Fermion Mass Hierarchies and Modulus Stabilisation in Modular-Invariant Models of Flavour

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Purpose: The Flavour Problem

Understanding the origins of flavour in both quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing and of CP violation in the quark and lepton sector, is one of the most challenging fundamental problems in contemporary particle physics.

"Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."

From Model Physicist, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 24 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

In what follows we will discuss a new approach to the flavour problem within the three family framework.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

- **Why** $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$ **(**$m_{\nu_j} \lesssim 0.5$ **eV,** $m_l \geq 0.511$ **MeV,** $m_q \gtrsim 2$ **MeV);**

- **The origins of the patterns of**
i) neutrino mixing of 2 large and 1 small angles ($\theta^l_{12} = 33.65^\circ$, $\theta^l_{23} = 47.1^\circ$, $\theta^l_{13} = 8.49^\circ$**),**
and of ii) Δm^2_{ij}**, i.e., of** $\Delta m^2_{21} \ll |\Delta m^2_{31}|$, $\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30$.

- **The origin of the hierarchical pattern of charged lepton masses:**
$m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The quark Flavour Problem

Consists of two basic elements (sub-problems), namely, understanding:

- **The origin(s) of the observed patterns of up- and down-type quark masses characterized by strong hierachies.**

$$m_d \ll m_s \ll m_b\,,\ \frac{m_d}{m_s} = 5.02 \times 10^{-2}\,,\ \frac{m_s}{m_b} = 2.22 \times 10^{-2}\,,\ m_b = 4.18\ \text{GeV};$$

$$m_u \ll m_c \ll m_t\,,\ \frac{m_u}{m_c} = 1.7 \times 10^{-3}\,,\ \frac{m_c}{m_t} = 7.3 \times 10^{-3}\,,\ m_t = 172.9\ \text{GeV};$$

- **The origin of the pattern of the quark mixing: the three quark mixing angles are small and hierarchical,** $\sin\theta^q_{13} \ll \sin\theta^q_{23} \ll \sin\theta^q_{12} \ll 1$**,** $\sin\theta^q_{12} \cong 0.22$**.**

Each of the considered sub-problems of the lepton and qaurk flavour problems is by itself a formidable problem. As a consequence, solutions to each individual problem have been proposed. However, a universal "elegant and convincing" solution, i.e., solution without significant "drawbacks", to the lepton and quark flavour problems is still lacking.

The Flavour Problem: Modular Invariance Approach

Modular invariance approach to the flavour problem was proposed in F. Feruglio, arXiv:1706.08749 and has been intensively developed in the last four years.

In this approach the flavour (modular) symmetry is broken by the vacuum expectation value (VEV) of a single scalar field - the modulus τ. The VEV of τ can also be the only source of violation of the CP symmetry.

Many (if not all) of the drawbacks of the widely studied alternative approaches are absent in the modular invariance approach to the flavour problem.

The first phenomenologically viable "minimal" (in terms of fields, i.e., without flavons) lepton flavour model based on modular symmetry appeared in June of 2018 (J.T. Penedo, STP, arXiv:1806.11040). Since then various aspects of this approach were and continue to be extensively studied – the number of publications on the topic exceeds 150.

The talk is based on the first two of the following articles.

1. P.P. Novichkov, J.T. Penedo and S.T. Petcov, "Fermion Mass Hierarchies, Large Lepton Mixing and Residual Modular Symmetries", JHEP 2104 (2021) 206 [arXiv:2102.07488].

2. P.P. Novichkov, J.T. Penedo and S.T. Petcov, "Modular Flavour Symmetries and Modulus Stabilisation", JHEP 2203 (2022) 149 [arXiv:2201.02020].

3. P.P. Novichkov, J.T. Penedo and S.T. Petcov, "Double cover of modular S_4 for flavour model building", Nucl. Phys. B 963 (2021) 115301 [arXiv:2006.03058].

4. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Generalised CP Symmetry in Modular-Invariant Models of Flavour", JHEP 1907 (2019) 165 [arXiv:1905.11970].

5. P.P. Novichkov, S.T. Petcov and M. Tanimoto, "Trimaximal Neutrino Mixing from Modular A4 Invariance with Residual Symmetries," Phys. Lett. B 793 (2019) 247 [arXiv:1812.11289].

6. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Modular A_5 symmetry for flavour model building," JHEP 1904 (2019) 174 [arXiv:1812.02158].

7. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Modular S_4 models of lepton masses and mixing," JHEP 1904 (2019) 005 [arXiv:1811.04933].

8. J.T. Penedo and S.T. Petcov, "Lepton Masses and Mixing from Modular S_4 Symmetry," Nucl. Phys. B 939 (2019) 292 [arXiv:1806.11040].

Matter Fields and Modular Forms

The matter(super)fields (charged lepton, neutrino, quark) transform under $\overline{\Gamma} \simeq PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{I, -I\}$ ($\Gamma \simeq SL(2,\mathbb{Z})$) as "weighted" multiplets:

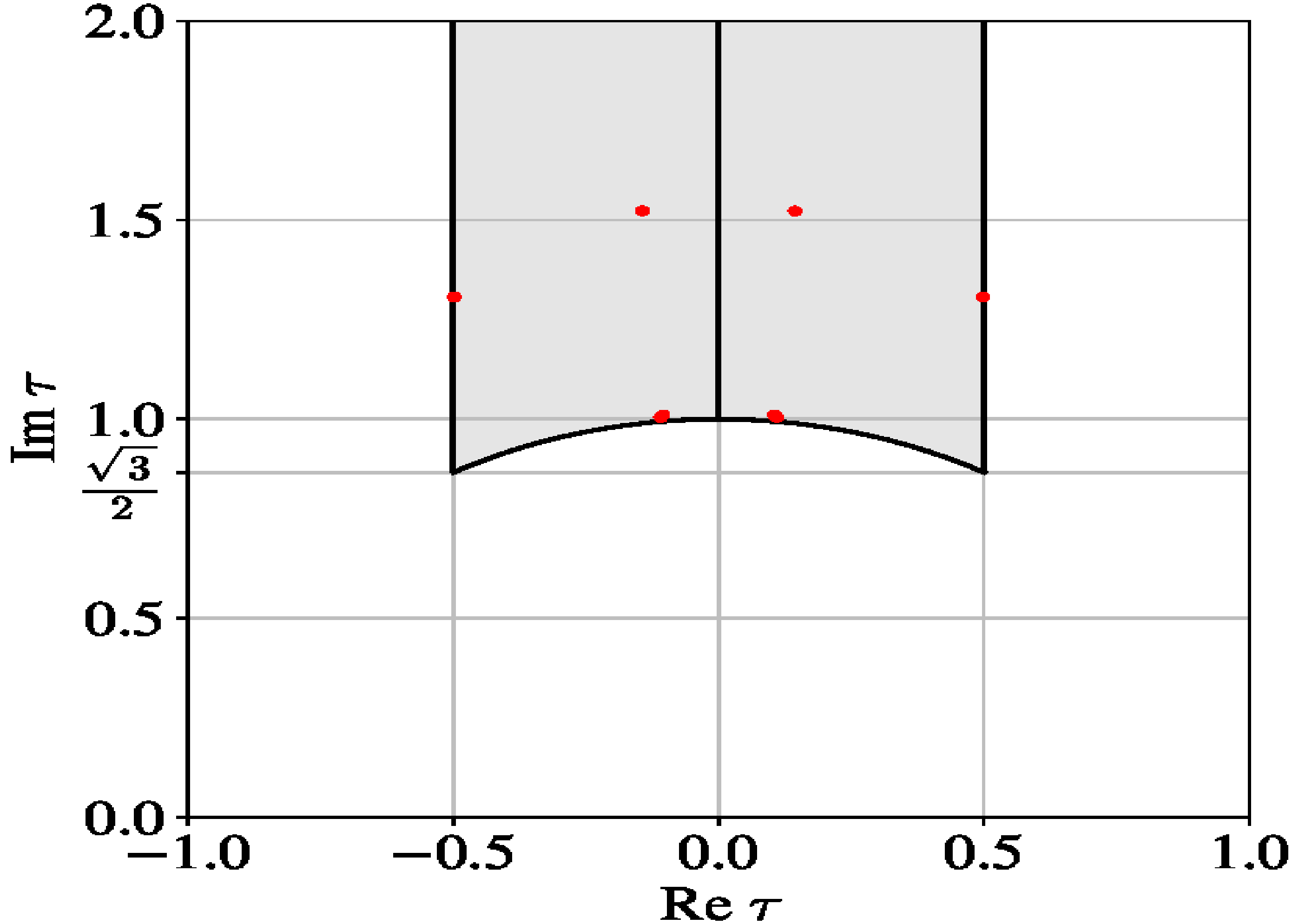
$$\psi_i \xrightarrow{\gamma} (c\tau + d)^{-k_\psi}\, \rho_{ij}(\tilde{\gamma})\psi_j\,, \gamma \in \overline{\Gamma}\ \ (\gamma \in \Gamma)\,,$$

$$\left(\gamma\tau = \frac{a\tau+b}{c\tau+d}\,, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, a,b,c,d \in \mathbb{Z}\,, ad - bc = 1\,, \mathrm{Im}\tau > 0\right)$$

k_ψ **is the weight of** ψ; $k_\psi \in \mathbb{Z}$ **(or rational number).**
$\Gamma(N)$ **- principal congruence (normal) subgroup of** $SL(2,\mathbb{Z})$.
$\rho(\tilde{\gamma})$ **is a unitary representation of the** *inhomogeneous (homogeneous)* **finite modular group** $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ **(**$\Gamma'_N = \Gamma/\Gamma(N)$**),** $\tilde{\gamma}$ **− representation of** γ **in** Γ_N **(**Γ'_N**)**

F. Feruglio, arXiv:1706.08749; S. Ferrara et al., Phys.Lett. B233 (1989) 147, B225 (1989) 363

As we have indicated in brackets, one can consider also the case of Γ **and** $\gamma \in \Gamma(N)$**. Then** $\rho(\gamma)$ **will be a unitary representation of the** *homogeneous* **finite modular group** Γ'_N**.**



The Fundamental Domain of $\overline{\Gamma}$ shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Remarkably, for $N \leq 5$, the inhomogeneous finite modular groups Γ_N are isomorphic to non-Abelian discrete groups widely used in flavour model building:
$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$.
Γ_N is presented by two generators S and T satisfying:

$$S^2 = (ST)^3 = T^N = I\,.$$

The group theory of $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$ is summarized, e.g., in P.P. Novichkov *et al.*, JHEP 07 (2019) 165, arXiv:1905.11970.

$\Gamma \simeq SL(2,\mathbb{Z})$ – *homogeneous* **modular group, $\Gamma(N)$ and the quotient groups $\Gamma'_N \equiv \Gamma/\Gamma(N)$ – homogeneous finite modular groups. For $N = 3, 4, 5$, Γ'_N are isomorphic to the double covers of the corresponding non-Abelian discrete groups:**
$\Gamma'_3 \simeq A'_4 \equiv T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$.
Γ'_N is presented by two generators S and T satisfying:

$$S^4 = (ST)^3 = T^N = I\,,\;\; S^2\,T = T\,S^2 \quad (S^2 = R)\,.$$

The group theory of $\Gamma'_3 \simeq A'_4$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$ for flavour model building was developed in X.-G. Liu, G.-J. Ding, arXiv:1907.01488 (A'_4);
P.P. Novichkov et al., arXiv:2006.03058 (S'_4); C.-Y. Yao et al., arXiv:2011.03501 (A'_5).

Relevant sub-groups of Γ_N and Γ'_N: :

$\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$

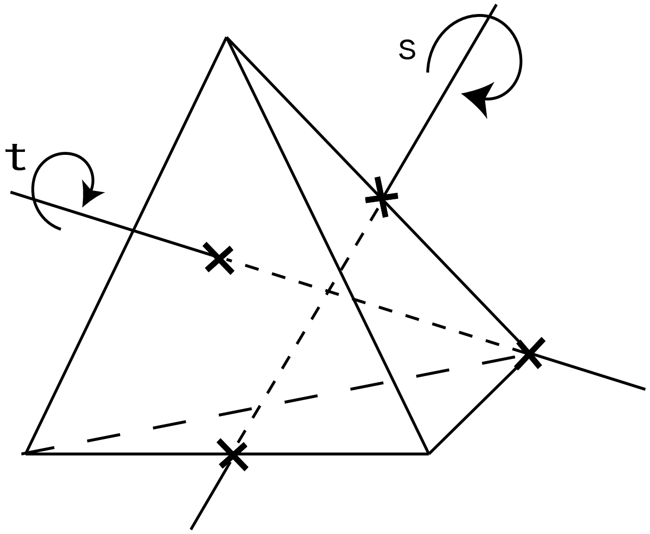
$\mathbb{Z}_N^{T} = \{I, T, (T)^2, ..., T^{N-1}\}$

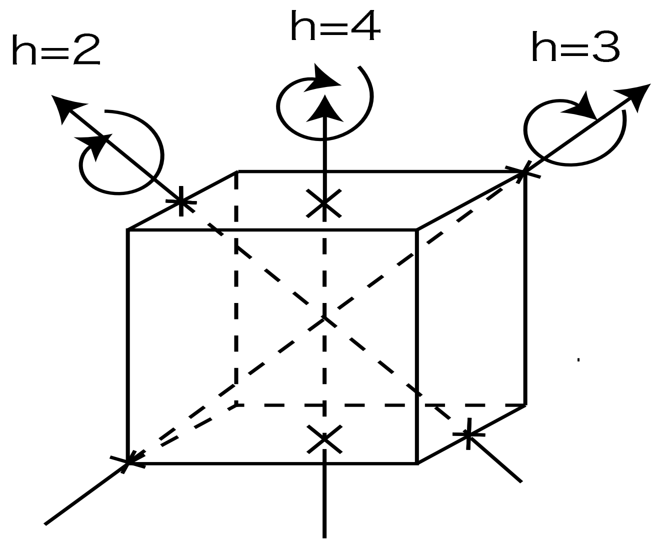
Γ_N: $\mathbb{Z}_2^{S} = \{I, S\}$

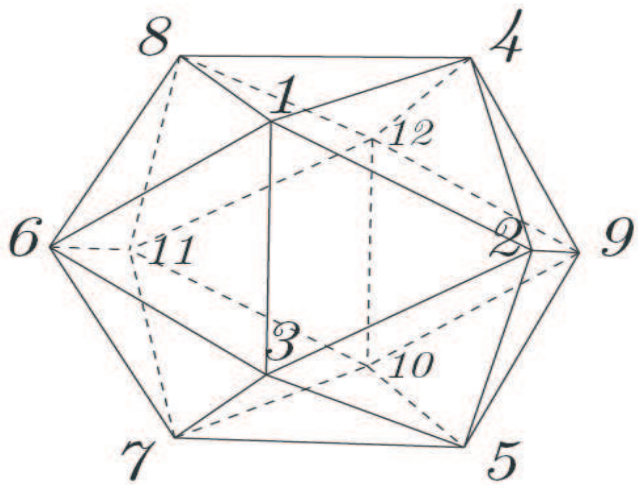
Γ'_N: $\mathbb{Z}_4^{S} = \{I, S, S^2, S^3\}$ ($R^2 = I$, $\mathbb{Z}_2^{R} = \{I, R\}$, $R\tau = \tau$)

Group	Number of elements	Generators	Irreducible representations
S_4	24	$S,\ T\ (U)$	$\mathbf{1},\ \mathbf{1}',\ \mathbf{2},\ \mathbf{3},\ \mathbf{3}'$
S'_4	48	$S,\ T\ (R)$	$\mathbf{1},\ \mathbf{1}',\ \mathbf{2},\ \mathbf{3},\ \mathbf{3}',\ \hat{\mathbf{1}},\ \hat{\mathbf{1}}',\ \hat{\mathbf{2}},\ \hat{\mathbf{3}},\ \hat{\mathbf{3}}'$
A_4	12	$S,\ T$	$\mathbf{1},\ \mathbf{1}',\ \mathbf{1}'',\ \mathbf{3}$
T'	24	$S,\ T\ (R)$	$\mathbf{1},\ \mathbf{1}',\ \mathbf{1}'',\ \mathbf{2},\ \mathbf{2}',\ \mathbf{2}'',\ \mathbf{3}$
A_5	60	$\tilde{S},\ \tilde{T}$	$\mathbf{1},\ \mathbf{3},\ \mathbf{3}',\ \mathbf{4},\ \mathbf{5}$
A'_5	120	$\tilde{S},\ \tilde{T}$	$\mathbf{1},\ \mathbf{3},\ \mathbf{3}',\ \mathbf{4},\ \mathbf{5},\ \hat{\mathbf{2}},\ \hat{\mathbf{2}}',\ \hat{\mathbf{4}},\ \hat{\mathbf{6}}.$

Number of elements, generators and irreducible representations of S_4, S'_4, A_4, $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.







Examples of symmetries: A_4, S_4, A_5.

From M. Tanimoto et al., arXiv:1003.3552

Modular Forms

Within the considered framework the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are expressed in terms of modular forms of a certain level N and weight k_f.

The modular forms are functions of a single complex scalar field – the modulus τ – and have specific transformation properties under the action of the modular group.

Both the Yukawa couplings and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma^{(\prime)}_N$.

Once τ acquires a VEV, the modular forms and thus the Yukawa couplings and the form of the mass matrices get fixed, and a certain flavour structure arises.

Quantitatively and barring fine-tuning, the magnitude of the values of the non-zero elements of the fermion mass matrices and therefore the fermion mass ratios are determined by the modular form values (which in turn are functions of the τ's VEV).

Modular Forms (contd.)

The key elements of the considered framework are modular forms $f(\tau)$ of *weight* k_f *and level* N **− holomorphic functions of τ, which transform under $\overline{\Gamma}$ (Γ) as follows:**

$$f\,(\gamma\tau) = (c\tau + d)^{k_f}\ f(\tau)\,, \quad \gamma \in \overline{\Gamma(N)}\ (\gamma \in \Gamma(N))\,,$$

In the case of $\overline{\Gamma}$ (Γ) *non-trivial* **modular forms exist only for positive even integer (positive integer) weight k_f.**
For given k, N (N is a natural number), the modular forms span a linear space of finite dimension:
of weight k and level 3, $\mathcal{M}_k(\Gamma_3^{(\prime)} \simeq A_4^{(\prime)})$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4^{(\prime)} \simeq S_4^{(\prime)})$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5^{(\prime)} \simeq A_5^{(\prime)})$, is $5k+1$.
Thus, $\dim\mathcal{M}_1(\Gamma'_3 \simeq A'_4) = 2$, $\dim\mathcal{M}_1(\Gamma'_4 \simeq S'_4) = 3$, $\dim\mathcal{M}_1(\Gamma'_5 \simeq A'_5) = 6$.
One can find a basis $F(\tau) \equiv (f_1(\tau), f_2(\tau), \ldots)^T$ in each of these spaces such that for any $\gamma \in \overline{\Gamma}$ ($\gamma \in \Gamma$), $F(\gamma\tau)$ belongs to the same space and transforms according to a unitary irreducible representation $\mathbf{r}$ of Γ_N (Γ'_N):

$$F\,(\gamma\tau) = (c\tau + d)^{k_F}\,\rho_{\mathbf{r}}(\tilde{\gamma})F(\tau)\,, \quad \gamma \in \overline{\Gamma} \quad (\gamma \in \Gamma)\,.$$

This result is at the basis of the modular invariance approach to the flavour problem proposed in F. Feruglio, arXiv:1706.08749.

Following arXiv:1706.08749, it was of highest priority and of crucial importance for model building to find the basis of modular forms of the lowest weight 2 (weight 1) transforming in irreps of Γ_N (Γ'_N).

Multiplets of Γ_N (Γ'_N) of higher weight modular forms can be constructed from tensor products of the lowest weight 2 (weigh 1) multiplets (they represent homogeneous polynomials of the lowest weight modular forms).

The modular forms of level $N = 2, 3, 4$ for $\Gamma_{2,3,4} \simeq S_3, A_4, S_4$ have been constructed first by use of the (log derivatives of) Dedekind eta function, $\eta(\tau)$,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,, \qquad q = e^{i2\pi\tau}\,.$$

$\eta(\tau)$ has the following q-expansion:

$$\eta(\tau) = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{n(3n-1)/2}\,.$$

For ($\Gamma_3 \simeq A_4$), the generating (basis) modular forms of weight 2 were shown to form a 3 of A_4 (expressed in terms of log derivatives of Dedekind η-function η'/η of 4 different arguments).

F. Feruglio, arXiv:1706.08749

For ($\Gamma_2 \simeq S_3$), the two basis modular forms of weight 2 were shown to form a 2 of S_3 (expressed in terms of η'/η of 3 different arguments).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

For ($\Gamma_4 \simeq S_4$), the 5 basis modular forms of weight 2 were shown to form a 2 and a $3'$ of S_4 (expressed in terms of η'/η of 6 different arguments).

J. Penedo, STP, arXiv:1806.11040

For ($\Gamma_5 \simeq A_5$), the 11 basis modular forms of weight 2 were shown to form a 3, a $3'$ and a 5 of A_5 (expressed in terms of Jacobi theta function $\theta_3(z(\tau), t(\tau))$ for 12 different sets of $z(\tau), t(\tau)$).

P.P. Novichkov et al., arXiv:1812.02158; G.-J. Ding et al., arXiv:1903.12588

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N = 4$ (i.e., S_4), multiplets of weight 4 (weight $k \leq 10$) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N = 3$ (i.e., A_4) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N = 5$ (i.e., A_5), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

More elegant constuction: modular forms for A_4', S_4', A_5', and A_4, S_4, A_5.

For ($\Gamma_3' \simeq A_4'$), the generating (basis) modular forms of weight 1 were shown to form a 2 of A_4', expressed in terms of two functions of the Dedekind eta function:

$$\hat{e}_1 = \frac{\eta^3(3\tau)}{\eta(\tau)}, \quad \hat{e}_2 = \frac{\eta^3(\tau/3)}{\eta(\tau)}.$$

X.-G. Liu, G.-J. Ding, arXiv:1907.01488

For ($\Gamma_4' \simeq S_4'$), the 3 basis modular forms of weight 1 were shown to form a $\hat{3}$ and of S_4', expressed in terms of two Jacobi constant functions (which are related to the Dedekind eta function, see further).

P.P. Novichkov et al., arXiv:2006.03058

For ($\Gamma_5' \simeq A_5'$), the 6 basis modular forms of weight 1 were shown to form a $\hat{6}$ of A_5' and are expressed in term of two functions of τ (which are related to the Dedekind eta function).

C.-Y. Yao et al., arXiv:2011.03501

In each of three cases of A_4', S_4' and A_5' the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the lowest weight 1 multiplets, were shown to be expressed in terms of only two independent functions of τ .

These pairs of functions are different for the three different groups; but they all are related (in a different way) to the Dedekind eta function and have similar $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

The modular forms of level $N = 2,3,4,5$ for $\Gamma^{(\prime)}_{2,3,4,5} \simeq S_3, A_4^{(\prime)}, S_4^{(\prime)}, A_5^{(\prime)}$ have been constructed by use of the of Dedekind eta function, $\eta(\tau)$.

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,, \qquad q = e^{i2\pi\tau}\,.$$

In the cases of $\Gamma^{(\prime)}_5 \simeq A_5^{(\prime)}$ two "Jacobi theta constants" (functions of τ) are also used.
Modular forms of level $N = 4$ for $\Gamma'_4 \simeq S'_4$ ($\Gamma_4 \simeq S_4$) – in terms of $\theta(\tau)$ and $\varepsilon(\tau)$:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,,\ \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1,2,4$, is the Dedekind eta.
Modular forms of level $N = 3$ for $\Gamma'_3 \simeq A'_4$ ($\Gamma_3 \simeq A_4$) – in terms of $\hat{e}_1$ and $\hat{e}_2$:

$$\hat{e}_1 = \frac{\eta^3(3\tau)}{\eta(\tau)}\,, \quad \hat{e}_2 = \frac{\eta^3(\tau/3)}{\eta(\tau)}\,.$$

Modular forms of level $N = 5$ for $\Gamma'_3 \simeq A'_5$ ($\Gamma_3 \simeq A_4$) – in terms of $\theta_5(\tau)$ and $\varepsilon_5(\tau)$:
$\theta_5(\tau) = \exp(-\,i\pi/10)\,\Theta_{\frac{1}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$, $\varepsilon_5(\tau) = \exp(-\,i\,3\pi/10)\,\Theta_{\frac{3}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$.

In each of three cases of A'_4, S'_4 and A'_5 the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the lowest weight 1 multiplets, were shown to be expressed in terms of only two independent functions of τ.

These pairs of functions are different for the three different groups; but they all are related (in different ways) to the Dedekind η-function (in the case of A'_5 (A_5) - to two Jacobi theta constants also) and have similar (fastly converging) $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

Thus, in the case of a flavour symmetry described by a finite modular group $\Gamma^{(\prime)}_{N}$, $N = 2, 3, 4, 5$, the elements of the matices of the Yukawa couplings in the considered approach represent homogeneous polynomials of various degree of only two (holomorphic) functions of τ. They include also a limited (relatively small) number of constant parameters.

Example: S_4'

P.P. Novichkov, J.T. Penedo. S.T.P., arXiv:2006.03058

Weight 1 modular forms furnishing a $\hat{3}$ of S_4':

$$Y^{(1)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \sqrt{2}\,\varepsilon\,\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix}$$

Modular S_4 lowest-weight 2 multiplets furnish a 2 and a $3'$ irreducible representations of S_4 (S_4') and are given by: :

$$Y^{(2)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4+\varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y^{(2)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4-\varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

At weight $k=3$, a non-trivial singlet and two triplets exclusive to S_4' arise:

$$Y^{(3)}_{\hat{\mathbf{1}}'}(\tau) = \sqrt{3}\left(\varepsilon\,\theta^5-\varepsilon^5\,\theta\right),$$

$$Y^{(3)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \varepsilon^5\,\theta+\varepsilon\,\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\,\varepsilon^2\,\theta^4-\varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6-5\,\varepsilon^4\,\theta^2\right) \end{pmatrix}, \qquad Y^{(3)}_{\hat{\mathbf{3}}'}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\,\theta^3 \\ \theta^6+3\,\varepsilon^4\,\theta^2 \\ -3\,\varepsilon^2\,\theta^4-\varepsilon^6 \end{pmatrix}.$$

At weight $k=4$ one again recovers the S_4 result: the modular forms furnish a 1, 2, 3 and $3'$ irreducible representations of S_4 (S_4').

$$Y^{(4)}_{\mathbf{1}}(\tau) = \frac{1}{2\sqrt{3}}\left(\theta^8+14\,\varepsilon^4\,\theta^4+\varepsilon^8\right), \qquad Y^{(4)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8-10\,\varepsilon^4\,\theta^4+\varepsilon^8\right) \\ \sqrt{3}\left(\varepsilon^2\,\theta^6+\varepsilon^6\,\theta^2\right) \end{pmatrix},$$

$$Y^{(4)}_{\mathbf{3}}(\tau) = \frac{3}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\left(\varepsilon^2\,\theta^6-\varepsilon^6\,\theta^2\right) \\ \varepsilon^3\,\theta^5-\varepsilon^7\,\theta \\ -\varepsilon\,\theta^7+\varepsilon^5\,\theta^3 \end{pmatrix}, \qquad Y^{(4)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8-\varepsilon^8\right) \\ \frac{1}{2\sqrt{2}}\left(\varepsilon\,\theta^7+7\,\varepsilon^5\,\theta^3\right) \\ \frac{1}{2\sqrt{2}}\left(7\,\varepsilon^3\,\theta^5+\varepsilon^7\,\theta\right) \end{pmatrix},$$

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ are given by:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,, \qquad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1, 2, 4$, is the Dedekind eta function.

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions - power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$ ($\mathrm{Im}(\tau) \geq \sqrt{3}/2$, $|q_4| \lesssim 0.26$) :

$$\theta(\tau) = 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2\,q_4^4 + 2\,q_4^{16} + \ldots\,,$$

$$\varepsilon(\tau) = 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2\,q_4 + 2\,q_4^9 + 2\,q_4^{25} + \ldots\,.$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, $\theta \to 1$, $\varepsilon \to 0$.
In this limit $\varepsilon \sim 2\,q_4$ and ε can be used as an expansion parameter instead of q_4.
Due to quadratic dependence in the exponents of q_4, the $q-$expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_4| \leq \exp(-\pi\sqrt{3}/4) \simeq 0.26$.

Similar conclusions are valid for the pair of functions in terms of which the lowest weight 1 modular forms, and thus all higher weight modular forms of A_4' and A_5' are expressed.

Example: A'_5

C.-Y. Yao et al., arXiv:2011.03501

Weight 1 modular forms furnishing a $\hat{6}$ of A'_5:

$$Y^{(1)}_{\hat{\mathbf{6}}}(\tau) = \left(2\,\varepsilon_5^5 + \theta_5^5, 2\,\theta_5^5 - \varepsilon_5^5, 5\,\varepsilon_5\,\theta_5^4, 5\sqrt{2}\,\varepsilon_5^2\,\theta_5^3, -5\sqrt{2}\,\varepsilon_5^3\,\theta_5^2, 5\,\varepsilon_5^4\,\theta_5\right)^T .$$

The functions $\theta_5(\tau)$ and $\varepsilon_5(\tau)$ are related to the Dedekind eta function and the Jacobi theta constants and have the following $q-$expansions:

$$\begin{aligned}\theta_5(\tau) &= 1 + \frac{3}{5}\,q_5^5 + \frac{2}{25}\,q_5^{10} - \frac{28}{125}q_5^{15} + \ldots\,,\\ \varepsilon_5(\tau) &= q_5\left(1 - \frac{2}{5}\,q_5^5 + \frac{12}{25}\,q_5^{10} + + \frac{37}{125}q_5^{15} + \ldots\right),\;\; q_5 \equiv \exp(i2\pi\tau/5)\,.\end{aligned}$$

In the "large volume" limit $\mathrm{Im}\,\tau \rightarrow \infty$, similar to the S'_4 two functions, $\theta_5 \rightarrow 1$, $\varepsilon_5 \rightarrow 0$.

In this limit $\varepsilon_5 \sim q_5$ and ε_5 can be used as an expansion parameter instead of q_5.

The q_5-expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_5| \leq \exp(-\pi\sqrt{3}/5) \simeq 0.34$.

The Framework

$\mathcal{N}=1$ **rigid (global) SUSY, the matter action** $\mathcal{S}$ **reads:**

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\ K(\tau,\overline{\tau},\psi,\overline{\psi}) + \left(\int \mathrm{d}^4x\, \mathrm{d}^2\theta\ W(\tau,\psi) + \text{h.c.}\right),$$

K **is the Kähler potential,** W **is the superpotential,** ψ **denotes a set of chiral supermultiplets** ψ_i**,** θ **and** $\overline{\theta}$ **are Grassmann variables;**
τ **is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation** τ **for the lowest complex scalar component of the modulus superfield and call this component also "modulus").**
τ **and** ψ_i **transform under the action of** $\overline{\Gamma}$ **(**Γ**) in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that** $\psi_i = \psi_i(x)$ **transform in a certain irrep** $\mathbf{r}_i$ **of** Γ_N **(**Γ'_N**), the transformations read:**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}\ (\Gamma): \qquad \begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d}, \\ \psi_i \to (c\tau+d)^{-k_i}\, \rho_{\mathbf{r}_i}(\gamma)\, \psi_i\,. \end{cases}$$

ψ_i **is not a modular form multiplet, the integer** $(-k_i)$ **can be** >0**,** <0**, 0.**
Invariance of $\mathcal{S}$ **under these transformations implies (global SUSY):**

$$\begin{cases} W(\tau,\psi) \to W(\tau,\psi)\,, \\ K(\tau,\overline{\tau},\psi,\overline{\psi}) \to K(\tau,\overline{\tau},\psi,\overline{\psi}) + f_K(\tau,\psi) + \overline{f_K}(\overline{\tau},\overline{\psi})\,. \end{cases}$$

The second line represents a Kähler transformation.
An example Kähler potential that is widely used in model building reads:

$$K(\tau,\overline{\tau},\psi,\overline{\psi}) = -\Lambda_0^2 \log(-i\tau + i\overline{\tau}) + \sum_i \frac{|\psi_i|^2}{(-i\tau + i\overline{\tau})^{k_i}}\,,$$

$\Lambda_0 > 0$ **having mass dimension one.**
More general $K(\tau,\overline{\tau},\psi,\overline{\psi})$ **and the possible consequences they can have for flavour model building are discussed in**
Mu-Chun Chen et al., arXiv:1909.06910 and 2108.02240; Y. Almumin et al., arXiv:2102.11286.

$$W(\tau,\psi) \to W(\tau,\psi)\,,$$

The superpotential can be expanded in powers of ψ_i:

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s}\left(Y_{i_1\ldots i_n,s}(\tau)\,\psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}\,,$$

1 stands for an invariant singlet of Γ_N (Γ'_N). For each set of n fields $\{\psi_{i_1},\ldots,\psi_{i_n}\}$, the index s labels the independent singlets. Each of these is accompanied by a coupling constant $g_{i_1\ldots i_n,s}$ and is obtained using a modular multiplet $Y_{i_1\ldots i_n,s}$ of the requisite weight. To ensure invariance of W under Γ_N (Γ'_N), $Y_{i_1\ldots i_n,s}(\tau)$ must transform as:

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)\,Y(\tau)\,,$$

r_Y is a representation of Γ_N (Γ'_N), and k_Y and r_Y are such that

$$k_Y = k_{i_1} + \cdots + k_{i_n}\,, \tag{1}$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \ldots \otimes \mathbf{r}_{i_n} \supset \mathbf{1}\,. \tag{2}$$

Thus, $Y_{i_1\ldots i_n,s}(\tau)$ represents a multiplet of weight k_Y and level N modular forms transforming in the representation r_Y of Γ_N (Γ'_N).

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma) \, \psi \quad (\rho(\gamma) \,, \; \Gamma_N^{(\prime)} \,, \; N = 2, 3, 4, 5) \,,$$
$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c_{r^c}(\gamma) \, \psi^c \,, \quad (\rho^c(\gamma) \,, \; \Gamma_N^{(\prime)}) \,.$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \,.$$

Inputs in the Analyses

Lepton sector: reference 3-ν mixing scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

ν_j, $m_j \neq 0$: **Majorana particles (assumed).**

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ **eV;**
the value of $\min(m_j)$ **and the "ordering" unknown.**

Δm^2_{21}, $|\Delta m^2_{31}|$ - **known.**

The PMNS matrix U - 3×3 **unitary:** θ_{12}, θ_{13}, θ_{23} - **known; CPV phases** δ, α_{21}, α_{31} - **unknown.**

Thus, 5 known + 4 unknown parameters + MO.

"Known" = measured; "unknown" = not measured.

m_e, m_μ, m_τ **also known - used as input.**

Example: Lepton Flavour Models Based on S_4 (Seesaw Models without Flavons)

P.P. Novichkov et al., arXiv:1811.04933

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism.

The fields involved:

- **two Higgs doublets** H_u **and** H_d**;**
- **three lepton** $SU(2)$ **doublets** L_1, L_2, L_3**;**
- **three neutral lepton gauge singlets** N_1^c, N_2^c, N_3^c**;**
- **three charged lepton** $SU(2)$ **singlets** E_1^c, E_2^c, E_3^c**.**

We work in a basis in which the S_4 generators S and T are represented by symmetric matrices for all irreducible representations r. In this basis the triplet irreps of S and T to be used read:

$$S = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad T = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$. **The plus (minus) corresponds to the irrep $\mathbf{3}$ ($\mathbf{3'}$) of S_4.**

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism. The superpotential in the lepton sector reads

$$W = \alpha\left(E^c L H_d f_E(Y)\right)_{\mathbf{1}} + g\left(N^c L H_u f_N(Y)\right)_{\mathbf{1}} + \Lambda\left(N^c N^c f_M(Y)\right)_{\mathbf{1}} \,,$$

a sum over all independent invariant singlets with the coefficients $\alpha = (\alpha, \alpha', \ldots)$, $g = (g, g', \ldots)$ and $\Lambda = (\Lambda, \Lambda', \ldots)$ is implied. $f_{E,N,M}(Y)$ denote the modular form multiplets required to ensure modular invariance.

We assume further:

- Higgs doublets H_u and H_d transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, and $k_u = k_d = 0$;
- lepton $SU(2)$ doublets L_1, L_2, L_3 furnish a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_L = 2$;
- neutral lepton gauge singlets N_1^c, N_2^c, N_3^c transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_N = 0$;
- charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1}' , \mathbf{1} , \mathbf{1}'$ and carry weights $k_{1,2.3} = 0, 2, 2$.

With these assumptions, we can rewrite the superpotential as

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c \, L \, f_{E_i}(Y)\right)_{\mathbf{1}} H_d + g\left(N^c \, L \, f_N(Y)\right)_{\mathbf{1}} H_u + \Lambda\left(N^c \, N^c \, f_M(Y)\right)_{\mathbf{1}}$$

By specifying the weights of the matter fields one obtains the weights of the relevent modular forms.
After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings, λ and $\mathcal{Y}$, as well as the Majorana mass matrix M for heavy neutrinos, are generated:

$$W = \lambda_{ij}\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}\, N_i^c\, L_j\, H_u + \frac{1}{2}\, M_{ij}\, N_i^c\, N_j^c\,,$$

a sum over $i,j = 1,2,3$ is assumed. After integrating out N^c and after EWS breaking, the charged lepton mass matrix M_e and the light neutrino Majorana mass matrix M_ν are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$M_e = v_d\, \lambda^\dagger\,, \quad v_d \equiv \mathrm{vev}(\mathrm{H_d^0})\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \quad v_u \equiv \mathrm{vev}(\mathrm{H_u^0})\,.$$

The Majorana mass term for heavy neutrinos

Assume $k_\Lambda = 0$, i.e., no non-trivial modular forms are present in $\Lambda\,(N^c\,N^c\,f_M\,(Y))_{\mathbf{1}}$, $k_N = 0$, and for both $\rho_N \sim \mathbf{3}$ or $\rho_N \sim \mathbf{3}'$

$$(N^c\,N^c)_{\mathbf{1}} = N_1^c\,N_1^c + N_2^c\,N_3^c + N_3^c\,N_2^c\,,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\,\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \textbf{for} \quad k_\Lambda = 0\,.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale Λ, which can be rendered real. The Majorana mass term conserves a "non-standard" lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

The neutrino Yukawa couplings

The lowest non-trivial weight, $k_L = 2$, leads to

$$g \left(N^c\, L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c\, L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u \,.$$

There are 4 possible assignments of ρ_N and ρ_L we consider. Two of them, namely $\rho_N = \rho_L \sim \mathbf{3}$ and $\rho_N = \rho_L \sim \mathbf{3'}$ give the following form of $\mathcal{Y}$:

$$\mathcal{Y} = g \left[\begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix} \right], \quad \textbf{for} \quad k_L + K_N = 2 \quad \textbf{and} \quad \rho_N = \rho_L \,.$$

The two remaining combinations, $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3'})$ and $(\mathbf{3'}, \mathbf{3})$, lead to:

$$\mathcal{Y} = g \left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix} \right], \quad \textbf{for} \quad k_L + k_N = 2 \quad \textbf{and} \quad \rho_N \neq \rho_L \,.$$

In both cases, up to an overall factor, the matrix $\mathcal{Y}$ depends on one complex parameter g'/g and the VEV of τ, vev(τ).

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\, \varepsilon^2\, \theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(\theta^4 - \varepsilon^4\right) \\ -2\, \varepsilon\, \theta^3 \\ -2\, \varepsilon^3\, \theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

The charged lepton Yukawa couplings

In the minimal (in terms of weights) viable possibility for $L_{1,2,3}$ furnishing a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carrying a weight $k_L = 2$, and $E^c_{1,2,3}$ transforming as singlets of Γ_4, $\rho_{1,2,3} \sim 1', 1, 1'$ (up to permutations) and carrying weights $k_{1,2.3} = 0, 2, 2$, the relevant part of W, W_e, can take 6 different forms which lead to the same matrix U_e diagonalising $M_e M_e^\dagger = v_d^2\, \lambda^\dagger \lambda$, and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to $\rho_L = \mathbf{3}$, $\rho_1 = \mathbf{1}'$, $\rho_2 = \mathbf{1}$, $\rho_3 = \mathbf{1}'$:

$$\alpha \left(E_1^c\, L\, Y^{(2)}_{\mathbf{3}'}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c\, L\, Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c\, L\, Y^{(4)}_{\mathbf{3}'}\right)_{\mathbf{1}} H_d \,.$$

This leads leads to

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1 Y_4 - Y_2 Y_5) & \beta\,(Y_1 Y_3 - Y_2 Y_4) & \beta\,(Y_1 Y_5 - Y_2 Y_3) \\ \gamma\,(Y_1 Y_4 + Y_2 Y_5) & \gamma\,(Y_1 Y_3 + Y_2 Y_4) & \gamma\,(Y_1 Y_5 + Y_2 Y_3) \end{pmatrix} ,$$

In this "minimal" example the matrix λ depends on 3 free parameters, α, β and γ, which can be rendered real by re-phasing of the charged lepton fields.

We recall that

$$M_e = v_d\, \lambda^\dagger \,, \qquad v_d \equiv \mathrm{vev}(\mathrm{H}_\mathrm{d}^0) \,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y} \,, \qquad v_u \equiv \mathrm{vev}(\mathrm{H}_\mathrm{u}^0) \,.$$

Parameters of the model: α, β, γ, g^2/Λ − real; g' and VEV of τ − complex, i.e., 6 real parameters + 2 phases for description of 12 observables (3 charged lepton masses, 3 neutrino masses, 3 mixing angles and 3 CPV phases). Excellent description of the data is obtained also for real g' (i.e., 6 real parameters + 1 phase, employing gCP).

The 3 real parameters $v_d\alpha$, β/α, γ/α − fixed by fitting m_e, m_μ and m_τ.
The remaining 3 real parameters and 2 (1) phases − $v_u^2 g^2/\Lambda$, $|g'/g|$, $|\tau|$ and $\arg(g'/g)$, $\arg\tau$ ($\arg\tau$) − describe the 9 ν observables, 3 ν masses, 3 mixing angles and 3 CPV phases.

The model considered leads to testable predictions for $\min(m_j)$ ($\sum_i m_i$), type of the ν mass spectrum (NO or IO), the CPV Dirac and Majorana phases, $|\langle m\rangle|$, the range of θ_{23}, as well as of correlations between different observables.

Seven real parameters (5 real couplings + the complex VEV of τ) − is the minimal number of parameters in the constructed so far phenomenologically viable lepton flavour models with massive Majorana neutrinos based on modular invariance.

Numerical Analysis

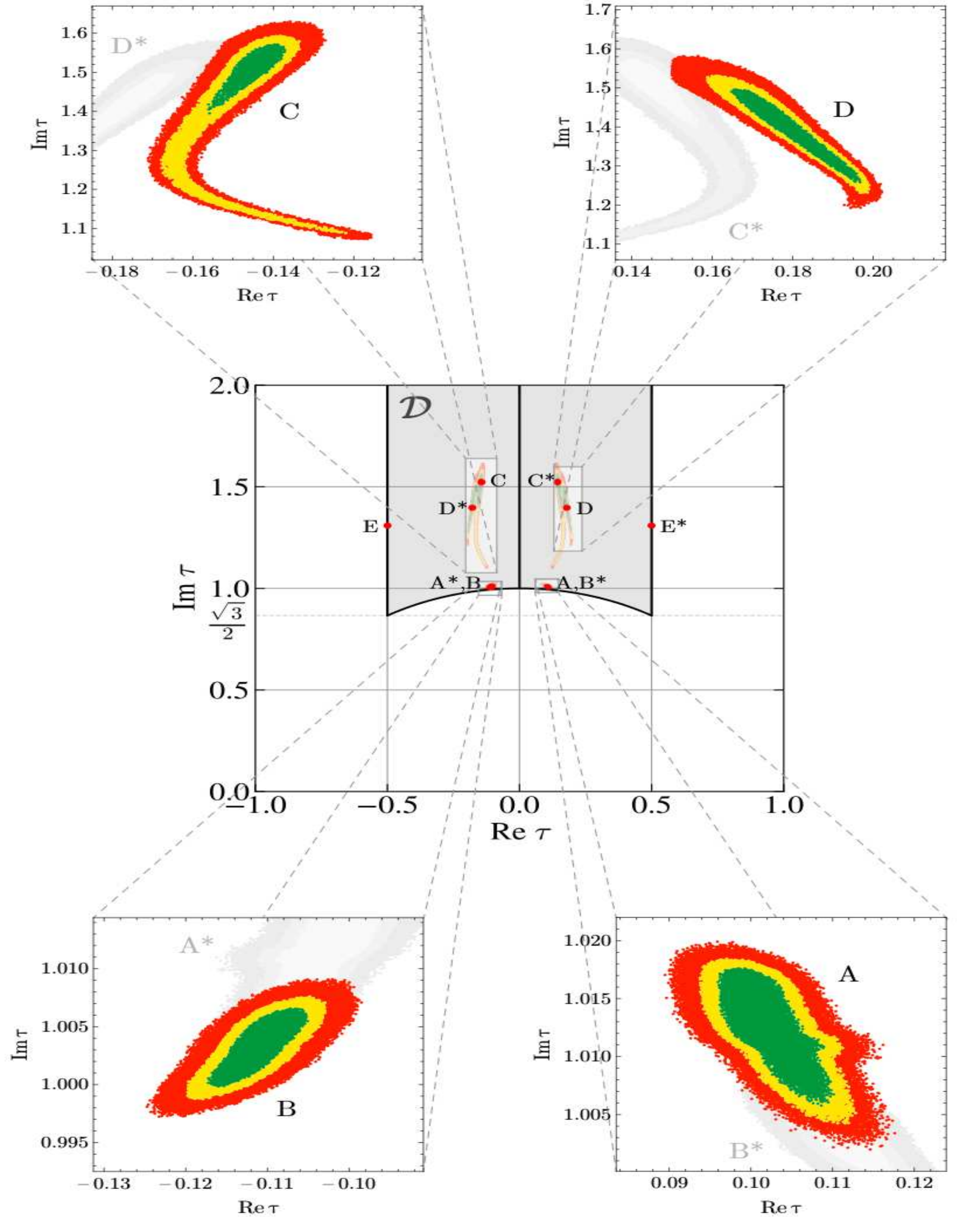
Each model depends on a set of dimensionless parameters

$$p_i = (\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots)\,,$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: $v_d\,\alpha$ for M_e and $v_u^2\,g^2/\Lambda$ for M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarized in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \text{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\|\Delta m^2\|/(10^{-3}\ \text{eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/\|\Delta m^2\|$	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2\theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

Best fit values and 1σ ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best fit value and 1σ range of δ did not drive the numerical searches here reported.



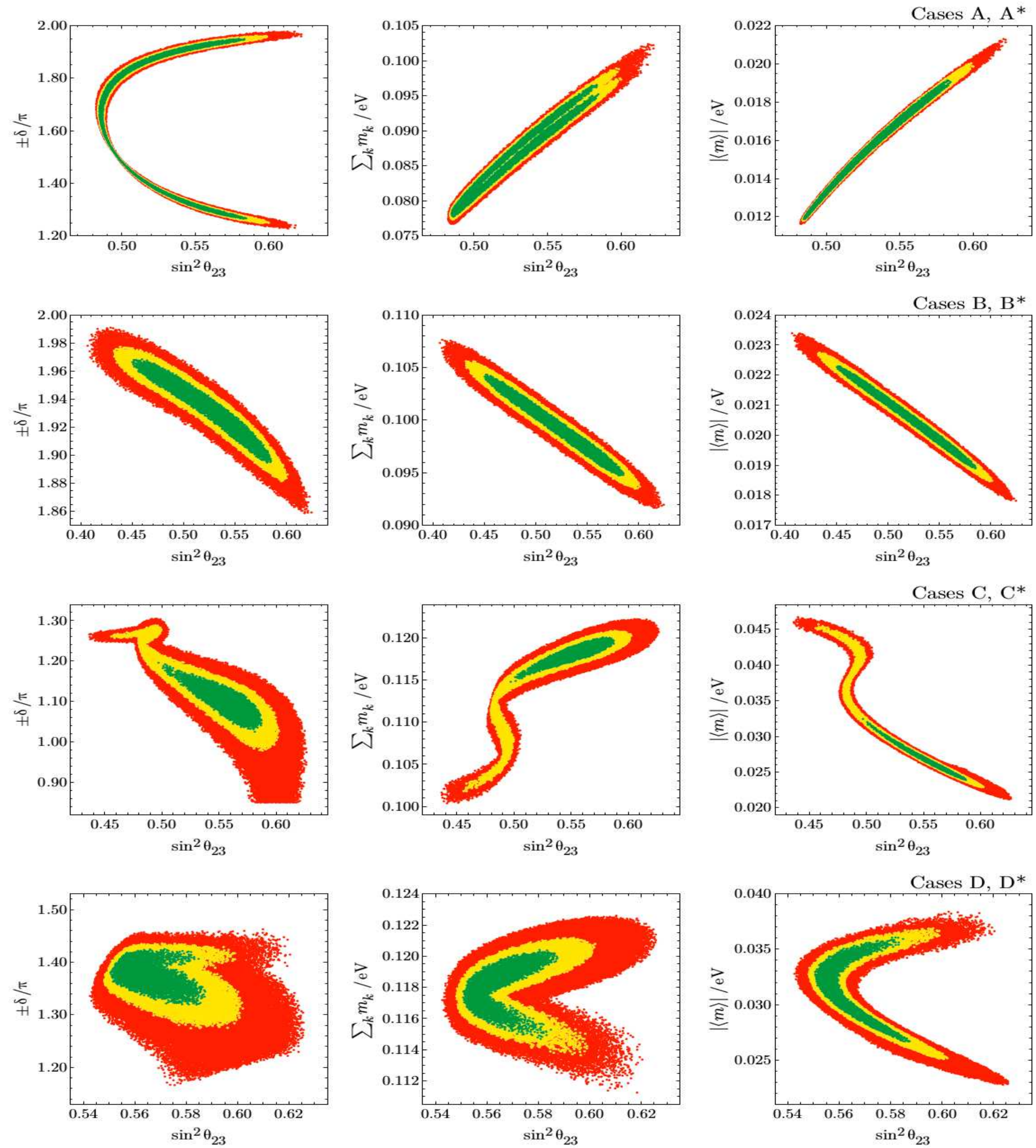
P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	2σ range	3σ range
$\text{Re}\,\tau$	± 0.1045	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
$\text{Im}\,\tau$	1.01	1.006 – 1.018	1.004 – 1.018
β/α	9.465	8.247 – 11.14	7.693 – 12.39
γ/α	0.002205	0.002032 – 0.002382	0.001941 – 0.002472
$\text{Re}\,g'/g$	0.233	$-0.02383 - 0.387$	$-0.02544 - 0.4417$
$\text{Im}\,g'/g$	± 0.4924	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d\,\alpha$ [MeV]	53.19		
$v_u^2\,g^2/\Lambda$ [eV]	0.00933		
m_e/m_μ	0.004802	0.004418 – 0.005178	0.00422 – 0.005383
m_μ/m_τ	0.0565	0.048 – 0.06494	0.04317 – 0.06961
r	0.02989	0.02836 – 0.03148	0.02759 – 0.03224
δm^2 [10^{-5} eV^2]	7.339	7.074 – 7.596	6.935 – 7.712
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	2.413 – 2.494	2.392 – 2.513
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3313	0.2656 – 0.3449
$\sin^2\theta_{13}$	0.02125	0.01988 – 0.02298	0.01912 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4846 – 0.5846	0.4838 – 0.5999
Ordering	NO		
m_1 [eV]	0.01746	0.01196 – 0.02045	0.01185 – 0.02143
m_2 [eV]	0.01945	0.01477 – 0.02216	0.01473 – 0.02307
m_3 [eV]	0.05288	0.05099 – 0.05405	0.05075 – 0.05452
$\sum_i m_i$ [eV]	0.0898	0.07774 – 0.09661	0.07735 – 0.09887
$\lvert\langle m\rangle\rvert$ [eV]	0.01699	0.01188 – 0.01917	0.01177 – 0.02002
δ/π	± 1.314	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
α_{21}/π	± 0.302	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
α_{31}/π	± 0.8716	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

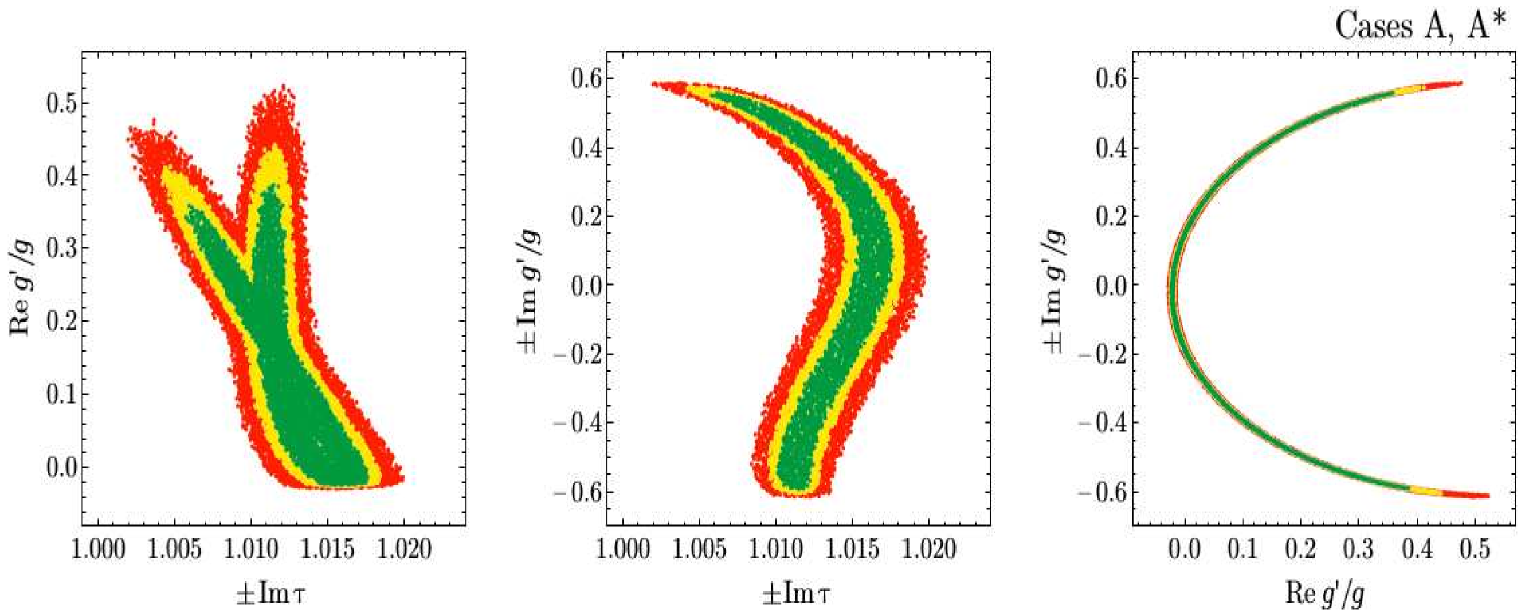
Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases A and A*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.1045 + i\,1.01$).

	Best fit value	2σ range	3σ range
$\text{Re}\,\tau$	∓ 0.109	$\mp(0.1051 - 0.1172)$	$\mp(0.103 - 0.1197)$
$\text{Im}\,\tau$	1.005	0.9998 – 1.007	0.9988 – 1.008
β/α	0.03306	0.02799 – 0.03811	0.02529 – 0.04074
γ/α	0.0001307	0.0001091 – 0.0001538	0.0000982 – 0.0001663
$\text{Re}\,g'/g$	0.4097	0.3513 – 0.5714	0.3241 – 0.5989
$\text{Im}\,g'/g$	∓ 0.5745	$\mp(0.5557 - 0.5932)$	$\mp(0.5436 - 0.5944)$
$v_d\,\alpha$ [MeV]	893.2		
$v_u^2\,g^2/\Lambda$ [eV]	0.008028		
m_e/m_μ	0.004802	0.004425 – 0.005175	0.004211 – 0.005384
m_μ/m_τ	0.05649	0.04785 – 0.06506	0.04318 – 0.06962
r	0.0299	0.02838 – 0.03144	0.02757 – 0.03223
δm^2 [10^{-5} eV^2]	7.34	7.078 – 7.59	6.932 – 7.71
$\|\Delta m^2\|$ [10^{-3} eV^2]	2.455	2.414 – 2.494	2.393 – 2.514
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3314	0.2662 – 0.3455
$\sin^2\theta_{13}$	0.02125	0.0199 – 0.02302	0.01914 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4503 – 0.5852	0.4322 – 0.601
Ordering	NO		
m_1 [eV]	0.02074	0.01969 – 0.02374	0.01918 – 0.02428
m_2 [eV]	0.02244	0.02148 – 0.02522	0.02101 – 0.02574
m_3 [eV]	0.05406	0.05345 – 0.05541	0.05314 – 0.05577
$\sum_i m_i$ [eV]	0.09724	0.09473 – 0.1043	0.0935 – 0.1056
$\|\langle m\rangle\|$ [eV]	0.01983	0.01889 – 0.02229	0.01847 – 0.02275
δ/π	± 1.919	$\pm(1.895 - 1.968)$	$\pm(1.882 - 1.977)$
α_{21}/π	± 1.704	$\pm(1.689 - 1.716)$	$\pm(1.681 - 1.722)$
α_{31}/π	± 1.539	$\pm(1.502 - 1.605)$	$\pm(1.484 - 1.618)$
$N\sigma$	0.02435		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases B and B*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.109 + i\,1.005$).



P.P. Novichkov et al., arXiv:1811.04933



P.P. Novichkov et al., arXiv:1811.04933

CP Symmetry in Modular Invariant Flavour Models

The formalism of combined finite modular and generalised CP (gCP) symmetries for theories of flavour was developed in P.P. Novichkov et al., arXiv:1905.11970.

gCP invariance was shown to imply that the constants g, which accompany each invariant singlet in the superpotential, must be real (in a symmetric basis of S and T and at least for $\Gamma_N^{(\prime)}$, $N \leq 5$). Thus, the number of free parameters in modular-invariant models which also enjoy a gCP symmetry gets reduced, leading to "minimal" models which have higher predictive power.

In these models, the only source of both modular symmetry breaking and CP violation is the VEV of the modulus τ.

The "minimal" phenomenologically viable modular-invariant flavour models with gCP symmetry constructed so far
– of the lepton sector with massive Majorana neutrinos (12 observables) contain $\geq$ 7 real parameters – 5 real couplings + the complex τ (6 real constants + 1 phase);
– of the quark sector contain $\geq$ 9 real parameters – 7 real coulplings + the complex τ;
– while the models of lepton and quark flavours (22 observables) have $\geq$ 15 real parameters - 13 real couplings + the complex τ.

See, e.g., B.-Y. Qu et al., arXiv:2106.11659

Under the CP transformatoion,

$$\tau \xrightarrow{\text{CP}} -\tau^* \,.$$

P.P. Novichkov et al., 1905.11970; A. Baur et al., 1901.03251 and 1908.00805

It was further demonstrated that CP is conserved for

$$\text{Re}\tau = \pm 1/2\,; \;\; \tau = e^{i\theta}\,,\; \theta = [\pi/3, 2\pi/3]\,; \text{Re}\tau = 0\,,\; \text{Im}\tau \geq 1\,.$$

i.e., for the values of τ's VEV at the boundary of the fundamental domain and on the imaginary axis.

Residual Symmetries

The breakdown of modular symmetry is parameterised by the VEV of τ.
There is no value of τ's VEV which preserves the full symmetry $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$).
At certain "symmetric points" $\tau = \tau_{\rm sym}$, $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$) is only partially broken, with the unbroken generators giving rise to residual symmetries.
The $R = S^2$ generator ($\Gamma_N^{(\prime)}$) is unbroken for any value of τ, thus a $\mathbb{Z}_2^R$ symmetry is always preserved.

There are only 3 inequivalent symmetric points in $\mathcal{D}$:

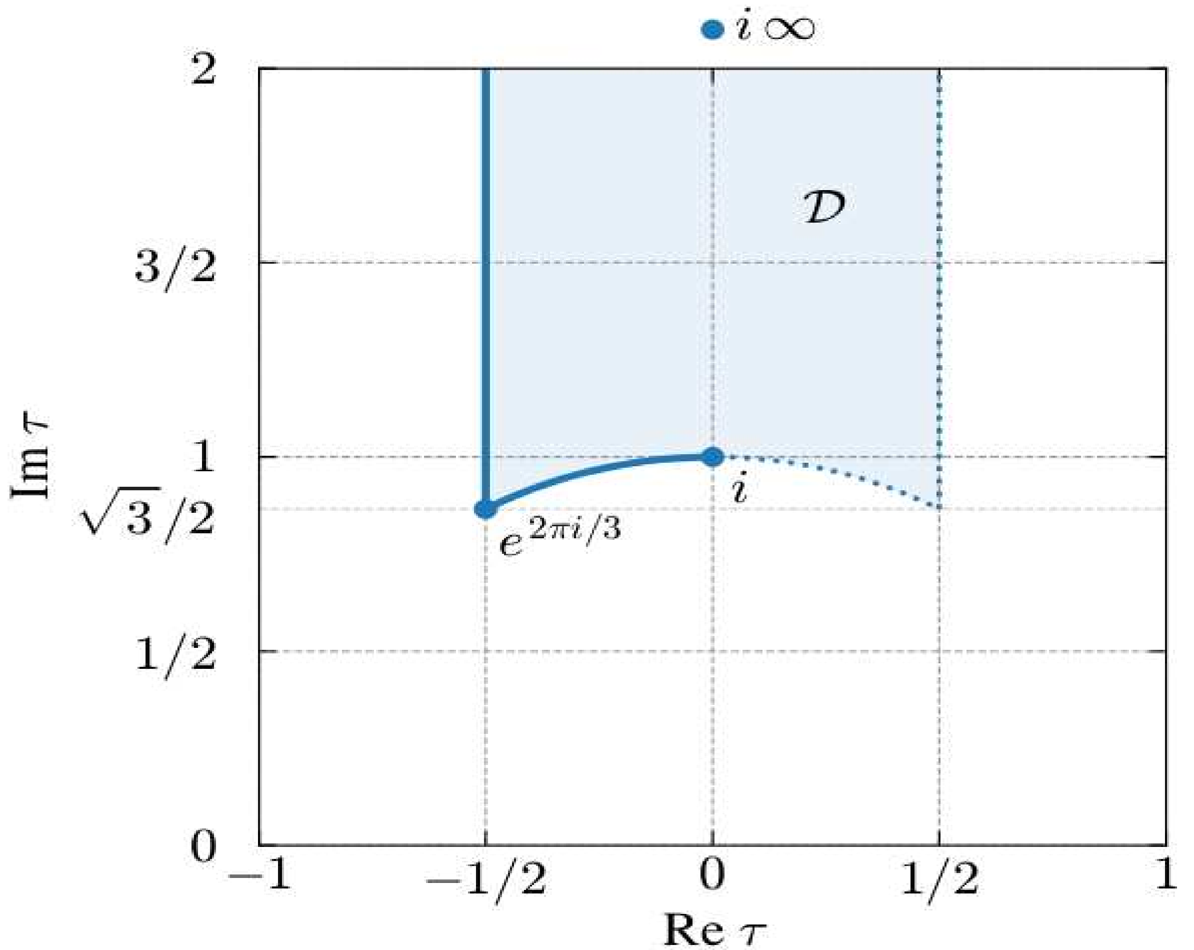
- $\tau_{\rm sym} = i\infty$, **invariant under T, preserving $\mathbb{Z}_N^T$;**
- $\tau_{\rm sym} = i$, **invariant under S, preserving $\mathbb{Z}_2^S$ ($\mathbb{Z}_4^S$, $S^2 = R$);**
- $\tau_{\rm sym} = \omega \equiv \exp(2\pi i/3)$, **invariant under ST, preserving $\mathbb{Z}_3^{ST}$.**

P.P. Novichkov et al., arXiv:1811.04933 and arXiv:2006.03058

These symmetric values of τ preserve the CP ($\mathbb{Z}_2^{CP}$) symmetry of a CP- and modular-invariant theory (e.g. a modular theory where the couplings satisfy a reality condition).

P.P. Novichkov et al., arXiv:1911.04933 and arXiv:2006.03058

The CP ($\mathbb{Z}_2^{\rm CP}$) symmetry is preserved for $\mathrm{Re}\,\tau = 0$ or for τ lying on the border of the fundamental domain $\mathcal{D}$, but is broken at generic values of τ.



The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points $\tau_{\text{sym}} = i\infty, i, \omega$. At the solid and dotted lines (which include the three points) CP is also preserved. The value of τ can always be restricted to $\mathcal{D}$ by a suitable modular transformation.

Figure from P.P. Novichkov et al., arXiv:2006.03058

Fermion Mass Hierarchies without Fine-Tuning

The $l-$ and $q-$ mass hierarchies in practically all modular flavour models proposed in the literarture before arXiv:2102.07488 – obtained with fine-tuning.

Fine-tuning:
i) high sensitivity of observables to model parameters, and/or
ii) unjustified hierarchies between model's parameters.

The flavour structure of the fermion mass matrices M_F can be severely constrained by the residual symmetries present at each of the 3 symmetry points,
$\tau_{\text{sym}} = i$,
$\tau_{\text{sym}} = \omega \equiv \exp(i\,2\pi/3) = -1/2 + i\sqrt{3}/2$, and
$\tau_{\text{sym}} = i\infty$:
residual symmetries may enforce the presence of multiple zeros in M_F.
The posibility to build viable flavour models with observed charged lepton (quark) mass hirarchies in the vicinity of the symmetry points was studied in H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al., 2101.08718 (see also G-J. Ding et al., 1910.03460).

As τ moves away from τ_{sym}, the zero entries in M_F will become non-zero. Their magnitude will be controlled by the size of the departure ϵ from τ_{sym} and by the field transformation properties under the residual symmetry group.

Thus, fine-tuning might be avoided in the vicinity of τ_{sym} as $l-$ and $q-$ mass hierarchies would follow from the properties of the modular forms present in the corresponding M_F rather than being determined by the values of the accompanying constants also present in M_F.

P.P. Novichkov, J.T. Penedo, STP, arXiv:2102.07488.

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c\, M(\tau)_{ij}\, \psi_j\,,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma)\, \psi \quad (\rho(\gamma)\,,\ \Gamma_N^{(\prime)}\,,\ N = 2,3,4,5)\,,$$

$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c_{r^c}(\gamma)\, \psi^c\,, \quad (\rho^c(\gamma)\,,\ \Gamma_N^{(\prime)})\,.$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger\,.$$

$\tau_{\text{sym}} = i\infty$

At $\tau_{\text{sym}} = i\infty$ we have $\mathbb{Z}_N^T$ symmetry ($\tau_{\text{sym}} = i\infty$ is invariant under T).

Consider T-diagonal basis for the group generators S and T.
In this basis $\rho^{(c)}(T) = \text{diag}(\rho_i^{(c)})$.

By setting $\gamma = T$ in the equation for $M(\gamma\tau)$ one finds

$$M_{ij}(T\tau) = (\rho_i^c \rho_j)^* \, M_{ij}(\tau)\,.$$

M_{ij} **is a function of $q \equiv \exp\left(2\pi i\tau/N\right)$ (recall the $q-$expansions) and**

$$\epsilon \equiv |q| = e^{-2\pi\,\text{Im}\,\tau/N}$$

parameterises the deviation of τ from the symmetric point.

The entries $M_{ij}(q)$ depend analytically on q. Further,
$q \xrightarrow{T} \zeta q$ ($T\tau = \tau + 1$), with $\zeta \equiv \exp\left(2\pi i/N\right)$. Thus, in terms of q,

$$M_{ij}(\zeta q) = (\rho_i^c \rho_j)^* M_{ij}(q)\,.$$

Expanding both sides in powers of q, one finds

$$\zeta^n M_{ij}^{(n)}(0) = (\rho_i^c \rho_j)^* M_{ij}^{(n)}(0)\,, \tag{3}$$

$M_{ij}^{(n)}$ **is the n-th derivative of M_{ij} with respect to q. This means that $M_{ij}^{(n)}(0)$ can only be non-zero for values of n such that $(\rho_i^c \rho_j)^* = \zeta^n$.**

In the symmetric limit $q \to 0$, e.g., $M_{ij} = M_{ij}^{(0)}(0) \neq 0$ only if $\rho_i^c \rho_j = 1$.

More generally, if $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \leq l < N$,

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots$$

in the vicinity of the symmetric point.

Thus, the entry M_{ij} is expected to be $\mathcal{O}(\epsilon^l)$ whenever $\text{Im}\,\tau$ is large.
The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$.

Summary

$\tau_{\rm sym} = i\infty$, $\mathbb{Z}_N^T$ **symmetry: for** $(\rho_i^c \rho_j)^* = \zeta^l$ **with** $0 \leq l < N$, $\zeta \equiv \exp\,(2\pi i/N)$

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots\,, \;\; |q_N| = e^{-2\pi\,{\rm Im}\,\tau/N} \equiv \epsilon\,, e.g.\; |q_4| \leq 0.26\,,$$

in the vicinity of the symmetric point.
The entry $M_{ij}{\sim}\mathcal{O}(\epsilon^l)$ **whenever** ${\rm Im}\,\tau$ **is large;** $l = 0, 1, 2;\; 3;\; 4$ **for** $A_4^{(\prime)}$; $S_4^{(\prime)}$; $A_5^{(\prime)}$.
The power l **only depends on how the representations of** ψ **and** ψ^c **decompose under the residual symmetry group** $\mathbb{Z}_N^T$**. Thus, we can have, forexample:**
$m_\tau : m_\mu : m_1 \sim (1, \epsilon, \epsilon^2)$ **for** $A_4^{(\prime)}$; $m_\tau : m_\mu : m_1 \sim (1, \epsilon, \epsilon^3)$ **for** $S_4^{(\prime)}$.

$\tau_{\rm sym} = i$, $\mathbb{Z}_4^S$ **symmetry: for** $(i^{k^c} i^k \rho_i^c \rho_j)^* = (-1)^n$, $n = 0, 1, 2, ...,$
$M_{ij}^n(0) \neq 0$, $M_{ij}{\sim}\mathcal{O}(\epsilon^m)$, $m = 0, 1$, $\epsilon \equiv |s|$, $s \equiv (\tau - i)/(\tau + i)$. **Not sufficient to reproduce the** $l-$ **and** $q-$ **mass hierarchies!**
The power $m = 0, 1$ **depends on how the representations of** ψ **and** ψ^c **decompose under** $\mathbb{Z}_4^S$ **and on their respective weights** k^c **and** k^c.

$\tau_{\rm sym} = \omega$, $\omega \equiv exp(i2\pi/3)$, $\mathbb{Z}_3^{ST}$ **symmetry: for** $(\omega^{k^c} \rho_i^c \omega^k \rho_j)^* = \omega^{2n}$, $\omega^3 = 1$,
$M_{ij}^n(0) \neq 0$, $M_{ij}{\sim}\mathcal{O}(\epsilon^m)$, $m = 0, 1, 2$, $\epsilon \equiv |u|$, $u \equiv (\tau - \omega)/(\tau - \omega^2)$.
The power $m = 0, 1, 2$ **depends on how the representations of** ψ **and** ψ^c **decompose under** $\mathbb{Z}_3^{ST}$ **and on their respective weights** k^c **and** k^c**. In this case we can have:**
$m_\tau : m_\mu : m_1 \sim (1, \epsilon, \epsilon^2)$ **for** $A_4^{(\prime)}$, $S_4^{(\prime)}$ **and** $A_5^{(\prime)}$.

Decomposition under Residual Symmetries

As τ departs from τ_{sym}, the entries M_{ij} of M_F are of $\mathcal{O}(\epsilon^l)$, where ϵ parameterises the deviation of τ from τ_{sym}.
The powers l are extracted from products of factors which, correspond to representations of the residual symmetry group.
One can systematically identify these residual symmetry representations for the different possible choices of Γ'_N representations of matter fields. This knowledge can be exploited to construct hierarchical M_F via controlled corrections to entries which are zero in the symmetric limit.

The matter fields ψ furnish 'weighted' representations $(\mathbf{r}, k)$ of Γ'_N.
When a residual symmetry is preserved by the value of τ,
ψ decompose into unitary representations of the residual symmetry group.
Modulo a possible $\mathbb{Z}_2^R$ factor, these groups are $\mathbb{Z}_N^T$, $\mathbb{Z}_4^S$, and $\mathbb{Z}_3^{ST}$.
A cyclic group $\mathbb{Z}_n \equiv \langle a \,|\, a^n = 1 \rangle$ has n inequivalent 1-dimensional irreps 1_k, $k = 0, \ldots, n-1$ is sometimes referred to as a "charge". The group generator a is represented by one of the n-th roots of unity,

$$\mathbf{1}_k : \quad \rho(a) = \exp\left(2\pi i \frac{k}{n}\right) .$$

For odd n, the only real irrep of $\mathbb{Z}_n$ is the trivial one, 1_0; for even n, there is one more real irrep, $1_{n/2}$. All other irreps are complex, and split into pairs of conjugated irreps: $(\mathbf{1}_k)^* = \mathbf{1}_{n-k}$.

Consider as an example a $(3,k)$ triplet ψ of S_4'.
It transforms under the unbroken $\gamma = ST$ at $\tau = \omega$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j = \omega^k \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j \,.$$

The eigenvalues of $\rho_3(ST)$ are 1, ω and ω^2.
So, in a ST-diagonal basis the transformation rule explicitly reads

$$\psi \xrightarrow{ST} \omega^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \psi = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{k+1} & 0 \\ 0 & 0 & \omega^{k+2} \end{pmatrix} \psi \,,$$

Thus, ψ decomposes as $\psi \rightsquigarrow \mathbf{1}_k \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+2}$ under $\mathbb{Z}_3^{ST}$.

One can find the residual symmetry representations for any other multiplet of a finite modular group in a similar way. For a given level N, the decompositions of fields under a certain residual symmetry group only depend on the pair (r, k).

The decompositions of the weighted representations of Γ_N' ($N \leq 5$) under the three residual symmetry groups, i.e. the residual decompositions of the irreps of $\Gamma_2' \simeq S_3$, $\Gamma_3' \simeq A_4' = T'$, $\Gamma_4' \simeq S_4' = SL(2, \mathbb{Z}_4)$, and $\Gamma_5' \simeq A_5' = SL(2, \mathbb{Z}_5)$ are listed in Tables 6–9 of Appendix A in P.P. Novichkov et al., arXiv:2102.07488.

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}\otimes\mathbf{r}^c$
2	S_3	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{2}\oplus\mathbf{1}^{(\prime)}]\otimes[\mathbf{1}\oplus\mathbf{1}^{(\prime)}\oplus\mathbf{1}']$
3	A'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{1}_a\oplus\mathbf{1}_a\oplus\mathbf{1}'_a]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}''_b]$
			$\tau\simeq i\infty$	$[\mathbf{1}_a\oplus\mathbf{1}_a\oplus\mathbf{1}'_a]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}''_b]$ with $\mathbf{1}_a\neq(\mathbf{1}_b)^*$
4	S'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{3}_a$, or $\mathbf{2}\oplus\mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}^{(\prime)}]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}'_b]$
		$(1,\epsilon,\epsilon^3)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes[\mathbf{2}\oplus\mathbf{1}$, or $\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}']$, $\mathbf{3}'\otimes[\mathbf{2}\oplus\mathbf{1}'$, or $\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}']$, $\hat{\mathbf{3}}'\otimes[\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$, or $\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}']$, $\hat{\mathbf{3}}\otimes[\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$, or $\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}']$
5	A'_5	$(1,\epsilon,\epsilon^4)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes\mathbf{3}'$

Hierarchical mass patterns which can be realised in the vicinity of symmetric points. These patterns are unaffected by the exchange $\mathrm{r}\leftrightarrow\mathrm{r}^c$ and may only be viable for certain weights. Subscripts run over irreps of a certain dimension. Primes in parenthesis are uncorrelated.

Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular groups Γ'_N, $N = 2, 3, 4, 5$, i.e., for S_3, A'_4, S'_4 and A'_5 is given in Tables 10 - 13 of Appendix B in P.P. Novichkov et al., arXiv:2102.07488.

The number of cases which can lead to viable hierechical charged lepton or quark mass mass patterns is extremely limitted.

Table 10. Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric point ω for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma_3 \simeq S_3$. Spectra are insensitive to transposition, i.e. to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3 ("flavour blind" $k\,, k^c$ considered).

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k + k^c \equiv 0$	$k + k^c \equiv 1$	$k + k^c \equiv 2$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{2} \oplus \mathbf{1}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$

Table 11. Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma'_5 \simeq A'_4$. Spectra are insensitive to transposition, i.e. to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$: $k+k^c \equiv 0$	$\tau \simeq \omega$: $k+k^c \equiv 1$	$\tau \simeq \omega$: $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}''$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$	$(1,1,\epsilon)$
$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,1,\epsilon)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,1,1)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(1,1,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(\epsilon,\epsilon^2,\epsilon^2)$	$(1,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}''\oplus\mathbf{1}''\oplus\mathbf{1}'$	$(\epsilon,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^2)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{1'} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$
$\mathbf{1'} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1''} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1}$	$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1'}$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1''} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1'} \oplus \mathbf{1'} \oplus \mathbf{1'}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$\mathbf{1''} \oplus \mathbf{1''} \oplus \mathbf{1''}$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$

Table 12. Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma'_4 \simeq S'_4$. Spectra are insensitive to transposition, i.e. to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\hat{\mathbf{3}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}'$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{3}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}'$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	$\tau \simeq i\infty$
$\hat{\mathbf{3}}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}'$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	 $k+k^c \equiv 1$	 $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{3}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{3}}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\hat{\mathbf{3}}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\hat{\mathbf{3}}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{2}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$			$\tau \simeq i\infty$
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon^3, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon^3, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon^3, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon^3, \epsilon^3, \epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(\epsilon, \epsilon, \epsilon)$

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	$\tau \simeq i\infty$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^3,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$

Table 13. Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma'_5 \simeq A'_5$. Spectra are insensitive to transposition, i.e. to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k + k^c \equiv 0$	 $k + k^c \equiv 1$	 $k + k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{3'}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3'}$	$\mathbf{3'}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3'}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$

A_5' Model with $L \sim \mathbf{3}$, $E^c \sim \mathbf{3}'$, $N^c \sim \hat{\mathbf{2}}'$

$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3}', k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$; vicinity of $\tau = i\infty$.

We consider first the most 'structured' series of hierarchical models, i.e. the case with both fields L, E^c furnishing complete irreps of the finite modular group.

At level $N = 5$ the only such possibility arises in the vicinity of $\tau = i\infty$ when L and E^c are different triplets of A_5'.
For neutrino masses generated via a type I seesaw, we have considered gauge-singlets N^c furnishing a complete irrep of dimension 2 or 3.

We performed a detailed search for a model which
i) is phenomenologically viable in the regime of interest,
ii) produces a charged-lepton spectrum which is not fine-tuned,
iii) involvs at most 8 effective parameters (including τ).

An observable O is typically considered fine-tuned with respect to some parameter p if **BG** $\equiv |\partial \ln O / \partial \ln p| \gtrsim 10$.

G. Giudice and R. Barbieri, 1987

Found one model satisfying these requirements:
$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3}', k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$.
The charged-lepton mass matrix has the following structure:

$$M_e^\dagger \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \epsilon \simeq q_5\,, \quad q_5 = \exp(i2\pi\tau/5)\,.$$

The predicted charged-lepton mass pattern is $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$.

S_4' Model with $L \sim \hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, $E^c \sim \hat{\mathbf{3}}'$, $N^c \sim \mathbf{3}$

$L \sim (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}, k_L = 2)$, $E^c \sim (\hat{\mathbf{3}}', k_E = 2)$, $N^c \sim (\mathbf{3}, k_N = 1)$; vicinity of $\tau = i\infty$.

In the second most 'structured' case, one of the fields L, E^c is an irreducible triplet, while the other decomposes into a doublet and a singlet of the finite modular group.
This possibility is realised at level $N = 4$ in the vicinity of $\tau = i\infty$.
For definiteness, we take $L = L_{12} \oplus L_3$ with $L_{12} \sim (\hat{\mathbf{2}}, k_L)$, $L_3 \sim (\hat{\mathbf{1}}, k_L)$, and $E^c \sim (\hat{\mathbf{3}}', k_E)$.

We have performed a systematic scan restricting ourselves to models involving at most 8 effective parameters (including τ) with no no limit on modular form weights.
Models predicting $m_e = 0$ are rejected.
N^c (when present) furnish a complete irrep of dimension 2 or 3.

Out of the 60 models thus identified, we have selected the only one which
i) is viable in the regime of interest and
ii) produces a charged-lepton spectrum which is not fine-tuned.
This model turns out to be consistent with the experimental bound on the Dirac CPV phase. It corresponds to $k_L = k_E = 2$ and $N^c \sim (\mathbf{3}, 1)$.

Using as expansion parameter $\epsilon \equiv \varepsilon/\theta \simeq 2q$, $\mathbf{q} = \exp(i\pi\tau/2)$, $M_e^\dagger$ is approximately given by:

$$M_e^\dagger \sim v_d \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \; ; \; M_e^\dagger \simeq \frac{\sqrt{3}}{2} v_d \alpha_1 \theta^8 \begin{pmatrix} \epsilon^2 & \frac{(\tilde{\alpha}_2 + \sqrt{3})}{2\sqrt{6}}\epsilon & \frac{(7\tilde{\alpha}_2 - \sqrt{3})}{2\sqrt{6}}\epsilon^3 \\ -\frac{\tilde{\alpha}_2}{6} & \frac{(7\sqrt{3}\tilde{\alpha}_2 + 9)}{6\sqrt{6}}\epsilon^3 & \frac{(\sqrt{3}\tilde{\alpha}_2 - 9)}{6\sqrt{6}}\epsilon \\ \tilde{\alpha}_3\epsilon^2 & -\frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon & \frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon^3 \end{pmatrix} , \; \tilde{\alpha}_{2(3)} \equiv \alpha_{2(3)}/\alpha_1 \, .$$

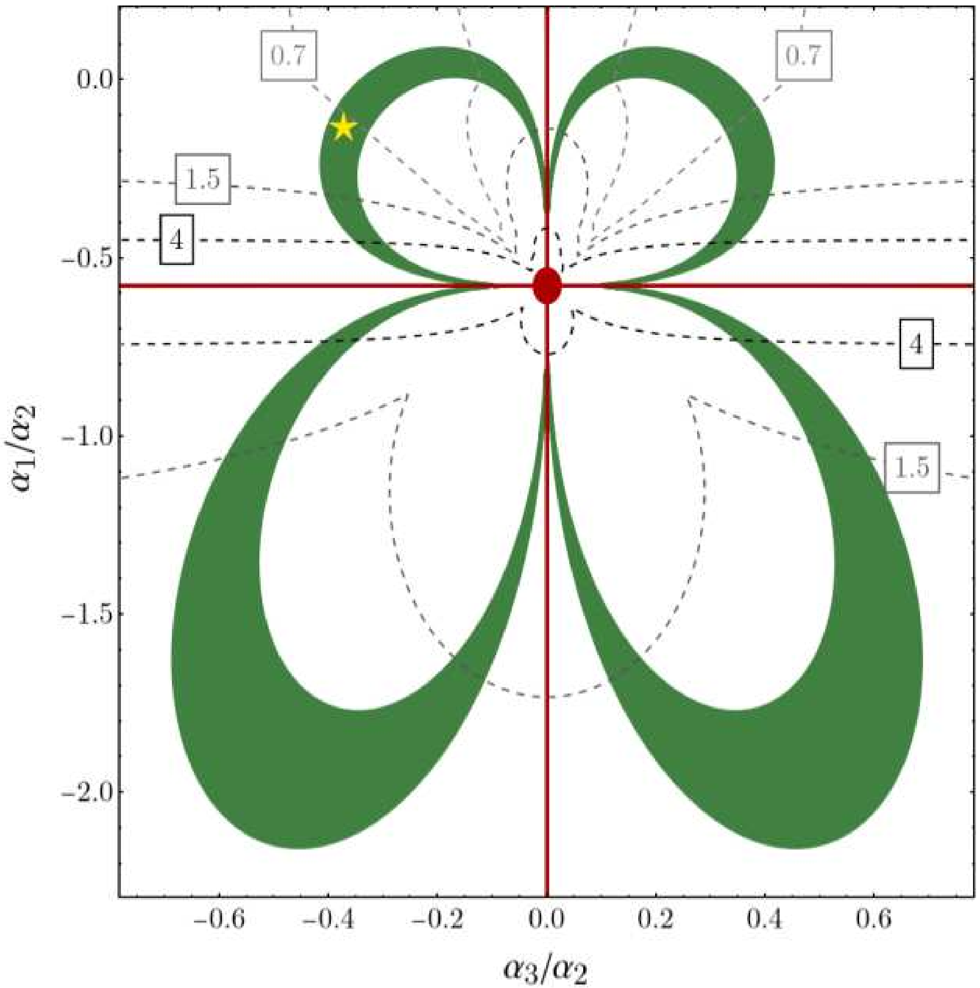
The charged-lepton mass pattern is predicted to be $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^3)$.

One can also find approximate expressions for the charged-lepton mass ratios:

$$\frac{m_e}{m_\mu} \simeq 18\sqrt{3}\frac{\left|\tilde{\alpha}_3(\tilde{\alpha}_2^2-3)\right|}{|\tilde{\alpha}_2|\left((\tilde{\alpha}_2+\sqrt{3})^2+12\tilde{\alpha}_3^2\right)}|\epsilon|^2 ,$$

$$\frac{m_\mu}{m_\tau} \simeq \sqrt{\frac{3}{2}}\frac{\sqrt{(\tilde{\alpha}_2+\sqrt{3})^2+12\tilde{\alpha}_3^2}}{|\tilde{\alpha}_2|}|\epsilon| .$$

These expressions isolate viable (ϵ-independent) regions in the plane of $\tilde{\alpha}_2^{-1} = \alpha_1/\alpha_2$ and $\tilde{\alpha}_3/\tilde{\alpha}_2 = \alpha_3/\alpha_2$.
These regions are shown in the next figure including contours quantifying the degree of fine-tuning involved in the relation between $l-$ mass ratios and constant parameters.
The model best-fit point corresponds to a small value of $\max(\mathbf{BG}) \simeq 0.74$.



Values of the charged-lepton Yukawa couplings of the S_4' model with "large" $\mathrm{Im}(\tau)$ which allow to reproduce charged-lepton mass ratios at 1σ C.L. (green). The red regions are not accessible due to an upper limit on $|\epsilon|$ within the fundamental domain. Contours refer to the Barbieri-Giudice measure of fine-tuning.The yellow star shows the location of the best-fit point for this model.

Model	A'_5	S'_4	S'_4
Re τ	$-0.47^{+0.037}_{-0.096}$	$0.0235^{+0.0019}_{-0.002}$	$-0.496^{+0.009}_{-0.016}$
Im τ	$3.11^{+0.26}_{-0.19}$	$2.65^{+0.05}_{-0.04}$	$0.877^{+0.0023}_{-0.024}$
α_2/α_1	$1.33^{+0.20}_{-0.18}$	$-7.43^{+2.76}_{-12.2}$	—
α_3/α_1	$3.07^{+0.21}_{-0.15}$	$2.76^{+5.27}_{-1.33}$	$2.45^{+0.44}_{-0.42}$
α_4/α_1	—	—	$-2.37^{+0.36}_{-0.3}$
α_5/α_1	—	—	$1.01^{+0.06}_{-0.06}$
g_2/g_1	$-0.0781^{+0.0228}_{-0.0346}$	$-0.407^{+0.0002}_{-0.0003}$	$1.5^{+0.15}_{-0.14}$
g_3/g_1	$0.57^{+0.0023}_{-0.0017}$	$0.321^{+0.02}_{-0.043}$	$2.22^{+0.17}_{-0.15}$
$v_d\,\alpha_1$, GeV	$0.404^{+0.303}_{-0.149}$	$1.73^{+1.8}_{-1.15}$	$4.61^{+1.32}_{-1.33}$
$v_u^2\,g_1/\Lambda$, eV	$0.778^{+1.13}_{-0.477}$	$42.5^{+9.88}_{-5.2}$	$0.268^{+0.057}_{-0.063}$
$\epsilon(\tau)$	$0.0998^{+0.0267}_{-0.0274}$	$0.0313^{+0.0021}_{-0.0022}$	$0.0186^{+0.0028}_{-0.0023}$
CL mass pattern	$(1,\epsilon,\epsilon^4)$	$(1,\epsilon,\epsilon^3)$	$(1,\epsilon,\epsilon^2)$
max(BG)	5.579	0.738	0.848
m_e/m_μ	$0.00474^{+0.00062}_{-0.0005}$	$0.00479^{+0.00058}_{-0.00056}$	$0.00475^{+0.00061}_{-0.00052}$
m_μ/m_τ	$0.0573^{+0.0111}_{-0.0137}$	$0.0574^{+0.0117}_{-0.013}$	$0.0556^{+0.0136}_{-0.0116}$
r	$0.0297^{+0.0021}_{-0.0021}$	$0.0298^{+0.0019}_{-0.0023}$	$0.0298^{+0.00196}_{-0.0023}$
δm^2, 10^{-5} eV^2	$7.33^{+0.39}_{-0.4}$	$7.38^{+0.34}_{-0.44}$	$7.38^{+0.35}_{-0.44}$
$\|\Delta m^2\|$, 10^{-3} eV^2	$2.47^{+0.04}_{-0.04}$	$2.48^{+0.05}_{-0.04}$	$2.48^{+0.05}_{-0.04}$
$\sin^2\theta_{12}$	$0.306^{+0.036}_{-0.028}$	$0.301^{+0.044}_{-0.034}$	$0.304^{+0.039}_{-0.036}$
$\sin^2\theta_{13}$	$0.0222^{+0.0021}_{-0.0018}$	$0.0223^{+0.0017}_{-0.0022}$	$0.0221^{+0.0019}_{-0.002}$
$\sin^2\theta_{23}$	$0.55^{+0.044}_{-0.097}$	$0.548^{+0.045}_{-0.107}$	$0.539^{+0.0522}_{-0.099}$
m_1, eV	$0.0493^{+0.00041}_{-0.00046}$	$0.0204^{+0.00042}_{-0.00035}$	0
m_2, eV	$0.05^{+0.00037}_{-0.00042}$	$0.0221^{+0.0003}_{-0.00028}$	$0.0086^{+0.0002}_{-0.00026}$
m_3, eV	0	$0.0542^{+0.00054}_{-0.00046}$	$0.0502^{+0.00046}_{-0.00043}$
$\Sigma_i m_i$, eV	$0.0993^{+0.0008}_{-0.0009}$	$0.0967^{+0.0013}_{-0.001}$	$0.0588^{+0.0002}_{-0.0002}$
$\|\langle m\rangle\|$, eV	$0.0197^{+0.002}_{-0.0031}$	$0.0181^{+0.0004}_{-0.0003}$	$0.00144^{+0.00035}_{-0.00033}$
δ/π	$1.88^{+0.37}_{-0.13}$	$1.44^{+0.01}_{-0.01}$	$1 \pm \mathcal{O}(10^{-6})$
α_{21}/π	$0.91^{+0.28}_{-0.09}$	$1.77^{+0.01}_{-0.01}$	0
α_{31}/π	0	$1.86^{+0.02}_{-0.02}$	$1 \pm \mathcal{O}(10^{-5})$
$N\sigma$	0.431	0.649	0.563

Large Mixing Angles without Fine-tuning
Viable PMNS matrix in the symmetric limit

Modular-symmetric model of lepton flavour with hierarchical charged-lepton masses is expected to be free of fine-tuning,
i.e., it is possible to have a PMNS matrix which is close to the observed one even in the symmetric limit, i.e., such that either none of its entries vanish, or only the (13) entry vanishes as $\epsilon \to 0$,
if it satisfies at least one of the four conditions:

1. $L \sim 1 \oplus 1 \oplus 1$, $E^c \sim 1 \oplus r$, **where 1 is some real singlet of the flavour symmetry, and r is some (possibly reducible) representation such that $r \not\supset 1$;**

2. $L \sim 1 \oplus 1 \oplus 1^*$, $E^c \sim 1^* \oplus r$, **where 1 is some complex singlet of the flavour symmetry, 1^* is its conjugate, and r is some (possibly reducible) representation such that $r \not\supset 1, 1^*$.**

3. **all charged-lepton masses vanish in the symmetric limit, i.e. the corresponding hierarchical pattern involves only positive powers of ϵ, e.g. $(\epsilon, \epsilon^2, \epsilon^3)$;**

4. **all light neutrino masses vanish in the symmetric limit, i.e. L decomposes into three (possibly identical) complex singlets none of which are conjugated to each other.**

P.P. Novichkov et al., arXiv:2102.07488

The first two of the conditions were formulated earlier in Y. Reyimuaji and A. Romanino, arXiv:1801.10530 (JHEP 03 (2018) 067) for arbitrary flavour symmetry groups.

One of the main conclusions: only a limited number of flavour symmetry representation choices for L and E^c give rise to a PMNS matrix which is viable in the symmetric limit (as defined above).

S_4' Models with $\tau \simeq \omega$

The most 'structured' lepton flavour models without fine-tuning:
these arise at level $N = 4$ in the vicinity of $\tau = \omega$ and correspond to E^c and L being a triplet (of weight 4) and the direct sum of three singlets (of weights 2) of the finite modular group S_4', respectively. The charged-lepton mass pattern is $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^2)$.
M_ν: **sessaw,** $N^c \sim (\mathbf{3}', 1)$; $m_1 = 0$.
The model: 7 real coupling constants + τ.

P.P. Novichkov et al., arXiv:2102.07488

In this model the b.f.v. and 3σ ranges of τ read:

$$\tau = -0.496^{+0.009}_{-0.016} + i\,0.877^{+0.0023}_{-0.024}; \quad u = \frac{\tau - \omega}{\tau - \omega^2}$$

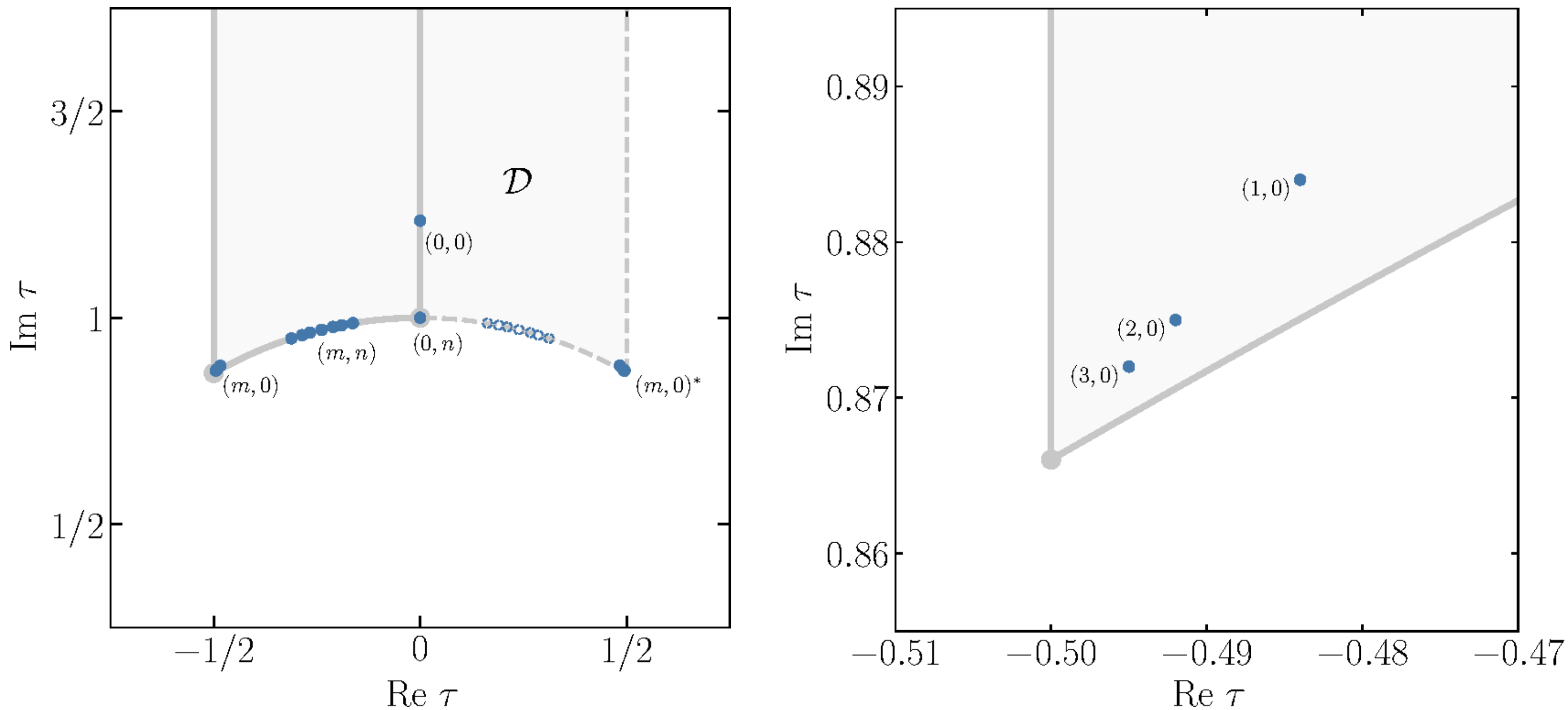
SUGRA potential: $V_{m,n}(\tau, \bar{\tau})$, $n, m \geq 0$ **integers, modular- and CP- invariant.**

M. Cvetic et al., Nucl. Phys. B361 (1991) 194; E. Gonzalo et al., arXiv:1812.06520

$V_{0,m}$: $u_{\rm min} \cong \frac{0.0145}{m + 0.0625} e^{-i2\pi/9}$ **– breaks modular and CP symmetries;**
$|u_{\rm min}|$- **from "Mexican hat"-like potential.**

P.P. Novichkov, J.T. Penedo, STP, arXiv:2201.02020

$$V_{0,2}: \quad u_{\rm min} \cong \frac{0.0145}{2 + 0.0625} e^{-i2\pi/9} \leftrightarrow \tau_{\rm min} = -0.492 + i\,0.875$$



Global minima of the potentials $V(\tau,\bar{\tau})_{m,n}$. Note that points on the right half of the unit arc, which are CP-conjugates of the (m,n) minima, are excluded as they lie outside the fundamental domain. The right panel shows the series $(m,0)$ in the vicinity of the left cusp in more detail.

P.P. Novichkov et al., arXiv:2201.02020

	$\mathbf{n = 0}$	$n = 1$	$n = 2$	$n = 3$
$m = 0$	$0.000 + 1.235i$	$0.000 + 1.000i$	$0.000 + 1.000i$	$0.000 + 1.000i$
$m = 1$	$\mathbf{\mp 0.484 + 0.884i}$	$-0.238 + 0.971i$	$-0.190 + 0.982i$	$-0.163 + 0.987i$
$m = 2$	$\mathbf{\mp 0.492 + 0.875i}$	$-0.286 + 0.958i$	$-0.239 + 0.971i$	$-0.211 + 0.978i$
$m = 3$	$\mathbf{\mp 0.495 + 0.872i}$	$-0.312 + 0.950i$	$-0.267 + 0.964i$	$-0.239 + 0.971i$

Values of the modulus τ at the global minima of the potential $V_{m,n}(\tau, \bar{\tau})$ for various m and n. The values of τ in red are CP-violating, those in blue are CP-conserving.
The minima of $V_{0,0}(\tau, \bar{\tau})$, $V_{1,1}(\tau, \bar{\tau})$ and $V_{0,3}(\tau, \bar{\tau})$ – considered in M. Cvetic et al., Nucl. Phys. B361 (1991) 194. M. Cvetic et al. conjectured: all minima of $V_{m,n}(\tau, \bar{\tau})$ lie on the border of D or on the imaginary axis in D (CP-conserving).
Not the case for $V_{m,0}(\tau, \bar{\tau})$.

P.P. Novichkov et al., arXiv:2201.02020

$$G(\tau,\overline{\tau}) = -\kappa^2\Lambda_K^2 \log(2\,\mathrm{Im}\,\tau) + \log\left|\kappa^3 W(\tau)\right|^2 ,$$

$\kappa^2 = 8\pi/M_P^2$, M_P being the Planck mass, Λ_K is a scale (mass dimension one),

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}},$$

Λ_W is a mass scale so that $H(\tau)$ is dimensionless.
Following M. Cvetic et al., Nucl. Phys. B361 (1991) 194 and E. Gonzalo et al., arXiv:1812.06520 (the most general H without singularities in the fundamental domain),

$$H(\tau) = (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3},$$

The Klein j function is invariant under the action of the modular group $SL(2,\mathbb{Z})$. Here, m and n are non-negative integers.

$$V(\tau,\overline{\tau}) = \frac{\Lambda_V^4}{(2\,\mathrm{Im}\,\tau)^{\mathfrak{n}}|\eta(\tau)|^{4\mathfrak{n}}}\left[\left|iH'(\tau) + \frac{\mathfrak{n}}{2\pi}H(\tau)\widehat{G}_2(\tau,\overline{\tau})\right|^2 \frac{(2\,\mathrm{Im}\,\tau)^2}{\mathfrak{n}} - 3|H(\tau)|^2\right], \quad \mathfrak{n} = 3,$$

$\Lambda_V = \left(\kappa^2\Lambda_W^6\right)^{1/4}$ is the mass scale of the potential, and $\widehat{G}_2$ is the non-holomorphic Eisenstein function of weight 2 given by

$$\widehat{G}_2(\tau,\overline{\tau}) = G_2(\tau) - \frac{\pi}{\mathrm{Im}\,\tau},$$

where G_2 is its holomorphic counterpart. G_2 can be related to the Dedekind function via

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi}G_2(\tau).$$

The potential $V(\tau,\bar{\tau})$ is modular- and CP- invariant.

Bottom-up modular invariance approaches to the lepton flavour problem have been exploited first using the finite modular groups

$\Gamma_3 \simeq A_4$ **(F. Feruglio, 1706.08479; J.C. Criado, F. Feruglio, 1807.01125);**

$\Gamma_2 \simeq S_3$ **(T. Kobayashi et al., 1803.10391);**

$\Gamma_4 \simeq S_4$ **(J.T. Penedo, S.T. Petcov, 1806.11040, minimal, no flavons).**

After these first studies, the interest in the approach grew significantly and models based on these and othere groups have been constructed and extensively studied:

$\Gamma_4 \simeq S_4$
(Novichkov:2018ovf,Kobayashi:2019mna,Okada:2019lzv,Kobayashi:2019xvz,Gui-JunDing:2019wap,Wang:2019ovr,Wang:2020dbp,Gehrlein:2020jnr);

$\Gamma_5 \simeq A_5$
(P.P. Novichkov et al., 1812.02158; Ding:2019xna,Gehrlein:2020jnr);

$\Gamma_3 \simeq A_4$
(Kobayashi:2018scp, Novichkov:2018yse, Nomura:2019jxj, Nomura:2019yft, Ding:2019zxk, Okada:2019mjf, Nomura:2019lnr, Asaka:2019vev, Gui-JunDing:2019wap, Zhang:2019ngf, Nomura:2019xsb, Kobayashi:2019gtp, Wang:2019xbo, Abbas:2020vuy, Okada:2020dmb, Ding:2020yen, Behera:2020sfe, Nomura:2020opk, Nomura:2020cog, Behera:2020lpd, Asaka:2020tmo, Nagao:2020snm, Hutauruk:2020xtk);

$\Gamma_2 \simeq S_3$ **(Okada:2019xqk,Mishra:2020gxg);**

$\Gamma_7 \simeq PSL(2, \mathbb{Z}_7)$ **(G.-J. Ding et al., 2004.12662).**

Similarly, attempts have been made to construct viable models of
quark flavour and of quark-lepton unification (including based on GUTs):

(H.Okada, M. Tanimoto, 1812.09677, 1905.13421; T. Kobayashi et al., 1906.10341; Kobayashi:2018wkl,Lu:2019vgm,Abbas:2020qzc, Okada:2020rjb,Du:2020ylx,Zhao:2021jxg,Chen:2021zty,Ding:2021eva,Ding:2021zbg).

The formalism of the interplay of modular and gCP symmetries has been developed and first applications made
(P.P. Novichkov et al., 1905.11970);
it was further extensively explored
(Kobayashi:2019uyt,Okada:2020brs,Yao:2020qyy,Wang:2021mkw,Qu:2021jdy),
as was the possibility of coexistence of multiple moduli
(P.P. Novichkov et al., 1811.04933 and 1812.11289 (pheno); deMedeiros-Varzielas:2019cyj,King:2019vhv,deMedeirosVarzielas:2020kji, Ding:2020zxw).

Modular invariant theories of flavour with more than one modulus, based on simplectic modular groups were also developed
(G.-J. Ding et al., 2010.07952 and 2102.06716).

The formalism of double covers Γ'_N has been developed and viable flavour models constructed for the cases of
$\Gamma'_3 \simeq T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$
(X.-G. Liu, G.-J. Ding, 1907.01488 (T'); P.P. Novichkov et al., 2006.03058 (S'_4): X. Wang et al., 2010.10159 (A'_5); Liu:2020akv, Yao:2020zml);
the formalism of metaplectic (two-fold) cover group of the modular group $SL(2,\mathbb{Z})$, involving half-integral (rational) weigth modular forms, has also been developed (X.-G. Liu et al., 2007.13706).

It was also realised that there esist three fixed (symmetry) points of the VEV of τ, $\tau_{\text{sym}} = \omega (= -1/2 + i\sqrt{3}/2)$, $i\infty$, i (in the mod. group fund. domain), at which the flavour (=finite modular) symmetry Γ_N (Γ'_N) is broken to non-trivial residual symmetries, $\mathbb{Z}_3^{ST}$, $\mathbb{Z}_N^T$ and $\mathbb{Z}_2^S$ ($\mathbb{Z}_4^S \times \mathbb{Z}_2^R$)

(P.P. Novichkov et al., 1811.04933 (2006.03058));
this fact was further exploited in flavour model building
(Novichkov:2018yse,Novichkov:2018nkm,Okada:2020brs)
and especially in connection with the posibility to build viable flavour models with observed charged lepton (quark) mass hirarchies in the vicinity of the symmetry points
(H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al., 2101.08718)
even without fine-tuning (P.P. Novichkov et al., 2102.07488).

It was shown also that one can have successful leptogeneis in theories with modular flavour symmetries
(T. Asaka et al., 1909.06520; X. Wnag, S. Zhou, 1910.09473; H. Okada et al., 2105.14292).

The bottom-up analyses are expected to eventually connect with the results of the top-down approach based on ultraviolet-complete theories
(Kobayashi:2018rad,Kobayashi:2018bff,deAnda:2018ecu,Baur:2019kwi,
Kariyazono:2019ehj,Baur:2019iai,Nilles:2020nnc,Kobayashi:2020hoc,
Abe:2020vmv,Ohki:2020bpo,Kobayashi:2020uaj,Nilles:2020kgo,Kikuchi:2020frp,
Nilles:2020tdp,Kikuchi:2020nxn,Baur:2020jwc,Ishiguro:2020nuf,Nilles:2020gvu,
Ishiguro:2020tmo,Hoshiya:2020hki,Baur:2020yjl,Kikuchi:2021ogn).

The presented list of publications is not exhaustive.

Instead of Conclusions

To summarise, there is still very important work to be done in the field of the modular invariance approach to the flavour problem. The stakes are high and worth the efforts: we are trying to develop The Theory of Flavour using the power and the beauty of the modular invariance.

Supporting Slides

Lepton and Quark Masses and Mixing

The observed patterns of the masses of up- and down-type quarks and of the charged leptons of the three families of SM are characterized by strong hierarchies:

$$m_d \ll m_s \ll m_b\,,\ \ \frac{m_d}{m_s} = 5.02\times 10^{-2}\,,\ \ \frac{m_s}{m_b} = 2.22\times 10^{-2}\,,\ \ m_b = 4.18~\text{GeV};$$
$$m_u \ll m_c \ll m_t\,,\ \ \frac{m_u}{m_c} = 1.7\times 10^{-3}\,,\ \ \frac{m_c}{m_t} = 7.3\times 10^{-3}\,,\ \ m_t = 172.9~\text{GeV};$$
$$m_e \ll m_\mu \ll m_\tau\,,\ \ \frac{m_e}{m_\mu} = 4.8\times 10^{-3}\,,\ \ \frac{m_\mu}{m_\tau} = 5.95\times 10^{-2}\,,\ \ m_\tau = 1776.86~\text{MeV}.$$

The three quark mixing angles are small and hierarchical,

$$\theta^q_{12} = 12.96^\circ\,,\quad \theta^q_{23} = 2.42^\circ\,,\quad \theta^q_{13} = 0.022^\circ,$$

while the lepton mixing is characterized by two large and one small angles,

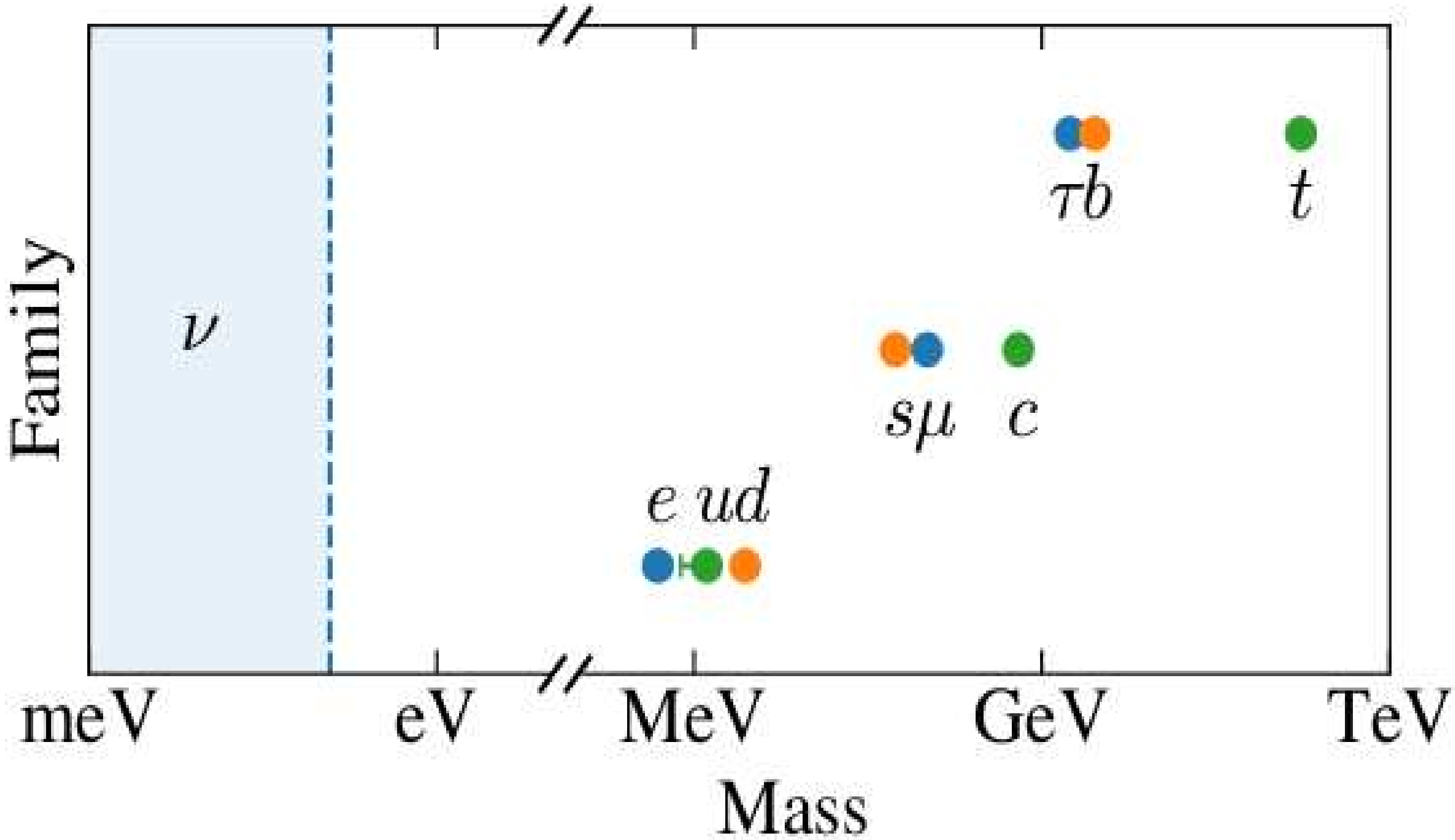
$$\theta^l_{12} = 33.65^\circ\,,\ \theta^l_{13} = 8.49^\circ\,,\ \theta^l_{23} = 47.1^\circ\ \ (45^\circ~\text{within}~1.5\sigma)\,.$$

The quoted values correspond to the standard" parametrisations of V_{CKM} and U_{PMNS}. The Dirac CPV phases in CKM and PMNS matrices read:

$$\delta_q = (73.5 - 5.1 + 4.2)^\circ\,,\qquad \delta_l = (1.37 - 0.16 + 0.18)\times 180^\circ\,(?)\,.$$

F. Capozzi et al. (Bari Group), arXiv:1804.09678.

Quark observables: RG running must be accounted for after one chooses the scale of modular symmetry (the RG running of the lepton observables is relatively small).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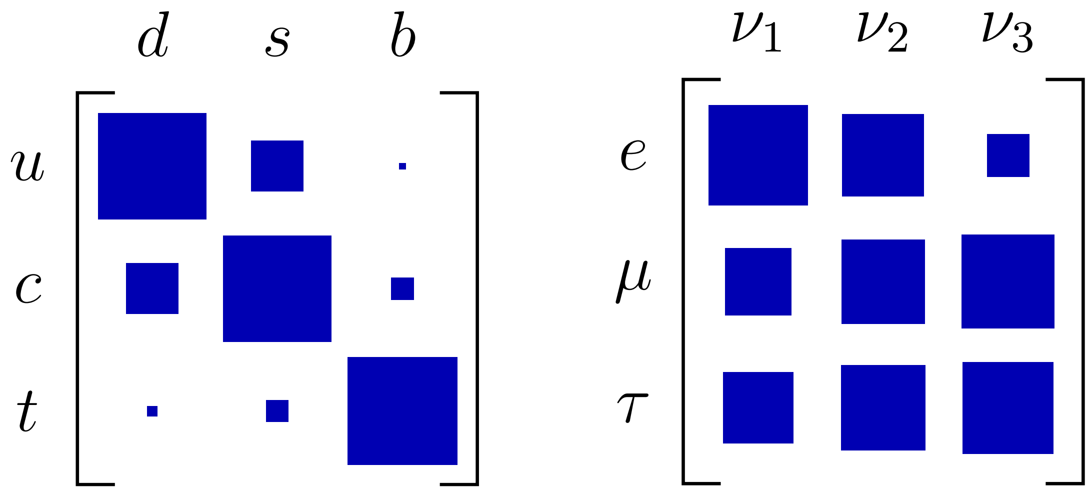

Figures by P. Novichkov

Considered Solutions to the Lepton and Quark Flavour Problems

• $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$:
seesaw mechanism, Weinberg operator, radiative ν mass generation, extra dimensions.
However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

• The origin of the hierarchical pattern of charged lepton and quark masses.

The best qualitative explanation is arguably provided by the Frogatt-Nielsen mechanism based on $U(1)_{FN}$ flavour symmetry and its generalisations.
Problems: predictions suffer from uncertainties; most naturally accomodates small mixing angles, while two lepton mixing angles are large.

• The origins of the patterns of neutrino mixing of 2 large and 1 small angles.

Arguably the most elegant and natural explanation is obtained within the non-Abelian discrete flavour symmetry approach to the problem.
However, the symmetry breaking in the lepton and quark flavour models based on non-Abelian discrete symmetries is impressively cumbersome: it requires the introduction of a plethora of "flavon" scalar fields having elaborate potentials, which in turn require the introduction of a number of "driving fields" and large shaping symmetries to ensure the requisite breaking of the symmetry leading to correct mass and mixing patterns.

Combining the proposed individual "solutions" of the related sub-problems it is difficult, if not impossible, to avoid the drawbacks of each of the "ingredient" sub-problem "solutions". In some cases this can be achieved at the cost of severe fine-tuning.