Universal inverse seesaw and radiative neutrino masses.

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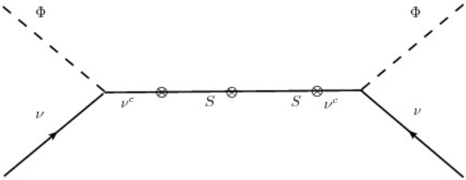
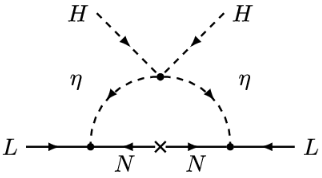
Introduction

Some reasons to extend the SM are:

1. SM charged fermion mass hierachy.
2. Tiny masses of light active neutrinos.
3. CKM and PMNS matrices very close and very different to the identity matrix, respectively.
4. Dark matter.
5. Number of SM fermion families.
6. Lepton asymmetry of the Universe.
7. $(g-2)_{e,\mu}$.
8. Lepton non universality.





Inverse seesaw

$$-\mathcal{L}_{mass}^{(\nu)} = \frac{1}{2}\begin{pmatrix} \overline{\nu_L^C} & \overline{N}_R & \overline{S}_R \end{pmatrix} \mathbf{M}_\nu \begin{pmatrix} \nu_L \\ N_R^C \\ S_R^C \end{pmatrix} + H.c$$

$$\mathbf{M}_\nu = \begin{pmatrix} 0_{3\times 3} & \mathbf{M}_1 & \mathbf{M_L} \\ \mathbf{M}_1^T & 0_{3\times 3} & \mathbf{M}_2 \\ \mathbf{M_L}^T & \mathbf{M}_2^T & \mu \end{pmatrix}$$

$$\mathbf{M_L} = 0_{3\times 3}$$

$$Q_{\nu_L}^{U(1)_L} = Q_{S_R}^{U(1)_L} = -Q_{N_R}^{U(1)_L} = 1$$

$$\widetilde{\mathbf{M}}_\nu = \mathbf{M}_1 \left(\mathbf{M}_2^T\right)^{-1} \mu \mathbf{M}_2^{-1} \mathbf{M}_1^T$$

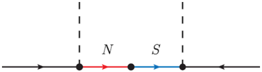
$$\mathbf{M}_\nu^{(1)} = -\frac{1}{2}\left(\mathbf{M}_2 + \mathbf{M}_2^T\right) + \frac{1}{2}\mu$$

$$\mathbf{M}_\nu^{(2)} = \frac{1}{2}\left(\mathbf{M}_2 + \mathbf{M}_2^T\right) + \frac{1}{2}\mu$$

One loop Ma radiative seesaw model

η and N are odd under a preserved Z_2

$$L\widetilde{\eta}N, \frac{\lambda_5}{2}\left(H^\dagger \cdot \eta\right)^2 + h.c$$



Linear seesaw:

$$\mu = 0_{3\times 3}$$

$$\widetilde{\mathbf{M}}_\nu = -\mathbf{M}_L \mathbf{M}_2^{-1} \mathbf{M}_1^T - \mathbf{M}_1 \left(\mathbf{M}_2^T\right)^{-1} \mathbf{M}_L^T$$

Universal Inverse seesaw mechanism.

The neutrino mass matrix for the IS in the basis $\left(\nu_L, \nu_R^C, N_R^C\right)$:

$$M_\nu = \begin{pmatrix} 0_{3\times 3} & m_{\nu D} & 0_{3\times 3} \\ m_{\nu D}^T & 0_{3\times 3} & M \\ 0_{3\times 3} & M^T & \mu \end{pmatrix}, \tag{1}$$

For charged fermions, in the basis $(\overline{f}_{1L}, \overline{f}_{2L}, \overline{f}_{3L}, \overline{F}_L, \overline{\widetilde{F}}_L)$-$(f_{1R}, f_{2R}, f_{3R}, F_R, \widetilde{F}_R)$ we might have:

$$M_F = \begin{pmatrix} 0_{3\times 3} & F_F & 0_{3\times 1} \\ G_F^T & 0 & X_F \\ 0_{1\times 3} & Y_F & m_F \end{pmatrix}, \tag{2}$$

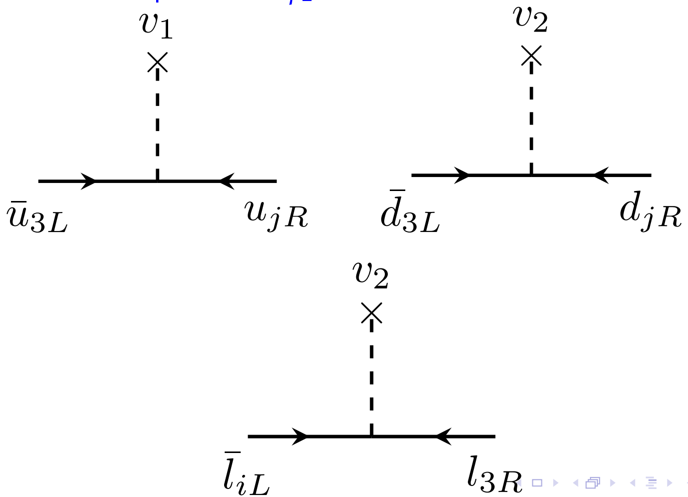
In the limit $m_F \to 0$ $(F = T, D, E)$

$$Q_{U(1)}\left(f_{iL}\right) = Q_{U(1)}\left(\widetilde{F}_L\right) = Q_{U(1)}\left(F_R\right) = a,$$
$$Q_{U(1)}\left(f_{iR}\right) = Q_{U(1)}\left(\widetilde{F}_R\right) = Q_{U(1)}\left(F_L\right) = b, \qquad a \neq b. \tag{3}$$

Extending the IS to the first and second families of SM charged fermions:

$$M_F = \begin{pmatrix} C_F + \Delta_F & F_F & 0_{3\times 1} \\ G_F^T & 0 & X_F \\ 0_{1\times 3} & Y_F & m_F \end{pmatrix}, \tag{4}$$

ϕ_1 gives mass to top whereas ϕ_2 to bottom and tau.



The second generation of SM charged fermions get their masses from the diagrams:

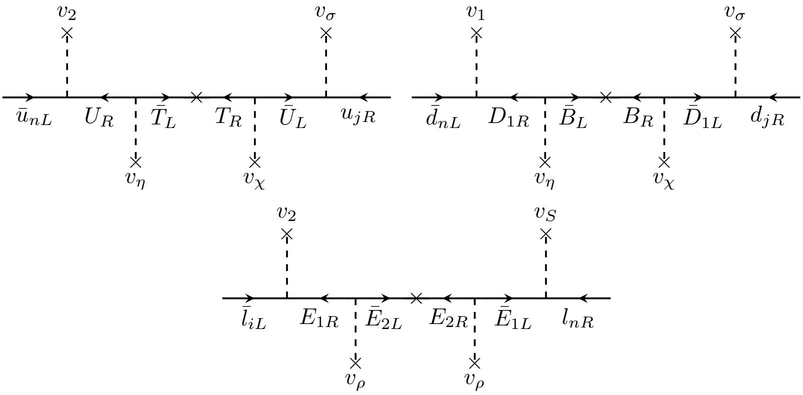


Figure: Tree level Feynman diagrams contributing to the entries of the SM charged fermion mass matrices. Here, $n = 1, 2$ and $i, j = 1, 2, 3$.

The subGeV mass scale of the second family of SM charged fermions, can be explained by $(F_Q)_n \left(G_Q^T\right)_i \sim (F_E)_i \left(G_E^T\right)_n \sim \mathcal{O}(10^{-2}\text{TeV}^2)$, $m_F \sim \mathcal{O}(10^{-1}\text{GeV})$

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_X$	Z_2	Z_4
ϕ_1	1	2	$\frac{1}{2}$	$\frac{1}{3}$	0	-1
ϕ_2	1	2	$\frac{1}{2}$	$\frac{2}{3}$	0	1
σ	1	1	0	$\frac{1}{3}$	0	-1
χ	1	1	0	$\frac{2}{3}$	0	-2
η	1	1	0	-1	0	2
ρ	1	1	0	2	0	-1
S	1	1	0	0	0	1
ζ_1^+	1	1	1	$\frac{2}{3}$	0	-1
ζ_2^+	1	1	1	1	0	0
ζ_3^+	1	1	1	$\frac{2}{3}$	0	1
φ_1	1	1	0	1	1	0
φ_2	1	1	0	0	1	1

Table: Scalar assignments under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times Z_2 \times Z_4$.

The $U(1)_X$ is local. The Z_4 is spontaneously broken whereas Z_2 is preserved. Notice that φ_1 and φ_2 are the only scalars charged under the preserved Z_2 symmetry.

The first family of SM charged fermions will get one loop level masses.

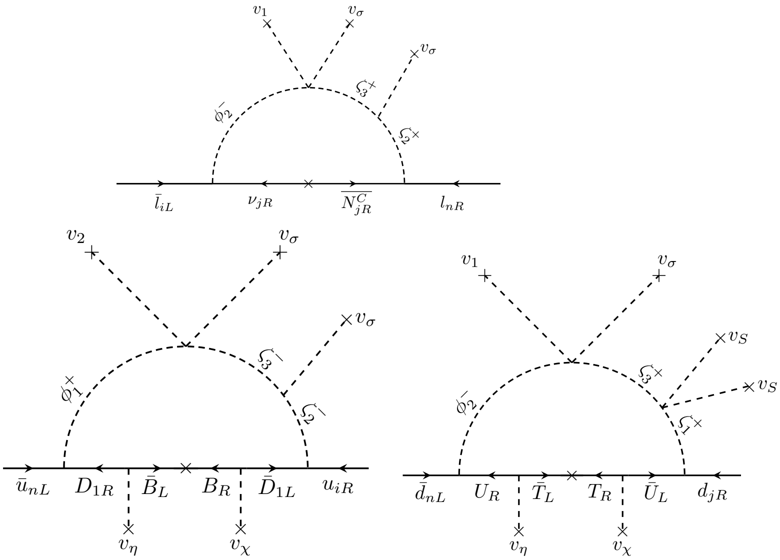


Figure: One loop Feynman diagrams contributing to the entries of the SM charged fermion mass matrices. Here, $n = 1, 2$ and $i, j = 1, 2, 3$.

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_X$	Z_2	Z_4
q_{nL}	3	2	$\frac{1}{6}$	0	0	1
q_{3L}	3	2	$\frac{1}{6}$	$\frac{1}{3}$	0	0
u_{iR}	3	1	$\frac{2}{3}$	$\frac{2}{3}$	0	1
d_{iR}	3	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	-1
U_L	3	1	$\frac{2}{3}$	$\frac{1}{3}$	0	2
U_R	3	1	$\frac{2}{3}$	$\frac{2}{3}$	0	2
T_L	3	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0
T_R	3	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0
D_{1L}	3	1	$-\frac{1}{3}$	0	0	2
D_{1R}	3	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	2
D_{2L}	3	1	$-\frac{1}{3}$	0	1	-1
D_{2R}	3	1	$-\frac{1}{3}$	$-\frac{1}{3}$	1	0
B_L	3	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	0
B_R	3	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	0

Table: Quark assignments under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times Z_2 \times Z_4$. Here $i = 1, 2, 3$.

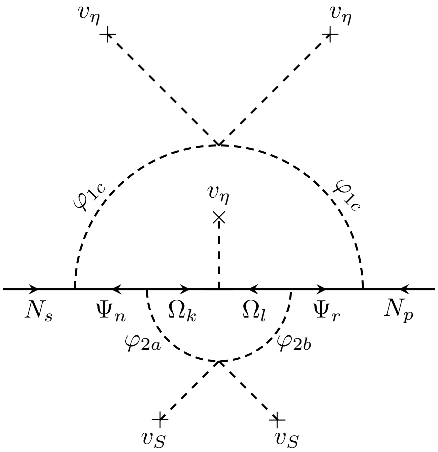


Figure: Two-loop Feynman diagram contributing to the Majorana neutrino mass submatrix μ. Here $n, k, l, r = 1, 2$, $s, p = 1, 2, 3$, $a, b, c = R, I$, with φ_{nR} and φ_{nI} corresponding to the CP even and CP odd parts of the scalar field φ_n, respectively.

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_X$	Z_2	Z_4
l_{iL}	1	2	$-\frac{1}{2}$	$-\frac{1}{3}$	0	1
l_{nR}	1	1	-1	-1	0	1
l_{3R}	1	1	-1	-1	0	0
E_{1L}	1	1	-1	-1	0	-2
E_{1R}	1	1	-1	-1	0	0
E_{2L}	1	1	-1	1	0	1
E_{2R}	1	1	-1	1	0	1
ν_{iR}^C	1	1	0	$-\frac{1}{3}$	0	0
N_{iR}	1	1	0	0	0	-1
Ψ_{nR}	1	1	0	1	1	1
Ω_{nR}	1	1	0	-1	0	0

Table: Lepton assigments under $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times Z_2 \times Z_4$. Here $i = 1, 2, 3$ and $n = 1, 2$.

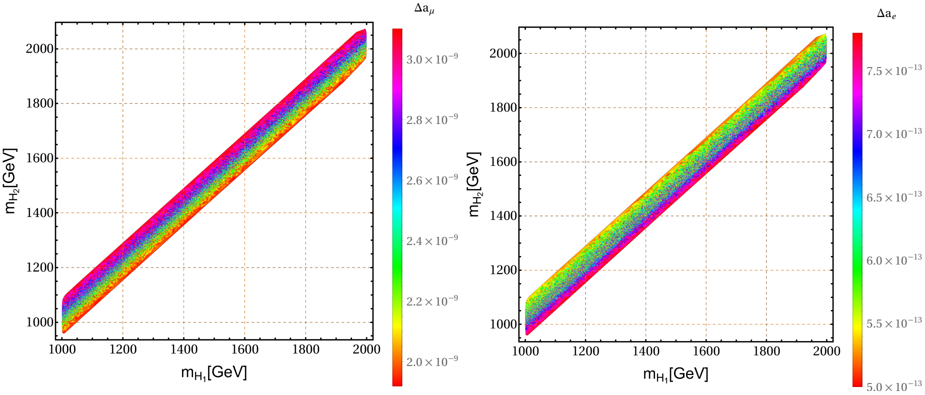


Figure: Correlation between the masses of the scalars H_1 and H_2 consistent with the muon and electron anomalous magnetic moments. Here $M_{A_1} = M_{A_2} = 1$ TeV, 0.5 TeV $\leqslant M_{H_i^\pm} \leqslant$ 1.5 TeV ($i = 1, 2, 3, 4$), 2 TeV $\leqslant m_{\widetilde{E}} \leqslant$ 3 TeV

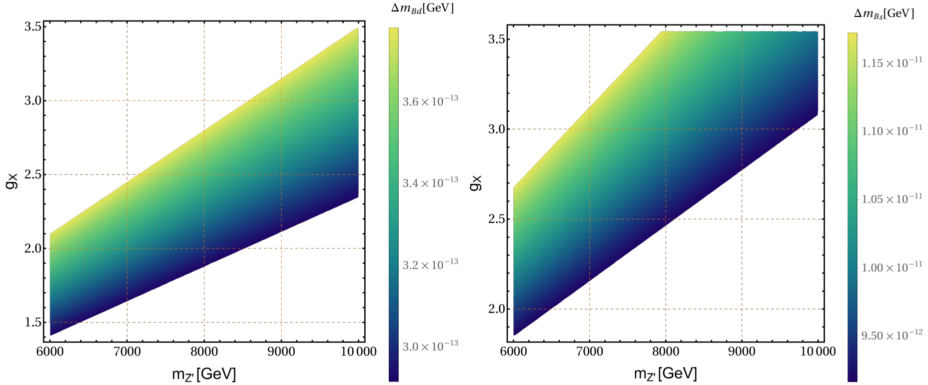


Figure: Allowed region in the $m_{Z'} - g_X$ plane consistent with the constraint arising from $B_d^0 - \bar{B}_d^0$ (left-plot) and $B_s^0 - \bar{B}_s^0$ (right-plot) mixings. Here we fix the couplings of the flavor violating neutral Yukawa interactions as 2×10^{-4} and 10^{-3} for the left and right plots, respectively. Furthermore, we have set $M_{H_1} = 1.2$ TeV, $M_{H_2} = 1.3$ TeV, $M_{A_1} = M_{A_2} = 1$ TeV.

$K^0 - \bar{K}^0$ mixing constraint is also fulfilled for a flavor violating Yukawa coupling of the order of 0.5×10^{-5}

The scalar DM φ_2 mainly annihilates into W^+W^-, ZZ, $t\bar{t}$, $b\bar{b}$, HH via a scalar portal interaction. The relic density is:

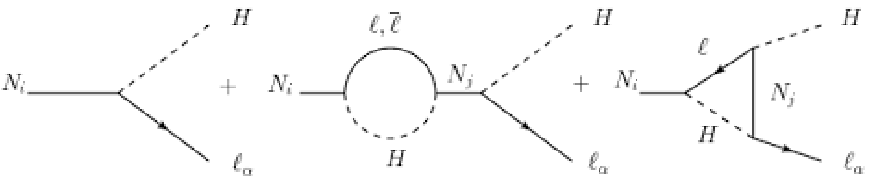
$$\Omega h^2 \simeq 0.1 \left(\frac{m_{\varphi_2}}{\lambda_{eff} \times 1.354\text{TeV}} \right)^2 , \tag{5}$$

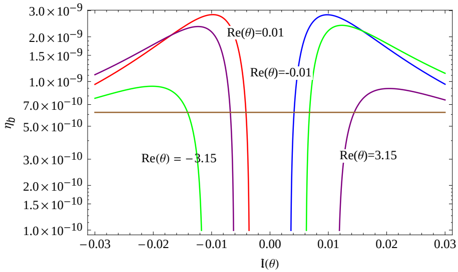
If the effective coupling is in the range $0.5 < \lambda_{eff} < 1.5$, the dark matter mass satisfies $0.75\text{TeV} < m_{\varphi_2} < 2.25\text{TeV}$.
Ψ_n annihilates into SM particles through the exchange of a new gauge boson. The relic density is given as

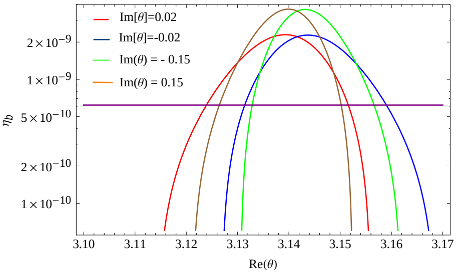
$$\Omega_\Phi h^2 \simeq 0.1\text{pb} \left(\frac{\alpha}{150\text{GeV}} \right)^{-2} \left(\frac{m_\Psi}{2.86\text{TeV}} \right)^2 , \tag{6}$$

where $\left(\frac{\alpha}{150\text{GeV}}\right)^2 \simeq 1\text{pb}$.

To study the lepton asymmetry we assume that the fermions Ψ_{nR} are heavier than the lightest pseudo-Dirac pair, $(N^{\pm})$. Thus, the lepton asymmetry parameter, is induced by decay process of $N^{\pm}$.







Scotogenic neutrino masses with GCU

Field	$SU(3)_c$	$SU(3)_L$	$U(1)_X$	$U(1)_N$	Q	$M_P = (-1)^{3(B-L)+2s}$
q_{iL}	$\mathbf{3}$	$\overline{\mathbf{3}}$	0	0	$(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})^T$	$(++-)^T$
q_{3L}	$\mathbf{3}$	$\mathbf{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})^T$	$(++-)^T$
u_{aR}	$\mathbf{3}$	$\mathbf{1}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$+$
d_{aR}	$\mathbf{3}$	$\mathbf{1}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$+$
U_{3R}	$\mathbf{3}$	$\mathbf{1}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	$-$
D_{iR}	$\mathbf{3}$	$\mathbf{1}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-$
l_{aL}	$\mathbf{1}$	$\mathbf{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$(0, -1, 0)^T$	$(++-)^T$
e_{aR}	$\mathbf{1}$	$\mathbf{1}$	-1	-1	-1	$+$
ν_{iR}	$\mathbf{1}$	$\mathbf{1}$	0	-4	0	$-$
ν_{3R}	$\mathbf{1}$	$\mathbf{1}$	0	5	0	$+$
Ω_{aL}	$\mathbf{1}$	$\mathbf{8}$	0	0	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} - & - & + \\ - & - & + \\ + & + & - \end{pmatrix}$
η	$\mathbf{1}$	$\mathbf{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$(0, -1, 0)^T$	$(++-)^T$
ρ	$\mathbf{1}$	$\mathbf{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$(1, 0, 1)^T$	$(++-)^T$
χ	$\mathbf{1}$	$\mathbf{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$(0, -1, 0)^T$	$(--+)^T$
ϕ	$\mathbf{1}$	$\mathbf{1}$	0	2	0	$+$
σ	$\mathbf{1}$	$\mathbf{1}$	0	1	0	$-$

Table: 3311 model field content ($a = 1, 2, 3$ and $i = 1, 2$ are family indices).

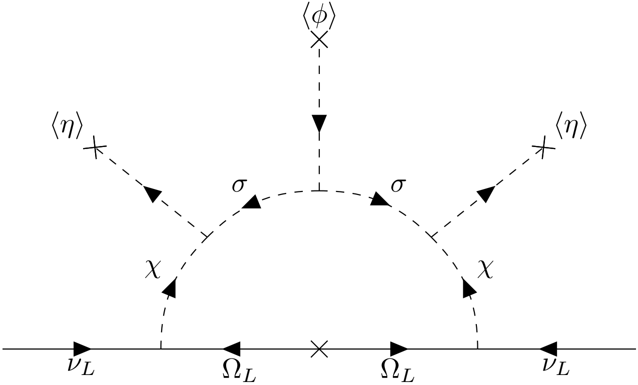


Figure: Feynman-loop diagram contributing to the light active Majorana neutrino mass matrix.

In the limit where the trilinear scalar interactions $\phi^\dagger\sigma^2$ and $(\eta^\dagger\chi)\sigma$ are absent, the model Lagrangian has an accidental $U(1)$ symmetry under which ϕ and σ have the same charge whereas the remaining fields are neutral under this symmetry.

$$Q = T_3 - \frac{T_8}{\sqrt{3}} + X, \qquad B - L = -\frac{2}{\sqrt{3}} T_8 + N, \tag{7}$$

$$q_{iL} = \begin{pmatrix} d_i \\ -u_i \\ D_i \end{pmatrix}_L \qquad q_{3L} = \begin{pmatrix} u_3 \\ d_3 \\ U_3 \end{pmatrix}_L \qquad l_{aL} = \begin{pmatrix} \nu_a \\ e_a \\ N_a \end{pmatrix}_L , \tag{8}$$

The gauged $B - L$ symmetry is spontaneously broken leaving a discrete remnant symmetry $M_P = (-1)^{3(B-L)+2s}$.

$$\begin{aligned} \langle \eta \rangle &= \frac{1}{\sqrt{2}} (v_1, 0, 0)^T, \quad \langle \rho \rangle = \frac{1}{\sqrt{2}} (0, v_2, 0)^T, \quad \langle \chi \rangle = (0, 0, w)^T, \\ \langle \phi \rangle &= \frac{1}{\sqrt{2}} \Lambda, \quad \langle \sigma \rangle = 0. \end{aligned} \tag{9}$$

We assume $w, \Lambda, \gg v_1, v_2$, such that the SSB pattern of the model is

$$\begin{array}{c} SU(3)_C \times SU(3)_L \times U(1)_X \times U(1)_N \\ \downarrow w, \Lambda \\ SU(3)_C \times SU(2)_L \times U(1)_Y \times M_P \\ \downarrow v_1, v_2 \\ SU(3)_C \times U(1)_Q \times M_P . \end{array} \tag{10}$$

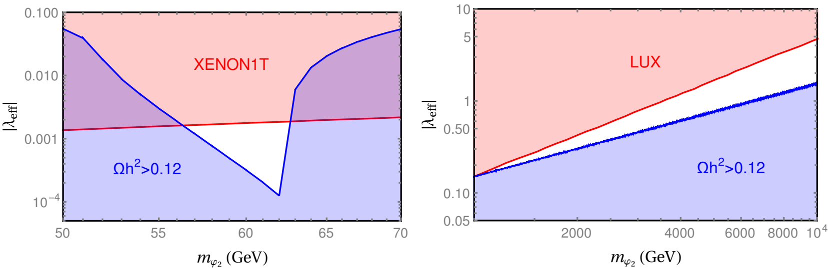


Figure: Viable mass regions where the field φ_2 of the simplified model described in the text behaves as a dark matter candidate. The red regions correspond to the current direct detection limits The blue regions represent values of the effective coupling $\lambda_{\rm eff}$ where the corresponding relic density is incompatible with the Planck measurement.

For the case of fermionic DM candidate, one has:

$$< \sigma v > \approx \left(\frac{\alpha}{150\,\text{GeV}}\right)^2 \left(\frac{M_\Omega}{3\,\text{TeV}}\right)^2 \approx \left(\frac{M_\Omega}{3\,\text{TeV}}\right)^2 \text{pb}, \tag{11}$$

$$\Omega_{DM} h^2 = \frac{0.1\,\text{pb}}{< \sigma v >}, \tag{12}$$

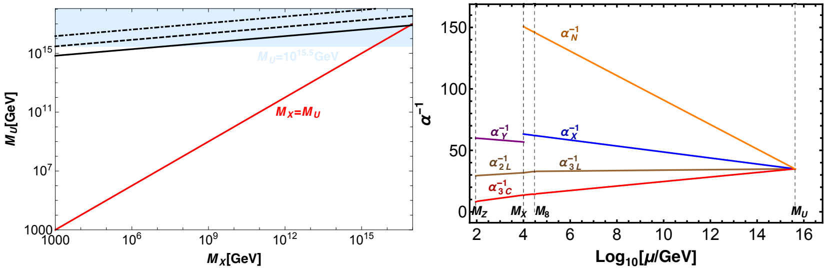


Figure: Left) Unification scale M_U as a function of the 3-3-1-1 symmetry breaking scale M_X, for three benchmark choices $M_8 = M_X$ (solid curve), $M_8 = 3M_X$ (dashed curve) and $M_8 = 10M_X$ (dot-dashed curve). (Right) An example of $\mathrm{SU(3)_c \times SU(3)_L \times U(1)_X \times U(1)_N}$ unification for a phenomenologically accessible 3-3-1-1 symmetry breaking scale $M_X = 10$ TeV and $M_8 = 3M_X = 30$ TeV.

Conclusions

- Inverse seesaw mechanism can explain the SM fermion mass hierarchy.
- Fermion masses and mixings, DM, $(g-2)_{e,\mu}$ anomalies, lepton and baryon asymmetries can be accounted for.
- Dark matter stability can arise from a residual matter-parity symmetry.
- Leptonic $SU(3)_L$ octets allow GCU and one loop scotogenic neutrino generation. DM can also be accounted for.

Acknowledgements

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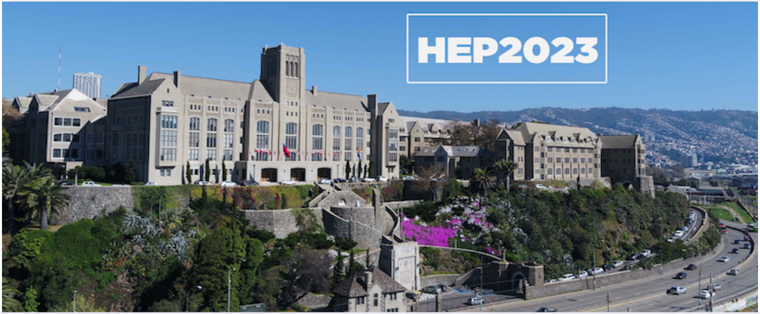

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Extra Slides

Thus, the lepton asymmetry parameter, is induced by decay process of $N^{\pm}$:

$$\begin{aligned} \epsilon_{\pm} &= \sum_{\alpha=1}^{3} \frac{\left\{\Gamma\left(N_{\pm} \to l_{\alpha} H^{+}\right) - \Gamma\left(N_{\pm} \to \bar{l}_{\alpha} H^{-}\right)\right\}}{\left\{\Gamma\left(N_{\pm} \to l_{\alpha} H^{+}\right) + \Gamma\left(N_{\pm} \to \bar{l}_{\alpha} H^{-}\right)\right\}} \\ &\simeq \frac{\Im\left[\left(y^{\nu}\right)^{\dagger} y^{\nu} \left(y^{\nu}\right)^{\dagger} y^{\nu}\right]_{11}}{8\pi A_{\pm}} \frac{r}{r^{2} + \frac{\Gamma_{\mp}^{2}}{m_{N_{\mp}}^{2}}}, \end{aligned} \tag{13}$$

where $r \equiv \frac{m_{N_{+}}^{2} - m_{N_{-}}^{2}}{m_{N_{+}} m_{N_{-}}}$, $\quad A_{\pm} = \left(\left(y^{\nu}\right)^{\dagger} y^{\nu}\right)_{11}$, $\quad \Gamma_{\pm} \equiv \frac{A_{\pm} m_{N_{\pm}}}{8\pi}$.

$$\eta_B = \frac{\epsilon_{N_\pm}}{g_*} \quad \text{for} \quad K^{eff}_{N_\pm} \ll 1,$$
$$\eta_B = \frac{0.3\epsilon_{N_\pm}}{g_* K^{eff}_{N_\pm} (\ln K^{eff}_{N_\pm})^{0.6}} \quad \text{for} \quad K^{eff}_{N_\pm} \gg 1. \tag{14}$$

Here, $g_* \simeq 118$ defined as

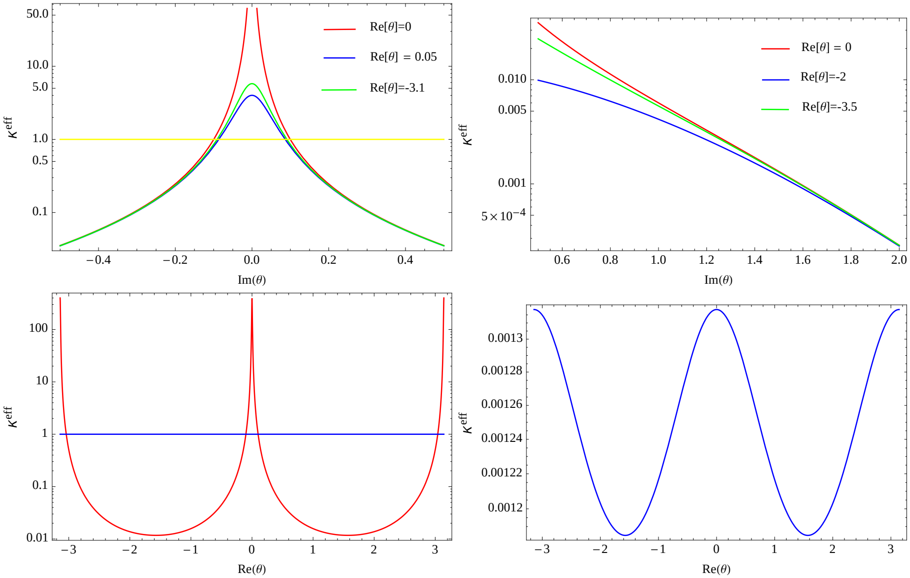
$$K^{eff}_{N_\pm} \simeq \left(\frac{\Gamma_+ + \Gamma_-}{H}\right)\left(\frac{m_{N^+} - m_{N^-}}{\Gamma_\pm}\right)^2 \tag{15}$$

where $\mathrm{H} = \sqrt{\frac{4\pi^3 g_*}{45}} \frac{T^2}{M_{Pl}}$ is the Hubble constant.

$$y^\nu = \frac{v_\sigma}{v_2}\left(\mathrm{U_{PMNS}} \mathrm{M}_\nu^{\frac{1}{2}} \mathrm{R} \mu^{-\frac{1}{2}} y^N\right). \tag{16}$$

$$R = \begin{pmatrix} c_y c_z & -s_x c_z s_y - c_x s_z & s_x s_z - c_x s_y c_z \\ c_y s_z & c_x c_z - s_x s_y s_z & -c_z s_x - c_x s_y s_z \\ s_y & s_x c_y & c_x c_y \end{pmatrix}, \tag{17}$$

$c_x = \cos x, s_x = \sin x$ and we set $x = 0, y = z = \Re[\theta] + i\Im[\theta]$,
$Y^{(N)} = \mathrm{Diag}(0.5, 0.9\mathrm{i}, 1.8)$, $v_2 = 24.6\mathrm{GeV}$, $v_\sigma \simeq 5 \times 10^3 \mathrm{GeV}$. Then $\mu \simeq 1\mathrm{keV}$.



Re[θ]=0
Re[θ] = 0.05
Re[θ]=-3.1
50.0
10.0
5.0
1.0
0.5
0.1
κ^eff
−0.4
−0.2
0.0
0.2
0.4
Im(θ)
Re[θ] = 0
Re[θ]=-2
Re[θ]=-3.5
0.010
0.005
0.001
5 × 10^−4
κ^eff
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Im(θ)
100
10
1
0.1
0.01
κ^eff
−3
−2
−1
0
1
2
3
Re(θ)
0.0013
0.00128
0.00126
0.00124
0.00122
0.0012
κ^eff
−3
−2
−1
0
1
2
3
Re(θ)

For the case of two heavy seesaw mediators, one has:

$$M = \begin{pmatrix} F_1 G_1 + X_1 Y_1 & F_1 G_2 + X_1 Y_2 & F_1 G_3 + X_1 Y_3 \\ F_2 G_1 + X_2 Y_1 & F_2 G_2 + X_2 Y_2 & F_2 G_3 + X_2 Y_3 \\ F_3 G_1 + X_3 Y_1 & F_3 G_2 + X_3 Y_2 & F_3 G_3 + X_3 Y_3 \end{pmatrix}, \tag{18}$$

$$M_{ij} = M^i_j = F_i G_j + X_i Y_j = F^i G_j + X^i Y_j \tag{19}$$

$$F^i = F_i, \qquad X^i = X_i \tag{20}$$

$$\det\left(M^i_j\right) = \frac{1}{3!}\varepsilon_{j_1 j_2 j_3}\varepsilon^{i_1 i_2 i_3} M^{j_1}_{i_1} M^{j_2}_{i_2} M^{j_3}_{i_3}, \qquad M^i_j = M_{ij} \tag{21}$$

$$\begin{aligned} \det\left(M^i_j\right) &= \frac{1}{3!}\varepsilon_{j_1 j_2 j_3}\varepsilon^{i_1 i_2 i_3}\left(F^{j_1} G_{i_1} + X^{j_1} Y_{i_1}\right) \\ &\times \left(F^{j_2} G_{i_2} + X^{j_2} Y_{i_2}\right)\left(F^{j_3} G_{i_3} + X^{j_3} Y_{i_3}\right) = 0 \end{aligned} \tag{22}$$

For the case of three fermionic seesaw mediators, one has:

$$M_{ij} = M^i_j = F_i G_j + X_i Y_j + R_i S_j = F^i G_j + X^i Y_j + R^i S_j \tag{23}$$

Then, it follows that:

$$\det\left(M^i_j\right) = \varepsilon_{j_1 j_2 j_3}\varepsilon^{i_1 i_2 i_3} F^{j_1} G_{i_1} X^{j_2} Y_{i_2} R^{j_3} S_{i_3} \neq 0 \tag{24}$$

provided that:

$$G_{i_1} \neq Y_{i_2} \neq S_{i_3}, \qquad F^{j_1} \neq X^{j_2} \neq R^{j_3} \tag{25}$$

Therefore, we have shown that in order to generate the masses of three fermion families via a seesaw mechanism, there should be at least three fermionic seesaw mediators. Furthermore, the number of the massless states obtained in a mass matrix resulting from a seesaw mechanism is $3 - n$, where n is the number of fermionic seesaw mediators.