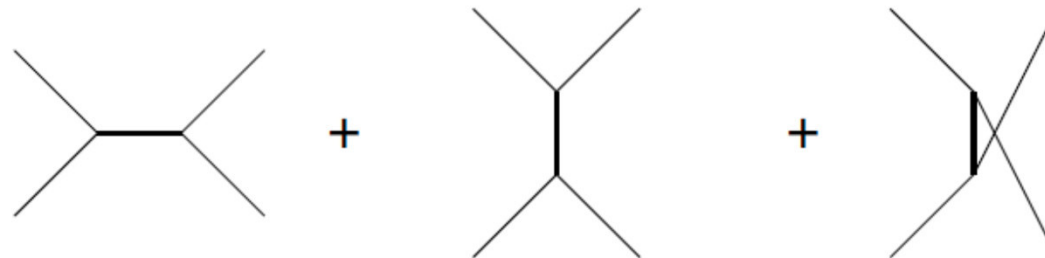
Thank You

Part 5: Intermediate resonance and bound state

Resonance S

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\left(\partial_\mu S\right)^2 - \frac{1}{2}M^2 S^2 + \frac{g}{2!}S\varphi^2,$$

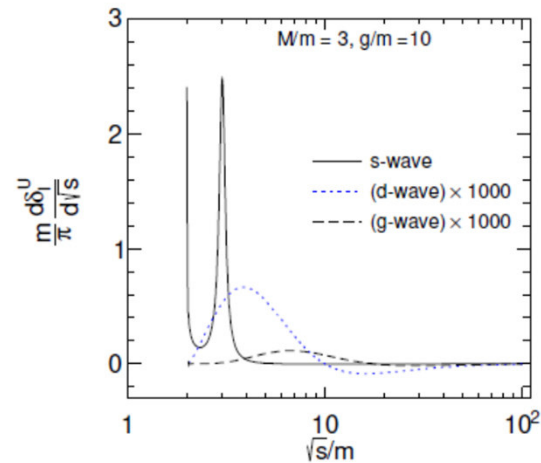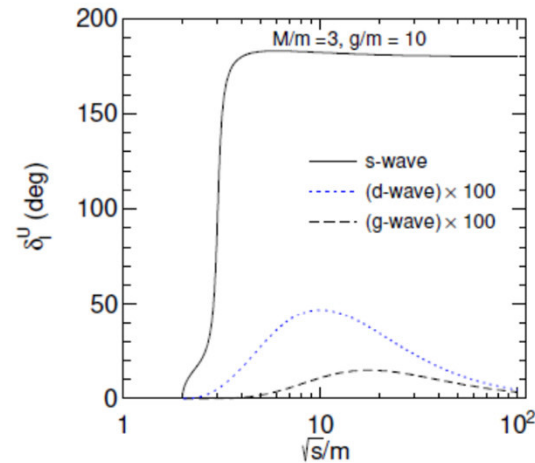$$A(s,t,u) = \frac{-g^2}{s - M^2 + i\epsilon} + \frac{-g^2}{t - M^2 + i\epsilon} + \frac{-g^2}{u - M^2 + i\epsilon}.$$

The three-level decay width (valid for $M > 2m$):

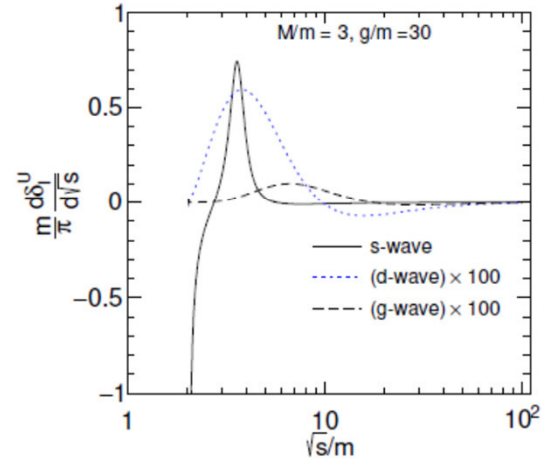$$\Gamma_S = \frac{g^2}{2} \frac{\sqrt{\frac{M^2}{4} - m^2}}{8\pi M^2}$$
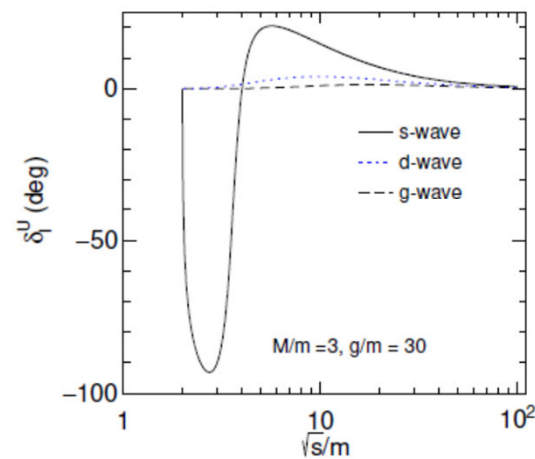
$$A_0(s) = \frac{1}{2} \int_{-1}^{+1} d\xi A(s,\theta) = -\frac{g^2}{s - M^2 + i\epsilon} + 2g^2 \frac{\log\left[1 + \frac{s-4m^2}{M^2}\right]}{s - 4m^2 + i\epsilon}$$

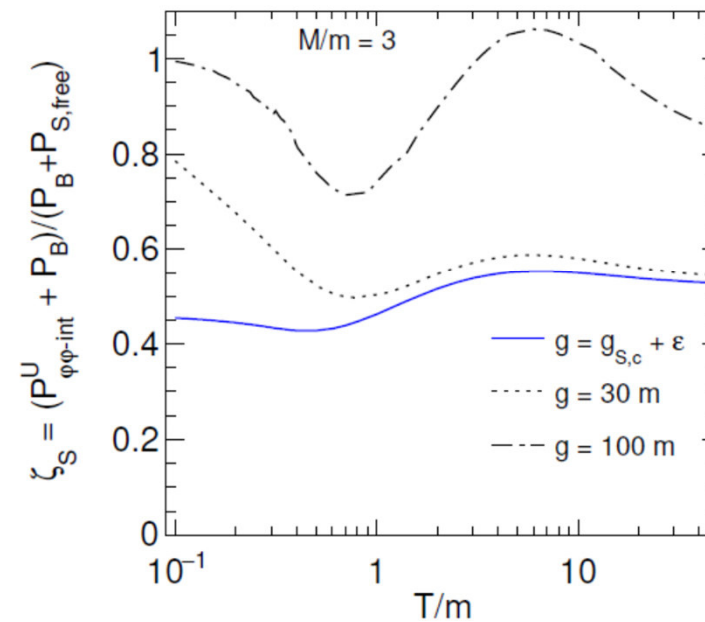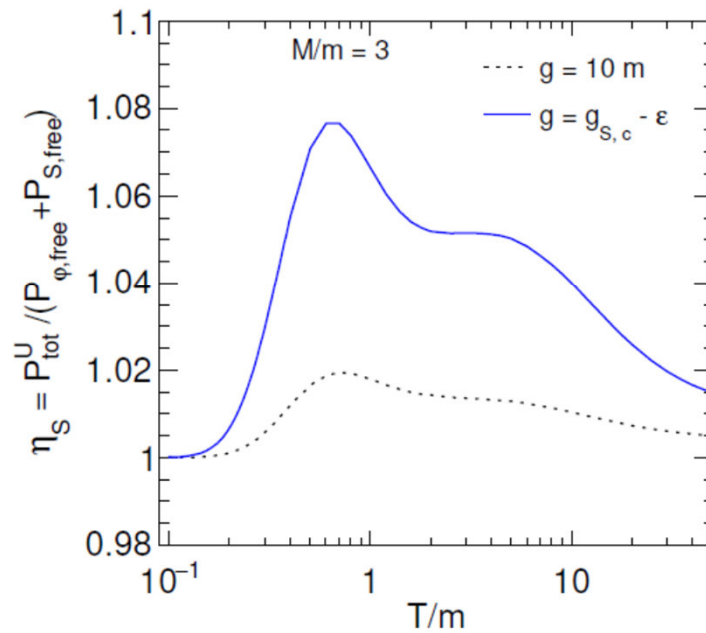# Phase shift and its derivative ($M > 2m$)

$g < g_c$

$g > g_c$

★ Resonance peak is observed near $\sqrt{s} = 3m$

$$P_{tot}^{U} = \eta_S \left(P_{\varphi,free} + P_{S,free} \right)$$

$$\eta_S = \frac{P_{tot}^{U}}{P_{\varphi,free} + P_{S,free}}$$

$$P_{tot}^{U} = P_{\varphi,free} + \zeta_S \left(P_{S,free} + P_B \right)$$

$$\zeta_S = \frac{P_{\varphi\varphi-int}^{U} + P_B}{P_{S,free} + P_B}$$

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QM derivation of the phase-shift formula

As a last step, we discuss a simple way based on Quantum Mechanics. The radial wave function with angular momentum l of a particle scattered by central potential $U(r)$ is

$$\psi_l(r) \propto \sin[kr - l\pi/2 + \delta_l] , \tag{17}$$

where $k = \left|\vec{k}\right|$ is the length of the three-momentum, and δ_l is the phase shift due the interaction with the potential. If we confine our system into a sphere of radius R, the condition $kR - l\pi/2 + \delta_l = n\pi$ with $n = 0, 1, 2, ...$ must be met, since $\psi_l(r)$ has to vanish at the boundary. Conversely, the number of states n_0 that one can have by limiting k in the range $(0, k_0)$ is given by $n_0 = (k_0 R - l\pi/2 + \delta_l)/\pi$. Then, the density of state that one can place between k and $k + dk$ is given by

$$\frac{dn_l}{dk} = \frac{R}{\pi} + \frac{1}{\pi}\frac{d\delta_l}{dk} , \tag{18}$$

where the first term describes the density of states $\frac{dn_l^{free}}{dk}$ in absence of interactions, while the second term $\frac{1}{\pi}\frac{d\delta_l}{dk}$ describes the effect of the interacting potential. When translating the discussion from Quantum Mechanics to Quantum Field Theory, we replace the momentum k with the invariant mass M, the angular momentum l with the pair (I, J). Upon summing over the latter, one obtains the full density of states of an interacting pion gas as

$$\frac{dn}{dM} = \delta(M - M_\pi) + \sum_{I,J}\frac{1}{\pi}\frac{d\delta_{(I,J)}(M)}{dM} . \tag{19}$$

Phase shift formula - recall

Next, one has to obtain the partial waves.

$$A(s,t,u) = A(s,\theta) = \sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(\cos\theta)$$

where $P_l(\xi)$ are the Legendre polynomials with

$$\int_{-1}^{+1}d\xi P_l(\xi)P_{l'}(\xi) = \frac{2}{2l+1}\delta_{ll'} \ .$$

For identical particles, one has the following definition of the phase space of the l-th wave

$$\frac{e^{2i\delta_l(s)}-1}{2i} = ka_l(s) = \frac{1}{2}\cdot\frac{k}{8\pi\sqrt{s}}A_l(s)$$

Recall from scattering theory:

$$\frac{e^{2i\delta_k}-1}{2i} = a_k = \frac{-\sqrt{s}\Gamma(\sqrt{s})}{s-m^2+i\sqrt{s}\Gamma(\sqrt{s})}$$

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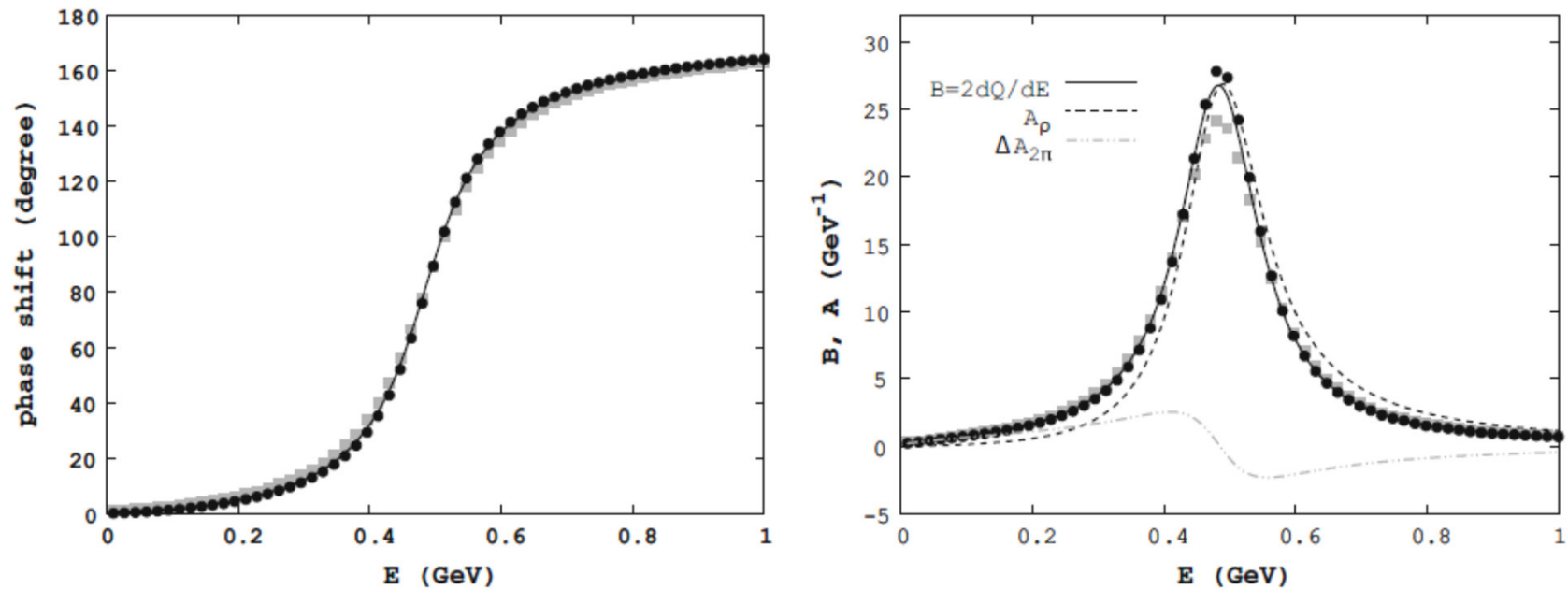
Additional results with the Lee model

Fig. 1 Phase shift computed from the Lee model (Eq. (36)) and the corresponding effective spectral functions (Eqs. (22), (23)) for the ρ-meson. E is the energy of the relative motion. Note that $B = A_\rho +$ $\Delta A_{2\pi}$. The points are the numerical results obtained from the momentum grid method: Grey squares indicate results based on Eq. (36), while black circles indicate results based on Eq. (48). See text

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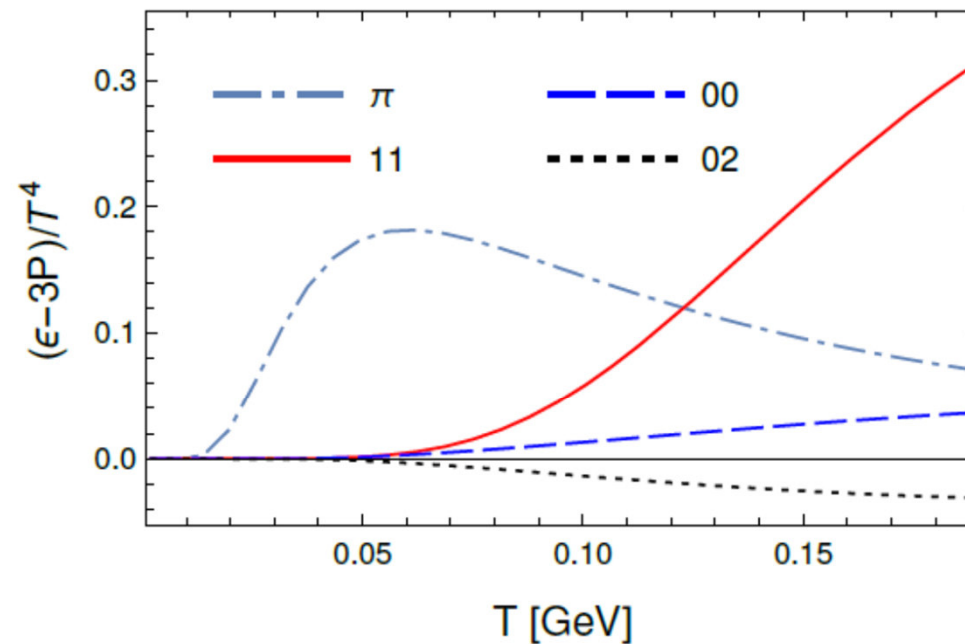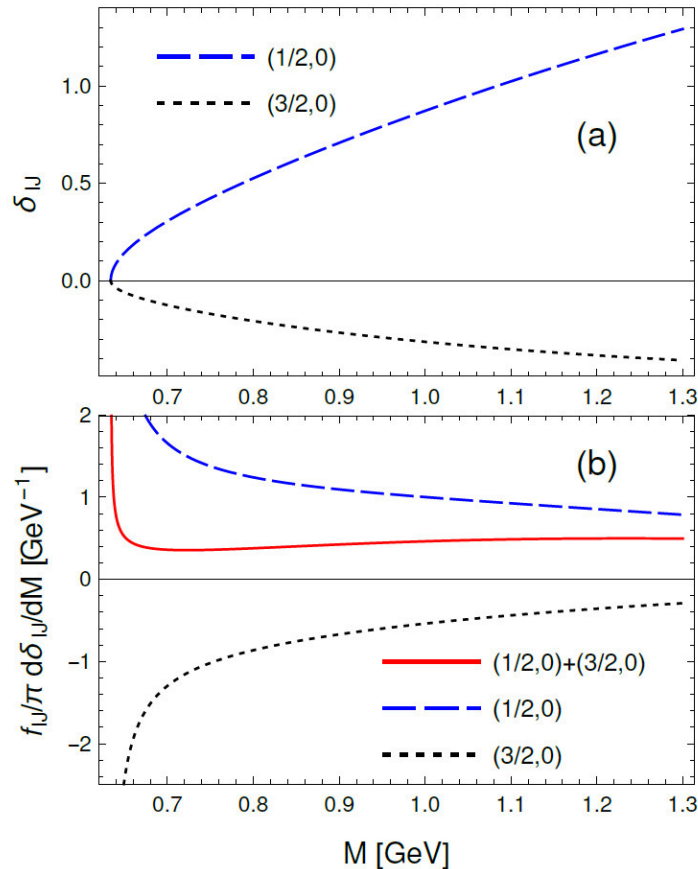# Rho, pion, and sigma: the trace anomaly

FIG. 3. (color online) Pion, ρ-meson, isoscalar-scalar, and isoscalar-tensor contributions to the volume density of the trace of the energy-momentum tensor divided by T^4, plotted as functions of T.

The scalar kaonic resonace K₀*(700): (partial) cancellation in thermal models

arxiv: 1506.01260

The total contribution from is the red curve:

$$\ln Z_{(1/2,0)} + \ln Z_{(3/2,0)}$$

cancellation is evident:
easiest thing to do is to forget about the k.
(Eventually, visible in correlations).

$$\ln Z_{(1/2,0^{++})} + \ln Z_{(3/2,0^{++})} = \int_0^{\Lambda_0} dM \left[2\frac{d\delta_{(1/2,0)}}{\pi dM} + 4\frac{d\delta_{(3/2,0)}}{\pi dM} \right] \int_p \ln \left[1 - e^{-\frac{\sqrt{p^2+M^2}}{T}} \right]^{-1}$$
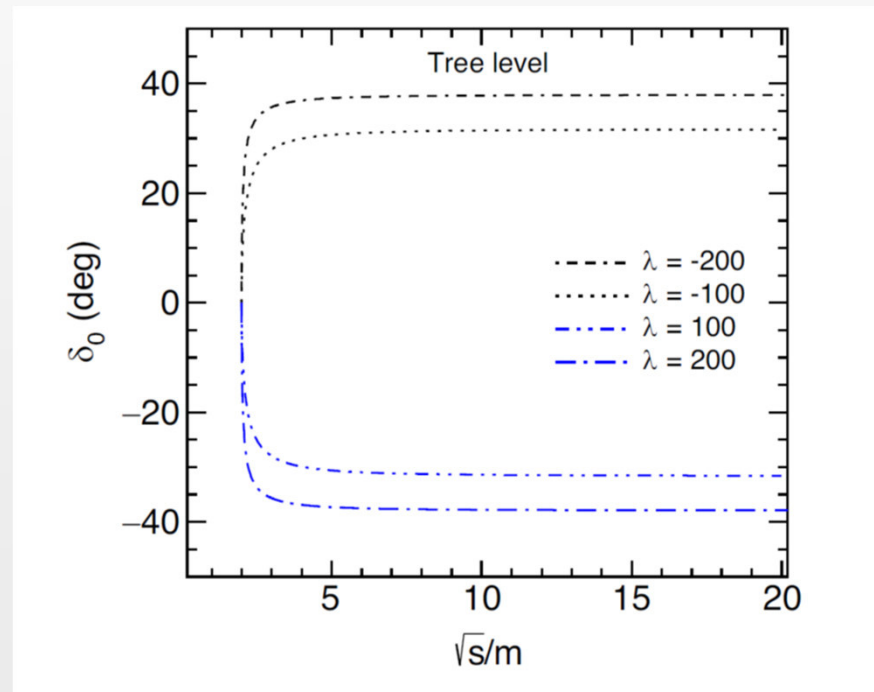
Phi^4 theory, some more details

$$A(s, t, u) = A(s, \theta) = \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(\cos \theta)$$

$$a_0^{\mathrm{SL}} = \frac{1}{2} \frac{A_0(s = 4m^2)}{8\pi\sqrt{4m^2}} = \frac{1}{2} \frac{-\lambda}{16\pi m}$$
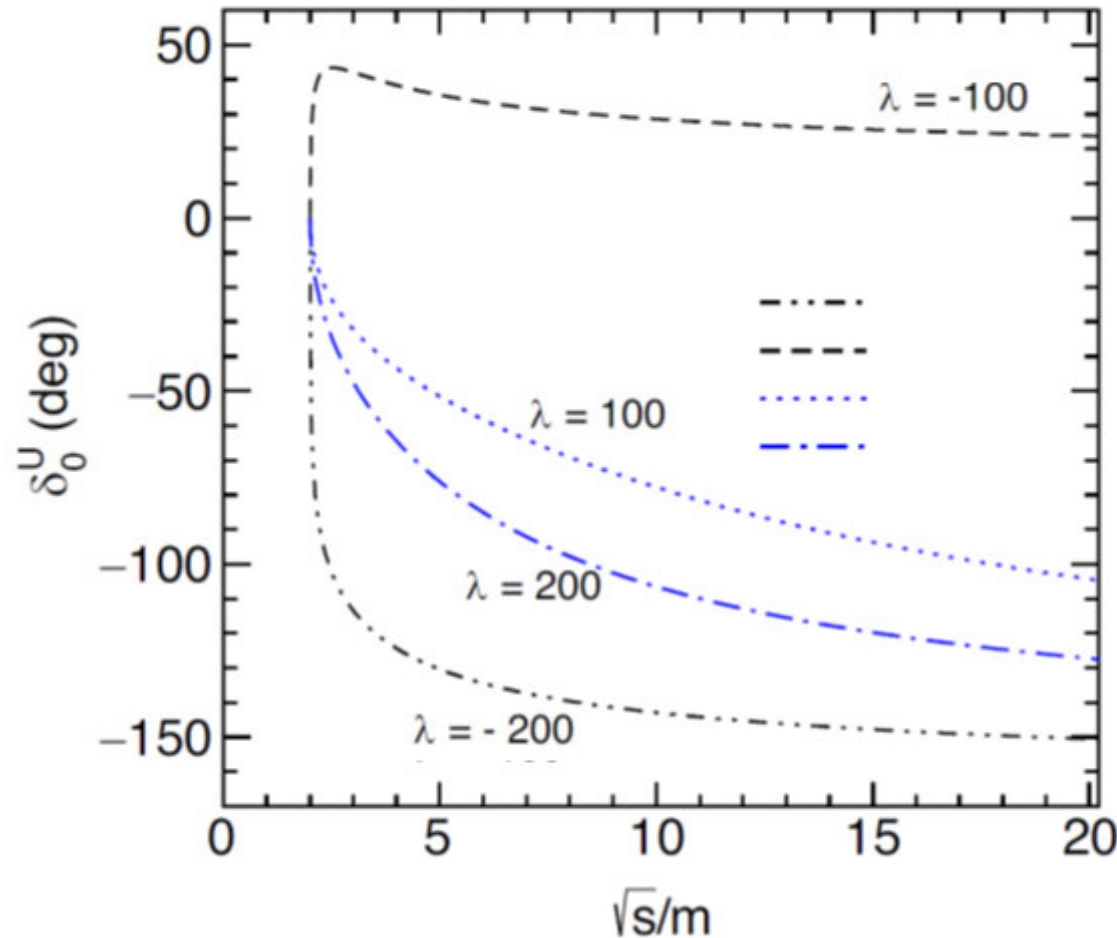
Tree-level phase shift

$$\delta_0(s) = \frac{1}{2} \arg \left[1 - \frac{1}{16\pi} \sqrt{\frac{4m^2}{s} - 1} A_0(s) \right]$$
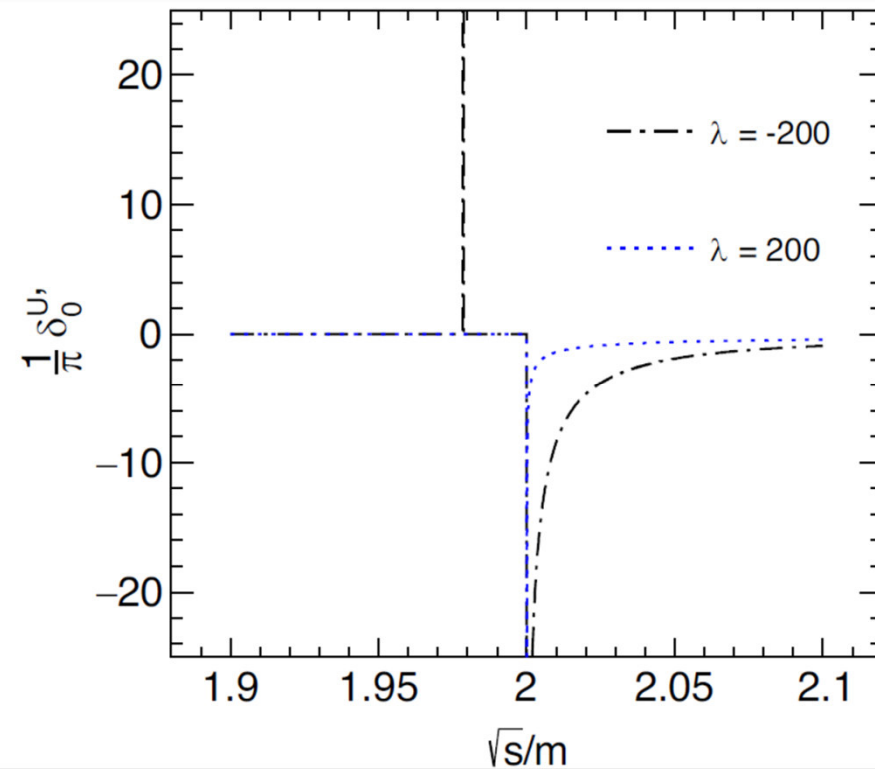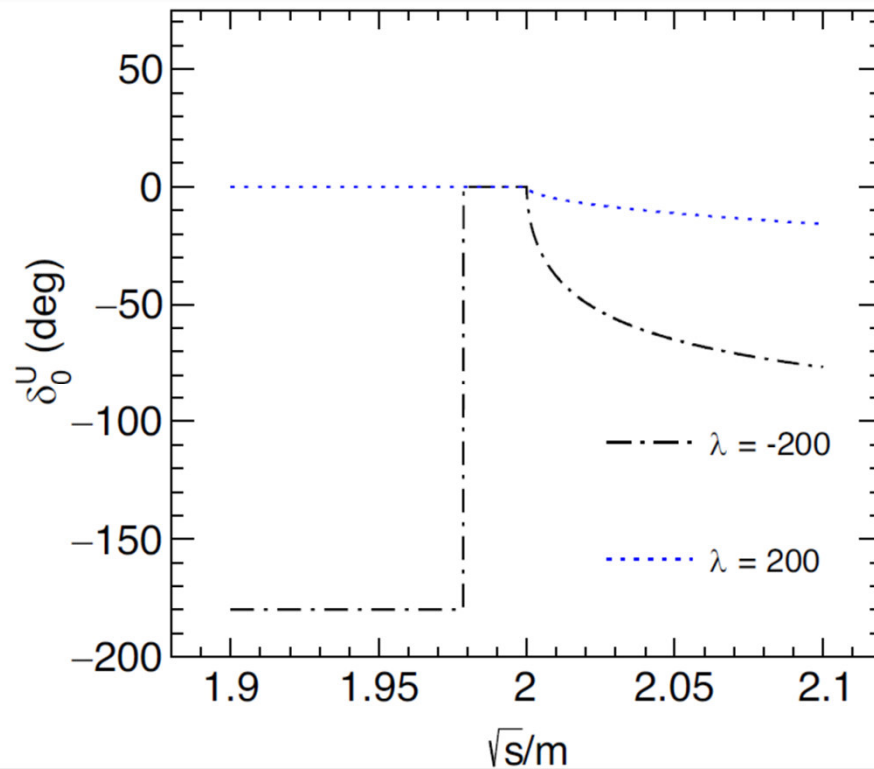

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Phase shift

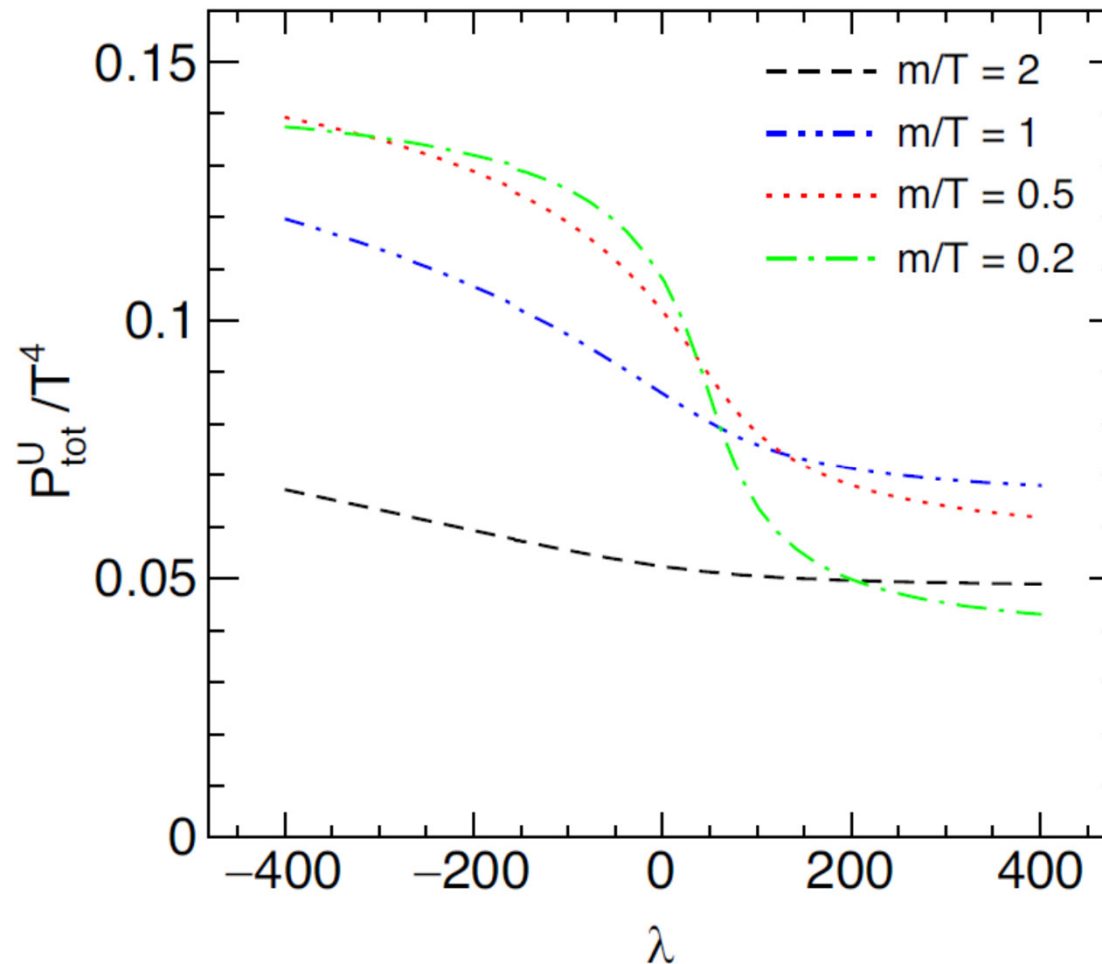$$\delta_0^U(s \to \infty) = 0 \quad \text{for } \lambda \in (\lambda_c, 0) \qquad \delta_0^U(s \to \infty) = -\pi \quad \text{for } \lambda < \lambda_c.$$

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Levinson's theorem

Unitarized phase shift (extension below threshold)

Pressure as function of the coupling λ

arxiv: 2009.13547

The pressure is continuous also at the critical value!

Loop and

$$\Sigma(s) = \frac{1}{2}\frac{1}{16\pi}\left(-\frac{1}{\pi}\sqrt{1 - \frac{4m^2}{s + i\epsilon}}\ln\frac{\sqrt{1 - \frac{4m^2}{s+i\epsilon}} + 1}{\sqrt{1 - \frac{4m^2}{s+i\epsilon}} - 1}\right) + \frac{1}{16\pi^2}$$

$$\mathrm{Re}\Sigma(s) = \frac{s}{\pi}P\int_{s_{th}}^{\infty}\frac{I(s')}{(s'-s)s'}.$$

$$\mathrm{Im}\Sigma(s) = \begin{cases} \frac{1}{2}\frac{\sqrt{\frac{s}{4}-m^2}}{8\pi\sqrt{s}} & \text{for } s > (2m)^2 \\ \varepsilon & \text{for } s < (2m)^2, \end{cases}$$

$$\mathrm{Re}\Sigma(s) = \frac{s}{\pi}P\int_{s_{th}}^{\infty}\frac{I(s')}{(s'-s)s'}.$$

Unitarized phase shifts

$$A_k^U(s) = [A_k^{-1}(s) - \Sigma(s)]^{-1}$$

$$A_0^U(s) = [A_0^{-1}(s) - \Sigma(s)]^{-1} = \frac{-\lambda}{1 + \lambda\Sigma(s)},$$

$$\frac{e^{2i\delta_0^U(s)} - 1}{2i} = \frac{1}{2} \cdot \frac{k}{8\pi\sqrt{s}} A_0^U(s).$$

$$a_0^{U,\text{SL}} = \frac{1}{2} \frac{1}{16\pi m} \frac{-\lambda}{1 + \frac{\lambda}{16\pi^2}}.$$

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Unitarization

$$I(s) = \mathrm{Im}\,\Sigma(s) = \frac{1}{2} \frac{\sqrt{\frac{s}{4} - m^2}}{8\pi\sqrt{s}} \quad \text{for } \sqrt{s} > 2m\ .$$

loop function $\Sigma(s)$ for a complex s is chosen by considering two subtractions:

$$\Sigma(s) = -\frac{(s - m^2)(s - 3m^2)}{\pi} \int_{4m^2}^{\infty} \frac{\frac{1}{2} \frac{\sqrt{\frac{s'}{4} - m^2}}{8\pi\sqrt{s'}}}{(s - s' + i\varepsilon)(s' - m^2)(s' - 3m^2)} ds'$$

⋆ The subtractions guarantee that $\Sigma(s = m^2) = 0$ and $\Sigma(s = 3m^2) = 0$.

⋆ In this way, the choice of $\Sigma(s)$
(i) preserves the pole corresponding to $s = m^2$
(ii) assures that the unitarized amplitude diverges at the branch point $s = 3m^2$ generated by the single-particle pole for m^2 along the t and u channels
($\log 0$ at the 2nd term of A_0)

Real and imaginary parts of $\Sigma(s)$
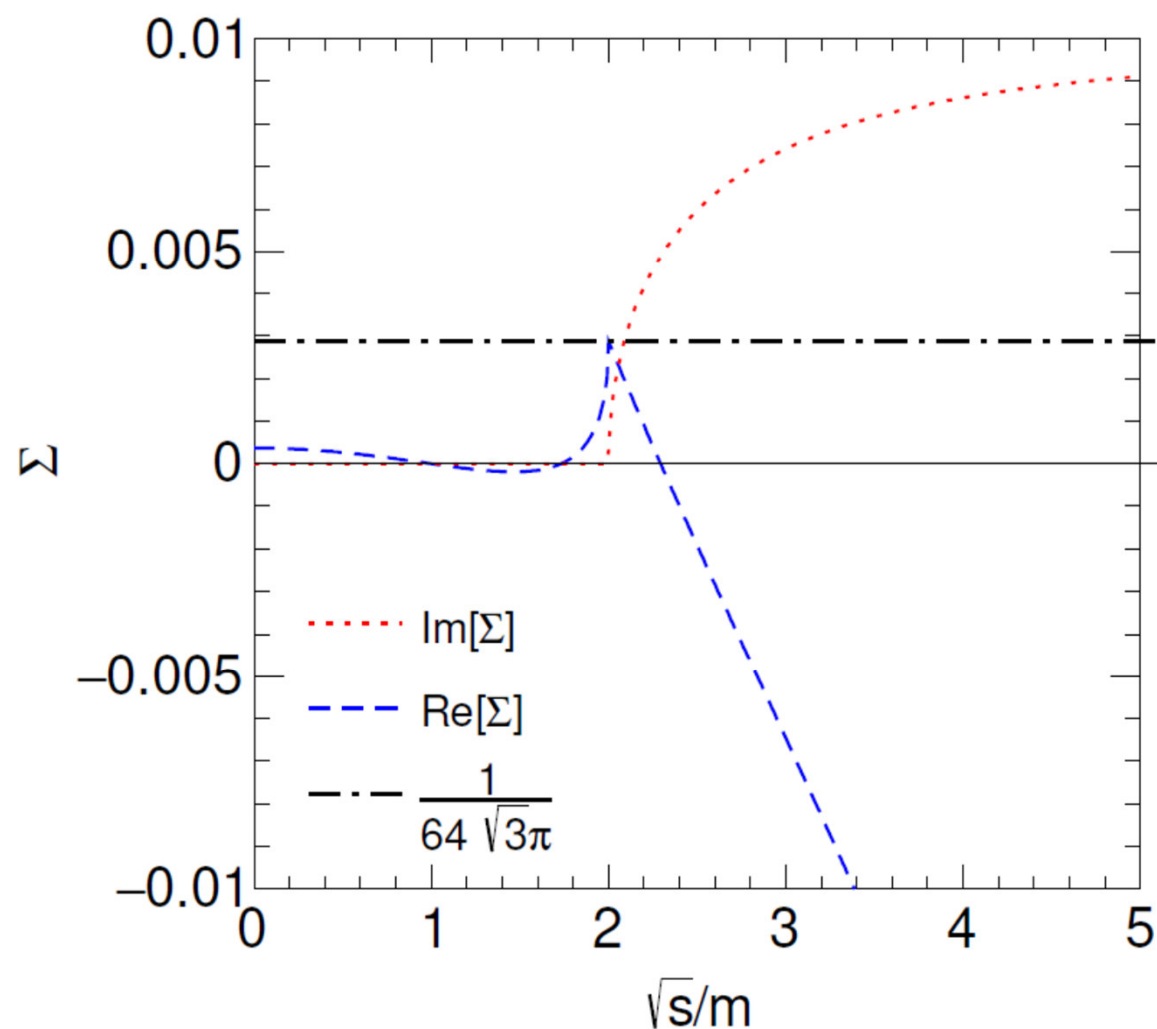
For $M < 2m$, S is stable

$$\Sigma(s) = -\frac{(s - M^2)(s - (4m^2 - M^2))}{\pi} \int_{4m^2}^{\infty} \frac{\frac{1}{2}\frac{\sqrt{\frac{s'}{4} - m^2}}{8\pi\sqrt{s'}}}{(s - s' + i\varepsilon)(s' - M^2)(s' - (4m^2 - M^2))} ds'$$

For $M > 2m$, S is a resonance
requirement: $Re[\Sigma(s = M^2)] = 0$
(note $Im[\Sigma(s)] \propto \Gamma_S$)

$$\Sigma(s) = -\frac{(s - (4m^2 - M^2))}{\pi} \int_{4m^2}^{\infty} \frac{\frac{1}{2}\frac{\sqrt{\frac{s'}{4} - m^2}}{8\pi\sqrt{s'}}}{(s - s' + i\varepsilon)(s' - (4m^2 - M^2))} ds' + C,$$