Transition Amplitudes and Holography in Coloured Boulatov-Ooguri Type Group Field Theory Models

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Literature:

→ G. Schmid: *On* 3*–Dimensional Quantum Gravity and Quasi-Local Holography in Spin Foam Models and Group Field Theory*. Master's Thesis, March 2022. arXiv:2205.05079 [gr-qc].

→ C. Goeller, D. Oriti and G. Schmid: *Transition Amplitudes and Quasi-Local Holography in the Coloured Boulatov-Ooguri Model*. To appear in July/August 2022.

Motivation: 3D Gravity, Holography and Coloured Tensor Models

→ 3d gravity has no local degrees of freedom and hence is a TOPOLOGICAL FIELD THEORY.

⇒ Due to its (almost) trivial dynamics it provides a simple toy model to study aspects of quantum gravity, as there are still many interesting phenomena (BTZ black holes, local defects, ...).

→ Guiding principle: HOLOGRAPHY, i.e. the idea of fully describing a theory in a region of space-time in terms of a *dual* theory solely living on its boundary.

→ Holography in the PONZANO-REGGE SPIN FOAM MODEL: (Ponzano, Regge, 1968)

(1) PR Model on the 3-ball is dual to two copies of the Ising model on its boundary 2-sphere. (Dittrich, Hnybida 2013; Bonzom, Costantino, Livine 2015)

(2) PR Model on the solid torus recovers the structure of the BMS group in the asymptotic limit. (Dittrich, Goeller, Livine, Riello 2017/19)

⇒ How to add a sum over topologies?

→ **Systematic Approach:** Consider GROUP FIELD THEORY (GFT) corresponding to a spin foam model, which organizes the amplitudes associated to different complexes in a systematic way.

→ Particularly useful: *coloured* version of GFT.

↪ at most point-like, isolated singularities ("*pseudomanifolds*") → more control over perturbative sum.

The Coloured Boulatov Model

The COLOURED BOULATOV MODEL (Boulatov 1992, Gurau 2011) for 3D Riemannian quantum gravity with $\Lambda = 0$ is a field theory based on four $\mathbb{C}$-valued bosonic scalar fields $\{\varphi_l\}_{l=0,\dots,3} \subset L^2(\mathrm{SU}(2)^3, \mathrm{d}g)$ satisfying

$$\forall h \in \mathrm{SU}(2) : \varphi_l(hg_1, hg_2, hg_3) = \varphi_l(g_1, g_2, g_3)$$

for all $g_1, g_2, g_3 \in \mathrm{SU}(2)$. The action of this model is given by

$$\mathcal{S}_\lambda[\varphi_l, \overline{\varphi}_l] := \sum_{l=0}^{3} \int_{\mathrm{SU}(2)^3} \left(\prod_{i=1}^{3} \mathrm{d}g_i \right) |\varphi_l(g_1, g_2, g_3)|^2$$

$$- \lambda \int_{\mathrm{SU}(2)^6} \left(\prod_{i=1}^{6} \mathrm{d}g_i \right) \varphi_0(g_1, g_2, g_3)\varphi_1(g_3, g_4, g_5)\varphi_2(g_5, g_2, g_6)\varphi_3(g_6, g_4, g_1) + c.c.$$

Note that we ***do not*** assume any symmetry under permutation of arguments or any reality condition!

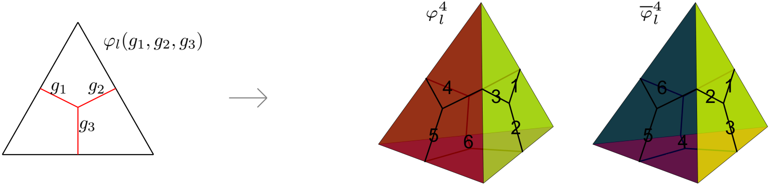
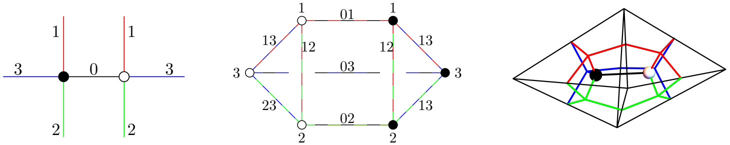


Figure: The fields describe triangles and the interaction terms produce tetrahedra with opposite orientation.

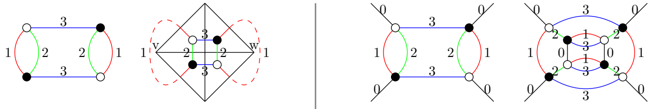
Crystallization Theory: Coloured Graphs

The *Feynman Diagrams* of the model turn out to be "*Coloured Graphs*", which are the central tools of CRYSTALLIZATION THEORY, a branch of geometric topology.

$\mathcal{G}$ $\partial\mathcal{G}$ $\Delta_{\mathcal{G}}$



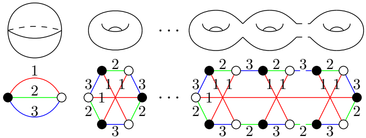
- → A "*coloured graph*" $\mathcal{G}$ represents a *unique* simplicial complex $\Delta_{\mathcal{G}}$.
- → The underlying graph of $\mathcal{G}$ is the *internal dual 1-skeleton* of the complex $\Delta_{\mathcal{G}}$.
- → From $\mathcal{G}$ we can construct the "*boundary graph*", which is the *boundary dual* 1*-skeleton*.
  - ↪ The complex dual to $\partial\mathcal{G}$ represents the *desingularized* boundary of $\Delta_{\mathcal{G}}$.



γ Δ_γ $|\Delta_\gamma| \cong S^2$ $\mathcal{G}_1$ $\partial\Delta_{\mathcal{G}_1} = \Delta_\gamma$ $\mathcal{G}_2$ $\partial\Delta_{\mathcal{G}_2} = \Delta_\gamma/_{v\sim w}$

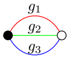
- → **Note:** The colouring encodes the gluing of higher-dimensional cells and their nested structure.
  - ↪ Topology can be fully reconstructed from the graph and its colouring!

Step 1: Take a boundary topology and a graph γ representing it.



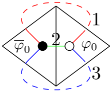
Step 2: Choose a boundary spin-network state.

$\psi \in L^2(\mathrm{SU}(2)^{\mathcal{E}_\gamma})$



$\psi(g_1, g_2, g_3)$

Step 3: Define boundary observables.

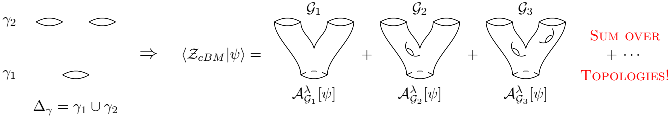


$\mathcal{O}_\psi[\varphi_0, \overline{\varphi}_0] \propto \int_{\mathrm{SU}(2)} \psi\varphi_0\overline{\varphi}_0$

The transition amplitudes, interpreted as the probability amplitudes, are defined as

$$\langle \mathcal{Z}_{\mathrm{cBM}}|\psi\rangle = \int \left(\prod_{l=0}^{3} \mathcal{D}\varphi_l \mathcal{D}\overline{\varphi_l} \right) e^{-\mathcal{S}_\lambda[\varphi_l, \overline{\varphi}_l]} \mathcal{O}_\psi[\varphi_0, \overline{\varphi_0}] = \sum_{\mathcal{G} \text{ with } \partial\mathcal{G}=\gamma} \frac{1}{\mathrm{sym}(\mathcal{G})} \mathcal{A}^\lambda_\mathcal{G}[\psi],$$

where the *amplitudes* $\mathcal{A}^\lambda_\mathcal{G}[\psi]$ are defined on the following slide.



↪ **Step 4**: Rewrite sum as *topological expansion* via a generalization of the rooting procedure used in the large N-limit of coloured tensor models (Gurau 2011).

Transition Amplitudes II: What are the amplitudes $\mathcal{A}_{\mathcal{G}}^{\lambda}[\psi]$?

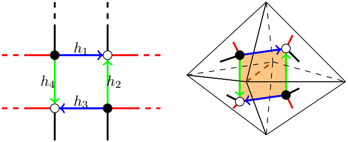
Consider a $(3+1)$-coloured graph $\mathcal{G}$ with boundary γ. The amplitude of $\mathcal{G}$ is given by

$$\mathcal{A}_{\mathcal{G}}^{\lambda}[\psi] := \langle \mathcal{A}_{\mathcal{G}}^{\lambda} | \psi \rangle_{L^2},$$

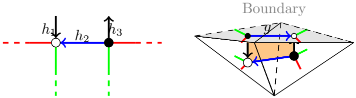
where $\mathcal{A}_{\mathcal{G}}^{\lambda}[\{g_e\}_{e\in\mathcal{E}_\gamma}]$ is the Ponzano-Regge Spin Foam Amplitude:

$$\mathcal{A}_{\mathcal{G}}^{\lambda}[\{g_e\}_{e\in\mathcal{E}_\gamma}] = (\lambda\overline{\lambda})^{\frac{|\mathcal{V}_{\mathcal{G},\mathrm{int}}|}{2}} \int_{\mathrm{SU}(2)^{|\mathcal{E}_{\mathcal{G}}|}} \Big(\prod_{e\in\mathcal{E}_{\mathcal{G}}} \mathrm{d}h_e\Big) \prod_{f\in\mathcal{F}_{\mathcal{G}}} \delta_{\mathrm{SU}(2)}\Big(\overrightarrow{\prod_{e\in f}} h_e^{\varepsilon(e,f)}[g]\Big),$$

where $\mathcal{F}_{\mathcal{G}}$ denotes the set of "*faces*" (=bicoloured paths in $\mathcal{G}$) and $\varepsilon(e,f) = \pm 1$ the relative orientation.



$\delta_{\mathrm{SU}(2)}(h_1 h_2^{-1} h_3 h_4^{-1})$



$\delta_{\mathrm{SU}(2)}(g^{-1} h_1 h_2^{-1} h_3)$

$\Rightarrow$ **Recall**: $\mathcal{A}_{\mathcal{G}}^{\lambda}$ can be understood as the discretized transition amplitude of $3d$ quantum gravity

$$\mathcal{Z}_{3\mathrm{d}}[\mathcal{M}, \partial\omega] := \int_{i^*\omega=\partial\omega} \mathcal{D}e\mathcal{D}\omega \exp\Big(i\int_{\mathcal{M}} \mathrm{tr}(e\wedge F[\omega])\Big)$$

on the fixed complex $\Delta_{\mathcal{G}}$ representing the manifold $\mathcal{M}$.

Simple Example: Spherical Boundary

Consider the simplest possible $(2+1)$-coloured graph γ representing the 2-sphere S^2.

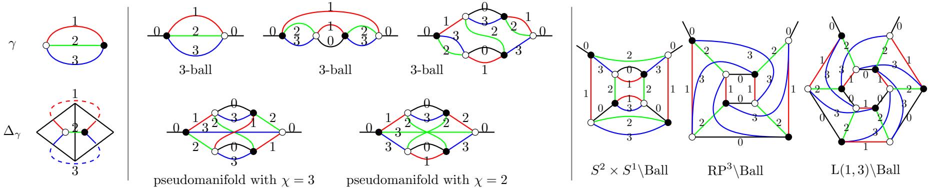


Figure: Boundary graph γ and some examples of $(3+1)$-coloured graphs with boundary γ.

Theorem

Let $\mathcal{G}$ be a $(3+1)$-coloured graph with boundary graph $\partial\mathcal{G} = \gamma$ representing a manifold. Then the amplitude $\mathcal{A}^{\lambda}_{\mathcal{G}}[\psi]$ is proportional to the *spin network evaluation* $\psi(\{g_i = \mathbb{1}\}_{i=1,2,3})$.

Hence, the transition amplitude with respect to boundary graph γ factorizes as

$$\langle \mathcal{Z}_{\mathrm{cBM}} | \psi \rangle|_{\text{manifolds}} = \underbrace{\mathcal{C}[\lambda, \overline{\lambda}]}_{\text{remaining bulk integrations \& factors of } \lambda\overline{\lambda}} \cdot \underbrace{\psi(\{g_i = \mathbb{1}\}_{i=1,2,3}).}_{\text{spin network evaluation}}$$

Further Work and Outlook

→ Next to trivial boundary topology: 2-torus $T^2 \cong S^1 \times S^1$ (work in progress).
↪ factorization into several terms, each representing a non-contractible cycle of the boundary.

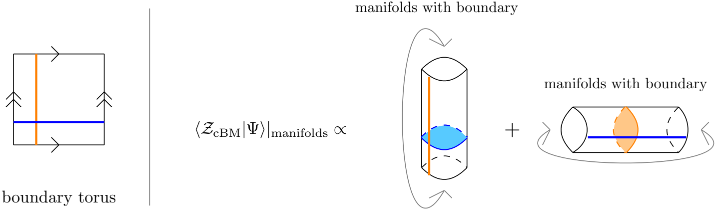


Figure: Factorization in the torus case.

Conjecture: The transition amplitude for some arbitrary boundary factorizes into several terms, each representing a non-contractible cycle on the boundary surface.

→ This provides an explicit proof for the holographic nature of the coloured Boulatov model!

→ **Final Goal:** Take an explicit choice of boundary state, calculate transition amplitude explicitly and study holographic dualities in this context.

→ Torus case: Boundary state with clear geometric interpretation should recover the structure of the *BMS group* in the asymptotic limit.