Dirac quantisation and deparametrisation in group field theory

Steffen Gielen
University of Sheffield

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Canonical quantisation of group field theory (GFT)

In textbook quantum field theories, *canonical quantisation* is based on a choice of coordinate time in the spacetime manifold. In background-independent quantum gravity there is no *a priori* spacetime or preferred notion of time (**problem of time**): canonical quantisation becomes less straightforward.

Two main alternatives:

- *Dirac quantisation*: build a kinematical Hilbert space, represent phase-space functions as operators, implement dynamics as constraints on states, define inner product through group averaging. (see standard LQG)
- *Deparametrisation*: identify a (matter) clock degree of freedom classically, proceed in standard textbook fashion. (see dust models etc in LQG)

Deparametrisation has been proposed in GFT using a scalar matter field as clock.

Two roads to canonical quantisation

Models for quantum gravity coupled to a scalar field χ, based on a real or complex group field $\varphi_J(\chi)$ where $J = (\vec{\jmath}, \vec{m}, \iota)$ is a Peter–Weyl multi-index.
For a complex field one can assume commutation relations [SG, Oriti & Sindoni 2013, Oriti 2016, . . .],

$$[\hat{\varphi}_J(\chi), \hat{\varphi}^\dagger_{J'}(\chi')] = \delta_{J,J'}\delta(\chi - \chi')$$

so that the fields act as creation and annihilation operators, which generate a (kinematical) Fock space. No dynamical information used so far.

Alternatively, define a canonical momentum [Wilson-Ewing 2019, . . .]

$$\pi_J(\chi) := \frac{\delta S}{\delta(\partial_\chi \varphi_J(\chi))}$$

using explicitly the matter field χ as a time variable. Then impose *at equal times*

$$[\hat{\varphi}_J(\chi), \hat{\pi}_{J'}(\chi)] = \mathrm{i}\delta_{J,J'}\,.$$

Dirac quantisation for GFT?

The usual proposal in the *timeless* approach is that we should really impose the equations of motion

$$\widehat{\frac{\delta S}{\delta \varphi_J(\chi)}}|\psi\rangle = 0\,, \tag{1}$$

but in practice we only do this on average,

$$\langle\psi|\widehat{\frac{\delta S}{\delta \varphi_J(\chi)}}|\psi\rangle = 0\,. \tag{2}$$

This is because we cannot solve (1) for interacting field theories. However even when interactions are neglected, the proposal is to solve only (2), often in a mean-field approximation.
If we decided to be more ambitious and solve (1), we would typically find states that are not normalisable in the kinematical inner product.

We need a different physical inner product and *group averaging* for GFT.

Dirac quantisation for (Klein–Gordon) QFT

In standard scalar QFT we do not have constraints, just dynamical equations.

To obtain a constrained formalism, add an extra “worldline” or “proper time” argument τ to a complex scalar field and define an action (in momentum space)

$$S[\Phi,\bar{\Phi},N]=\int\frac{\mathrm{d}^D p}{(2\pi)^D}\,\mathrm{d}\tau\left[\frac{\mathrm{i}}{2}\left(\bar{\Phi}\frac{\partial\Phi}{\partial\tau}-\Phi\frac{\partial\bar{\Phi}}{\partial\tau}\right)+N(p^2+m^2)|\Phi|^2\right].$$

Variation with respect to the “lapse” N now gives a constraint $(p^2+m^2)|\Phi|^2=0$; the other equations of motion are then $\partial\Phi/\partial\tau=\partial\bar{\Phi}/\partial\tau=0$. Hence classically this is equivalent to standard (free) field theory.
However we can now define kinematical field operators

$$[\hat{\Phi}(p),\hat{\Phi}^\dagger(p')]=(2\pi)^D\delta^{(D)}(p-p')$$

and we must impose the constraint $(p^2+m^2)\hat{\Phi}^\dagger(p)\hat{\Phi}(p)|\psi\rangle=0$, which means that physical states cannot contain any quanta with $p^2+m^2\neq 0$.

Dirac quantisation for (Klein–Gordon) QFT

Now we need to do group averaging to get a physical inner product, and go from the kinematical Fock space to a physical Fock space. For one-particle states,

$$\langle\phi_{\rm ph}|\psi_{\rm ph}\rangle \propto \int \frac{{\rm d}^D p}{(2\pi)^D}\delta(p^2+m^2)\overline{\phi(p)}\psi(p)$$

and we can extend this to the Fock space by defining "projections"

$$\hat{\Phi}(p) \mapsto \mathcal{P}\hat{\Phi}(p)\,, \quad \hat{\Phi}^\dagger(p) \mapsto \mathcal{P}\hat{\Phi}^\dagger(p)$$

such that the projected operators generate the physical one-particle Hilbert space when acting on a Fock vacuum $|0\rangle_{\rm ph}$. We then obtain (time-independent) creation and annihilation operators in the usual way.

Similar projections can be defined for all operators that are well-defined observables on the kinematical Fock space (e.g., number operator) $\Rightarrow$ standard QFT.

Dirac quantisation for (free) GFT

We can run the same arguments for a quadratic GFT action. We need to distinguish between Peter–Weyl modes for which the classical solutions are $\varphi_J \sim e^{\pm \mathrm{i} p\chi}$ and those for which the solutions are $\varphi_{\mathbf{J}} \sim e^{\pm P\chi}$ (the latter ones are relevant for a realistic cosmology, since they generate an expanding Universe).

Starting from an extended GFT action with dependence on a parameter τ,

$$[\hat{\varphi}_J(p), \hat{\varphi}^\dagger_{J'}(p')] = 2\pi\delta_{J,J'}\delta(p-p')\,,\ [\hat{\varphi}_{\mathbf{J}}(P), \hat{\varphi}^\dagger_{\mathbf{J}'}(P')] = 2\pi\delta_{\mathbf{J},\mathbf{J}'}\delta(P-P')$$

for these two types of modes, and we impose *strongly* that

$$\left(\mathcal{K}^{(0)}_J - \mathcal{K}^{(2)}_J p^2\right)\hat{\varphi}^\dagger_J(p)\hat{\varphi}_J(p)|\psi\rangle = \left(\mathcal{K}^{(0)}_{\mathbf{J}} + \mathcal{K}^{(2)}_{\mathbf{J}} P^2\right)\hat{\varphi}^\dagger_{\mathbf{J}}(P)\hat{\varphi}_{\mathbf{J}}(P)|\psi\rangle = 0\,.$$

(NB. We need a *momentum space* definition, which requires analytic continuation in χ for the "P" modes.)

Observables on physical Fock space

As in the case of the Klein–Gordon field, we obtain observables on the physical Fock space through "projection". For instance, we get a physical number operator

$$\hat{N} = \sum_J \left(\hat{a}_J^\dagger \hat{a}_J + \hat{b}_J^\dagger \hat{b}_J \right) + \sum_{\mathbf{J}} \left(e^{2m_{\mathbf{J}}\chi} \hat{A}_{\mathbf{J}}^\dagger \hat{A}_{\mathbf{J}} + e^{-2m_{\mathbf{J}}\chi} \hat{B}_{\mathbf{J}}^\dagger \hat{B}_{\mathbf{J}} \right) .$$

There is no "cross-term" as in mean-field approximations, which changes the resulting Friedmann equation (slightly), e.g., for a single $\mathbf{J}$ mode:

$$\left(\frac{\langle V'(\chi) \rangle}{\langle V(\chi) \rangle} \right)^2 = 4m_{\mathbf{J}}^2 \left(1 - 4 \frac{v_{\mathbf{J}}^2 \langle \hat{A}_{\mathbf{J}}^\dagger \hat{A}_{\mathbf{J}} \rangle \langle \hat{B}_{\mathbf{J}}^\dagger \hat{B}_{\mathbf{J}} \rangle}{\langle V(\chi) \rangle^2} \right)$$

Structure of these observables agrees mostly with deparametrised approach.
No χ operator corresponding to the clock itself, just as there is no time operator in standard QFT. Implications for operational definition of χ? Alternative construction (POVM's etc.)?

Summary

- Two alternative approaches to canonical quantisation of GFT have been proposed: one "timeless", one based on deparametrisation and closer to standard QFT. I reviewed attempts to connect the two.

- Usually no strong imposition of dynamics in timeless approach, but if we do impose dynamics strongly we can define Dirac quantisation and obtain a physical Hilbert space, at least for free approximation to GFT.

- Unclear whether any chance of explicit realisation of Dirac quantisation exists for interesting interacting GFTs.

- Differences between the approaches small in general, but have impact on, e.g., resulting cosmology, as unphysical states contribute in usual approach.

- Deparametrisation much easier to define, as in LQG, but suffers from usual drawback of loss of general covariance. Problem of time in GFT anyone??

Thank you!