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A new spin foam model of quantum geometry based on edge vectors

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Motivation and goal

Open issues: correctness of the imposition of simplicity constraints.

Unitary irreps of the Lorentz group.

Several strategies

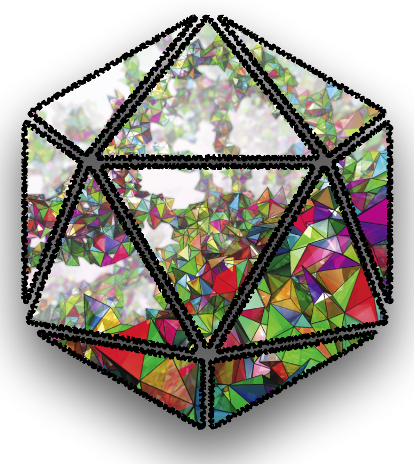
NO full control over the correctness of the quantum geometry.

Construct a spin foam model that manifestly encodes simplicial geometry. (Lorentzian geometry)

Usual spin foam: bivectors.

Edge vectors would do the job!

[Crane and Yetter, 2003]



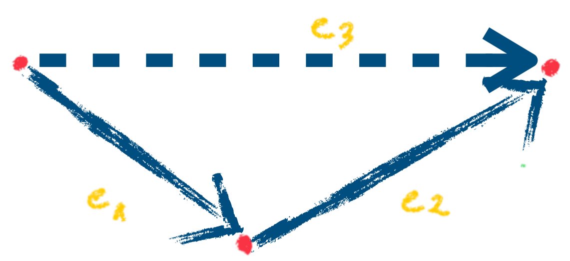
Outline

1. Quantum geometry of a triangle.
2. Quantum geometry of a tetrahedron.
3. New spin foam amplitude based on edge vectors.
4. Conclusion and outlook.

2. Quantum geometry of a triangle in 4d

A. The classical triangle in Minkowski space

Closure relation: $e_1 + e_2 = e_3$



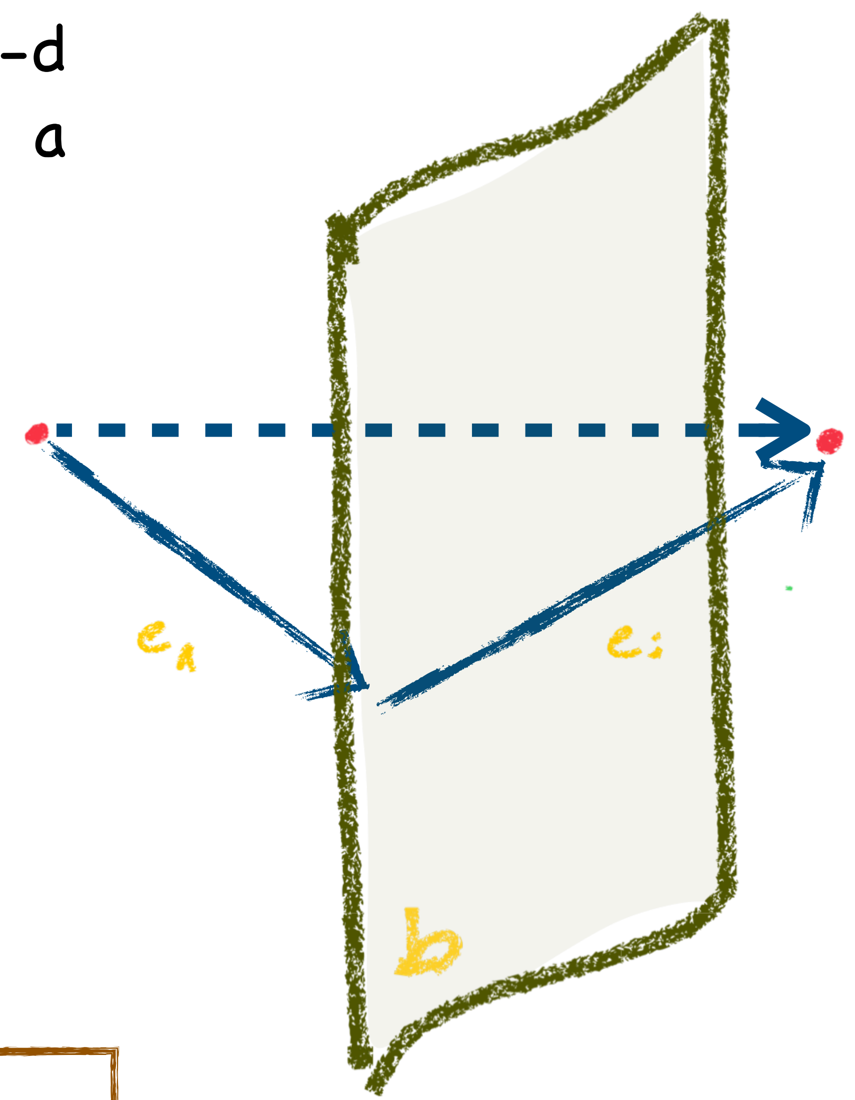
$e_1, e_2, e_3 \in M^4$

Encodes all the geometric properties of the triangle.

Closure relation + normal bivector (the 2-d surface orthogonal to the triangle is spanned by a normal bivector):

$$b := e_1 \wedge e_2 = e_1 \wedge e_3 = e_3 \wedge e_2 \,.$$

Same triangle but restricted to its skew symmetric part.



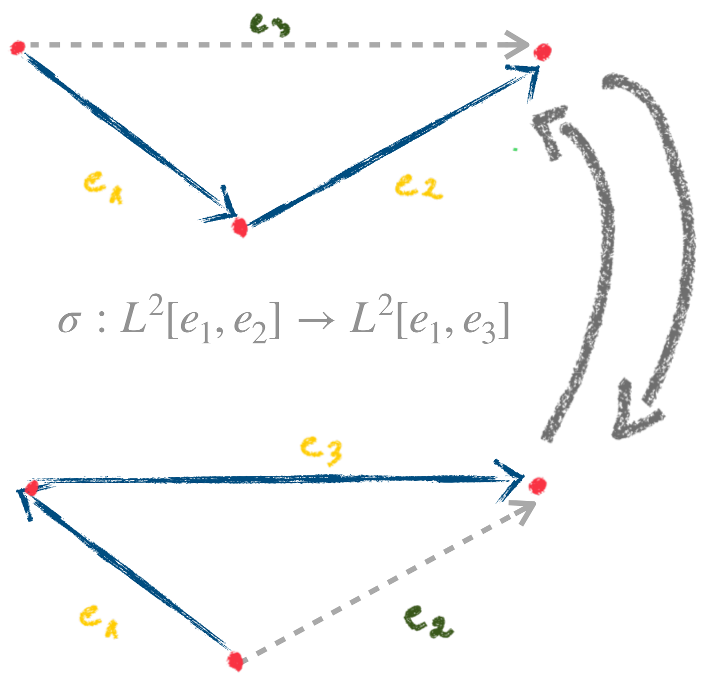
B. Quantization

Edge-based quantization

- **Quantization** of $e \in M^4$: $L^2[e_1, e_2] := L^2[e_1] \otimes L^2[e_2]$
- $\mathscr{H}$ of the translation group.

↓

The Hilbert space of the 4d harmonic oscillator



Bivector-based quantization

- Skew-symmetric part of the tensor product $T^A[e_1, e_2] = L^2[e_1] \wedge L^2[e_2]$
- **Quantization:** $b = *L \in so(3,1)^*$ + simplicity constraint

$$L^2[e_1] \wedge L^2[e_2] \cong F(so^*(1,3))$$

$R_{0,\rho}$ $\quad$ $R_{j,0}$

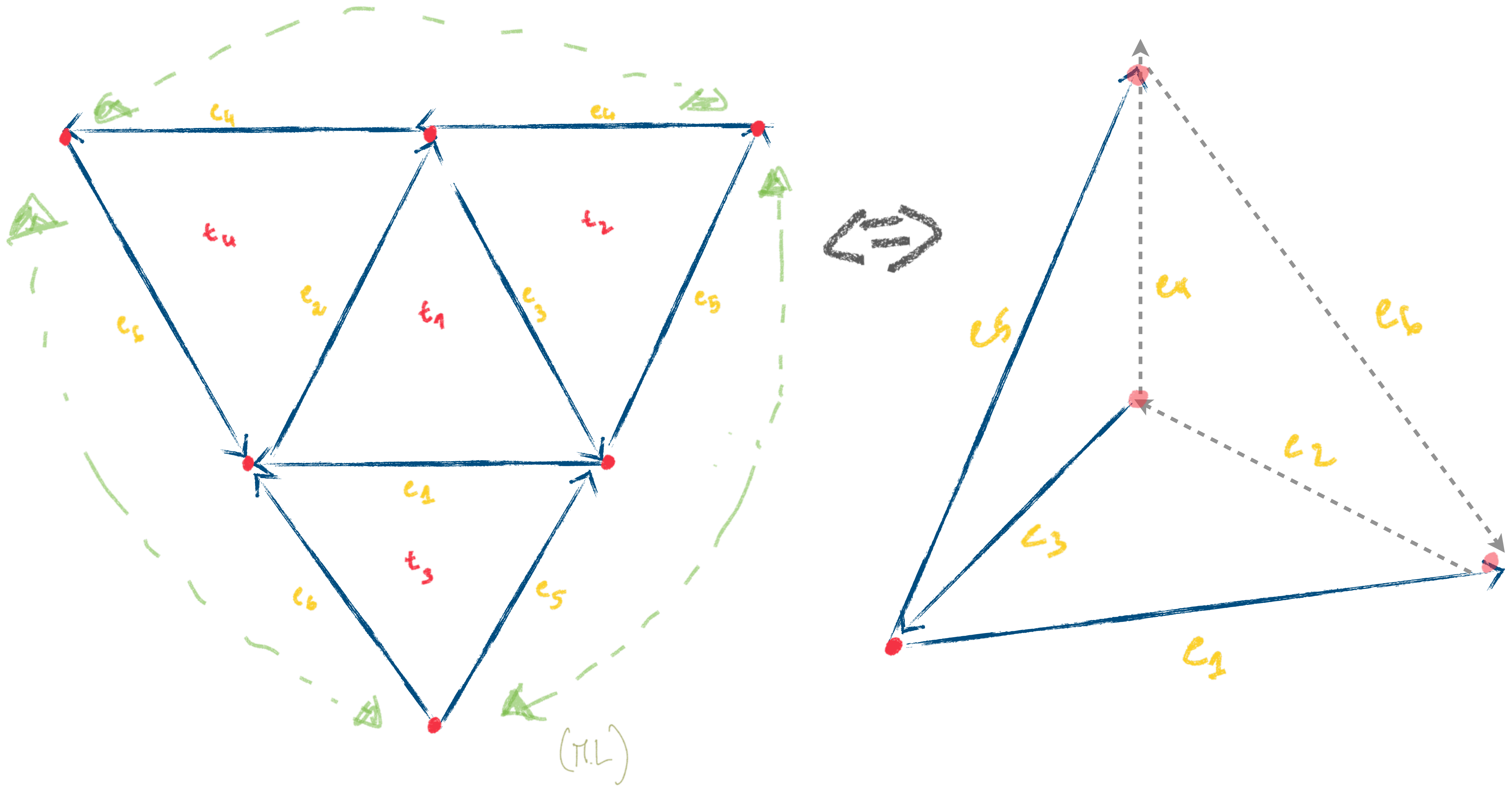
Balanced irreps!

$$\Psi_{n_r,\mu,\ell,m}(r,\eta,\theta,\phi) = \sum_{n_t,n_x,n_y,n_z} \int d^4\alpha_\nu e^{-\frac{1}{2}\sum_\nu |\alpha_\nu|^2} C^{n_t,n_x,n_y,n_z}_{n_r,\mu,\ell,m} \frac{\alpha_t^{*\,n_t}\alpha_x^{*\,n_x}\alpha_y^{*\,n_y}\alpha_z^{*\,n_z}}{\pi^4\sqrt{n_t!n_x!n_y!n_z!}} \Psi_{\alpha_t,\alpha_x,\alpha_y,\alpha_z}(t,x,y,z)$$

$$b_{e_1\wedge e_2} := -ia_1^\dagger \wedge a_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1)$$

3. Quantum tetrahedron

A. The classical tetrahedron via edge vectors



$$e_1 + e_2 + e_3 = 0\,, \quad e_4 + e_5 = e_3$$
$$e_1 + e_5 = e_6\,, \quad e_2 + e_4 + e_6 = 0\,.$$

Full geometric information

B. The classical tetrahedron via bivectors

b is based on the edge vectors e_i

1. **Dependence relation**: For each pair of bivectors $b_i, b_j \subset \{b_1, b_2, b_3, b_4\}$ $b_i \wedge b_j = 0$;
2. **Closure condition**: $b_1 + b_2 + b_3 + b_4 = 0$

Antisymmetric part of the geometry

C. The quantum tetrahedron

- Functions on the translation group on M^4: $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in F(M^4)^{\times 6}$

Closure constraint

$$\hat{C}_t(\lambda_1, \ldots, \lambda_6) = \delta(\lambda_1 + \lambda_2 + \lambda_3)\,\delta(-\lambda_3 + \lambda_4 + \lambda_5)\,\delta(-\lambda_5 + \lambda_6 - \lambda_1)\,\delta(-\lambda_6 - \lambda_4 - \lambda_2)\,,$$

Wave function: $(\hat{C}_t \star f)(\lambda_1, \ldots, \lambda_6)$

Skew-symmetric projection

Change of variables (via expansors)

- $F(so^*(1,3))^{\times 4}$: a sub-space of the tetrahedron Hilbert space $L^2[e_1, e_2, e_3, e_4, e_5, e_6]$

Closure constraint

2. $\hat{C}_{\tau_b}(x_1, x_2, x_3, x_4) = \delta(x_1 + x_2 + x_3 + x_4)$

Spacelike tetrahedra

Bivector wave function: $(\hat{C}_{\tau_b} \star f)(x_1, x_2, x_3, x_4)$

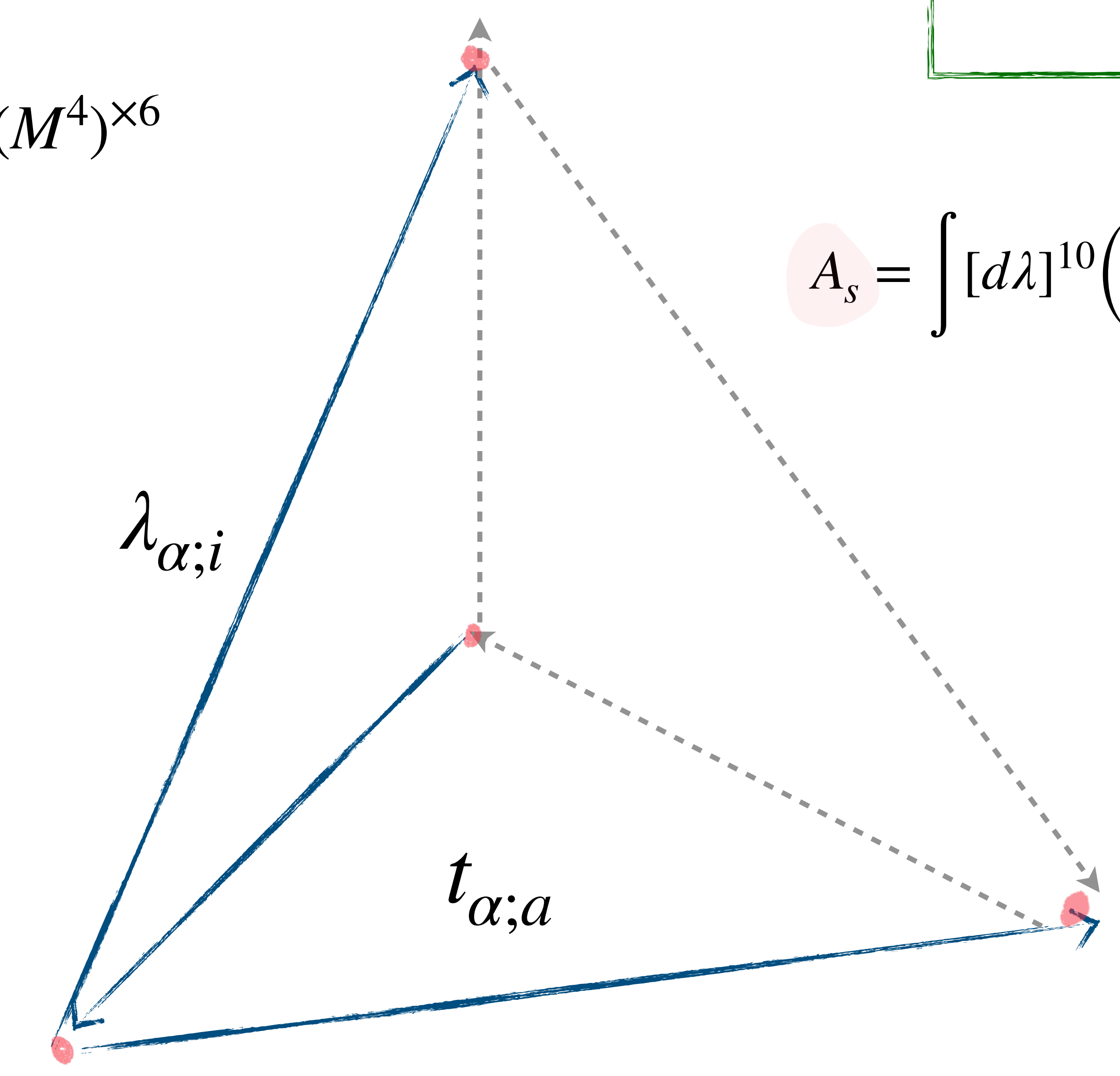
4. New spin foam model based on edge vectors

$f(\{\lambda_{\alpha;i}\}) \in F_\alpha(M^4)^{\times 6}$

$\alpha = 1,\ldots,5$
Tetrahedron

$i = 1,\ldots,6$
Edge

$a = 1,2,3,4$
Triangle



The 4-simplex amplitudes are combined together by identifying the edge decorations.

$$A_\Gamma = \prod_s A_s \prod_\tau A_\tau$$

τ Boundary tetrahedra

s Simplex

$$A_s = \int [d\lambda]^{10} \Big(\prod_{\alpha=1}^{3} \prod_{a=1}^{4-\alpha} \hat{C}_{t_{\alpha;a}}(\{\lambda_{\alpha;i}\}) \Big) \star \big(\delta(\lambda_{1;1}-\lambda_{2;4}) \star \delta(\lambda_{1;1}-\lambda_{4;5})\big) \big(\delta(\lambda_{1;2}-\lambda_{2;6}) \star \delta(\lambda_{1;2}-\lambda_{5;3})\big) \big(\delta(\lambda_{1;3}-\lambda_{2;2}) \star \delta(\lambda_{1;3}-\lambda_{3;6})\big) \big(\delta(\lambda_{1;4}-\lambda_{3;5}) \star \delta(\lambda_{1;4}-\lambda_{5;1})\big)$$
$$\big(\delta(\lambda_{1;5}-\lambda_{3;1}) \star \delta(\lambda_{1;5}-\lambda_{4;4})\big) \big(\delta(\lambda_{1;6}-\lambda_{4;3}) \star \delta(\lambda_{1;6}-\lambda_{5;2})\big) \big(\delta(\lambda_{2;1}-\lambda_{3;4}) \star \delta(\lambda_{2;1}-\lambda_{5;5})\big) \big(\delta(\lambda_{2;3}-\lambda_{3;2}) \star \delta(\lambda_{2;3}-\lambda_{4;6})\big)$$
$$\big(\delta(\lambda_{2;5}-\lambda_{4;1}) \star \delta(\lambda_{2;5}-\lambda_{5;4})\big) \big(\delta(\lambda_{3;3}-\lambda_{4;2}) \star \delta(\lambda_{3;3}-\lambda_{5;6})\big)$$

Amplitudes combinations: edge decorations:

$$A_\tau = \int [d\lambda]^6 \prod_{i=1}^{6} \delta(\lambda_{\alpha;i} - \lambda_{\beta;i})$$

Access the anti-symmetric data of the geometry:

$x_{\alpha;a}$

$$(\hat{C}_t \star f)(\lambda_1, \ldots, \lambda_6) \Rightarrow (\hat{C}_{\tau_b} \star f)(x_1, x_2, x_3, x_4)$$

The quantum Minkowskian time-like bivector

$$A_s = \int [dx]^{10} \Big(\prod_{\alpha=1}^{4} \hat{C}_{\tau_\alpha}(\{x_{\alpha;a}\}) \Big) \star \prod_{\alpha=1}^{4} \prod_{a=1}^{4} \delta(x_{\alpha;a} - x_{\alpha+a;5-a}),$$

- Recover the BC model as a **sector** of our more general one

$$A_s = \int [d\lambda]^{10} [dh]^5 d^5\mu \prod_{\alpha=1}^{5} \prod_{a=1}^{4} \prod_{i=1}^{6} \mu_\alpha^2 e_\star(h_\alpha, x_{\alpha;a}(\lambda_{\alpha;i}, \lambda_{\alpha;i})) D^{0,\mu_\alpha}_{0,0;0,0}(g(x_{\alpha;i}) g(x_{\alpha+i;5-i}))$$

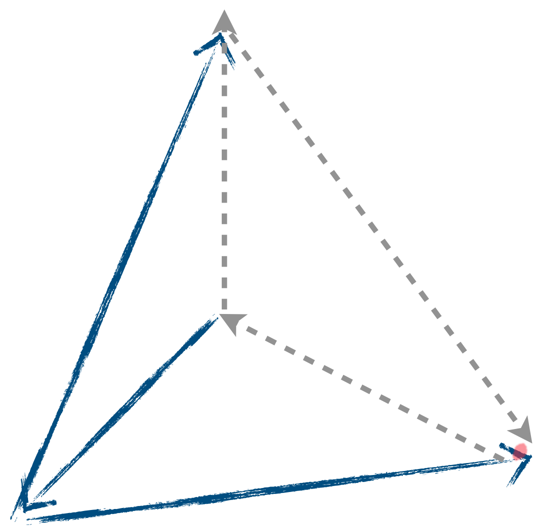
Gluing constraints; combination of BC amplitudes

- Full amplitude also via GFT formulation based on translation group

Conclusion

- Construction of a new SF model based on edge vectors.
- Algebraically, it is expressed in terms of irreps (and functions of) translation group;

→ Simplicial geometry is fully encoded and manifest;



→ Contains BC amplitudes, when expressed in terms of bivectors (harmonic oscillator/ translation group duality)

Outlook

- Analysis of amplitudes (divergences etc).
- Precise relation to BC model (what is encoded in the extra data?).
- Obtain an expression in terms of simplicial gravity action with edge vectors.
- Extracting physical consequences (e.g. in GFT cosmology)

Thank you for your attention!

Merci !

1. A bit of group theory: the interplay between the Lorentz group and translation group

A. Infinite dimensional unitary representations of the Lorentz group

- Finding unitary finite dimensional representations of the Lorentz group is still unsolved problem.
- Infinite dimensional ones are unitary and irreducible, studied by Dirac.

➡ Homogeneous realisation of the infinit dimensional irreps of the Lorentz group

- Lie group $SL(2,C)$ (double cover of the Lorentz group), its group element is given by: $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\alpha, \beta, \gamma, \delta \in C$ satisfying the relation $\alpha\delta - \beta\gamma = 1$

- The action of $R(g)$ on such homogeneous polynomials is $R(g)\,P(z_1, z_2) = P(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2)$ defines a realisation of the representation of $SL(2,C)$ and a Hilbert space D_λ

homogeneous polynomials P as a function of $z_1, z_2 \in C$ of order $n_1 - 1$ in z_1 and $n_2 - 1$ in z_2

- Define an appropriate scalar product with respect to which the operators $R(g)$ are unitary

$$\langle R(g) f_1(g), R(g) f_2(g) \rangle = \langle f_1(g), f_2(g) \rangle$$

Unitary reps but not yet an irreducible one!

- Use the homogeneous functions to compute the norm of a function belonging in the Hilbert D_λ

$$f(\sigma z_1, \sigma z_2) = \sigma^{\lambda_1} \bar{\sigma}^{\lambda_2} f(z_1, z_2) \Rightarrow ||f||^2 + |\sigma^{\lambda_1+\lambda_2+2}|\,.\,||f||^2 \qquad \Rightarrow \lambda_1 + \lambda_2 + 2 = 0$$

Unitary reps and irreducible one!

$R_{j\mu}(g)$: they are labelled by the half integer j and the real number $\mu \in R$ and again the $SL(2,C)$ transformations are specified by the action of $R_{i\bar{\mu}}$ on the polynomials of degree $(\frac{1}{2}(\mu + j), \frac{1}{2}(\mu - j))$.

X

Among the $SL(2,C)$ infinite dimensional representations, one can show that the unitary ones are those in the principal series

B. Expansors and the relation between Translation group and the Lorentz group

- Consider four real variables $\xi_0, \xi_1, \xi_2, \xi_3$ in Minkowski space
- A general vector in the product space will have coordinates A_{ijkh} which can be represented as the coefficients in a power series

$$P(\xi_\mu) = \sum_{ijkh} A_{ijkh}\, \xi_x^i \xi_y^j \xi_z^k \xi_t^{-1-h},$$

➡ The coefficient A is called expansor.
➡ These coefficients are regarded as the coordinates of vectors in a certain space of an infinite number of dimensions.

◉ For infinitesimal Lorentz transformation given by $\xi_0 = \xi_0' + \epsilon\xi_1', \quad \xi_1 = \xi_1' + \epsilon\xi_0', \quad \xi_2 = \xi_2', \quad \xi_3 = \xi_3'$

- This coordinate transformation induces the following expansors transformation:

$$\Sigma r!s!t!A_{rst}^{\prime 2} = \Sigma r!s!t!\left[A_{rst}^2 + 2(r+1)\epsilon A_{rt}A_{r+1,s-1,t} - 2(s+1)\epsilon A_{r-1,s+1,t}A_{rst}\right]$$

- Unitarity is enforced through the scalar product to be invariant $P_1 \cdot P_2 = \sum_{ijkh} A_{ijkh} B_{ijkh},$

The induced linear transformations on the expansors leave the square length invariant, presenting them as unitary representation of the Lorentz group.

The expansors present a quantisation of Minkowski space and the associated Hilbert space is then give by

$$L^2(M^4) \underset{X}{=} \oplus E^n$$

◉ Dirac idea: expansors can be interpreted as a tensor product of four harmonic oscillators, where the space components have positive energy whereas the time component has a negative one

$$x_a = \frac{1}{\sqrt{2}}\left(\xi_a + \frac{\partial}{\partial \xi_a}\right), \qquad \frac{\partial}{\partial x_a} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \xi_a} - \xi_a\right), \qquad a = x, y, z$$

$$x_t = \frac{1}{\sqrt{2}}\left(\xi_t - \frac{\partial}{\partial \xi_t}\right), \qquad \frac{\partial}{\partial x_t} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \xi_t} + \xi_t\right),$$

◉ The homogeneous polynomial on Minkowski space can be represented as a combination in the polynomials which are a general combination of four Hermite functions

$$\Psi_{ijkh}(t,x,y,z) = \frac{1}{\pi n!\sqrt{2^{i+j+k+h}}}\left(x_i - \partial_{x_i}\right)^i\left(x_j - \partial_{x_j}\right)^j\left(x_k - \partial_{x_k}\right)^k\left(x_h - \partial_{x_h}\right)^h e^{-\frac{1}{2}(x_t^2 + x_a x^a)}$$
$$= \psi_h(t)\,\psi_i(x)\,\psi_j(y)\,\psi_k(z)\,.$$

$$P(x_\mu) = \sum_{ijkh} A_{ijkh}\Psi_{ijkh}(x_\mu) \quad \Rightarrow \quad P(\xi_\mu) = \sum_{ijkh} A_{ijkh}\,\xi_x^i \xi_y^j \xi_z^k \xi_t^{-1-h},$$

- The alternative representation of the ξ variables that Dirac introduced is related to the theory of the four dimensional harmonic oscillator.
- The four x-parameters can be treated as the coordinates of a four-dimensional harmonic oscillator, whereas the respective four operators ∂_{x_μ} being the conjugate momenta p_{x_μ}.
- To illustrate further the duality between the expansors and the harmonic oscillator, a state of the oscillator with components $0,1,2,3$ occupying the ith, jth, kth, hth quantum states respectively is represented by Ψ. Following the map one can get back the ξ-representation and the function $\Psi_{ijkh}(x_\mu)$ goes over to $\xi_x^i \xi_y^j \xi_z^k \xi_t^{-1-h}$.
- In this sense, the state of the oscillator for which each of its components is in a quantum state is naturally identified with an expansor with one non-vanishing component, whereas a stationary states corresponds to a homogeneous expansor
- The degree of the expansor is this case represents the energy of the state.

• Recalling the expressions of the ladder operators associates to a four dimensional harmonic oscillator:

$$a_i^\dagger = \xi_i = \frac{1}{\sqrt{2}}(x_i - \partial_i)\,, \qquad a_0^\dagger = -\partial_{\xi_t} = \frac{1}{\sqrt{2}}(t - \partial_t)\,,$$

$$a_i = \partial_{\xi_i} = \frac{1}{\sqrt{2}}(x_i + \partial_i)\,, \qquad a_0 = \xi_t = \frac{1}{\sqrt{2}}(t + \partial_t)$$

Note that they are given by the inverse relation of

$$x_a = \frac{1}{\sqrt{2}}\left(\xi_a + \frac{\partial}{\partial \xi_a}\right), \qquad \frac{\partial}{\partial x_a} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \xi_a} - \xi_a\right),$$

$$x_t = \frac{1}{\sqrt{2}}\left(\xi_t - \frac{\partial}{\partial \xi_t}\right), \qquad \frac{\partial}{\partial x_t} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \xi_t} + \xi_t\right),$$

the space-like creation operators $a^\dagger$ are represented by the space coordinates of Minkowski space ξ_i but the time-like creation operator $a_0^\dagger$ is represented by its momentum (up to a sign), and vice-versa for the annihilation operators.

• Balanced representations of the Lorentz group in terms of representations of the translation group:

Derive the eigenstates of the Casimir operator (representations of the Lorentz group) as a combination of the eigenstates of the harmonic oscillator (representations of the translation group).

The set of homogeneous polynomials on Minkowski space can be derived as the general solution of the Schrödinger equation $H\,\Psi = E\,\Psi$ where the Hamiltonian operator $H = -\frac{1}{2}\Delta + \frac{1}{2}(t^2 - x^2 - y^2 - z^2)$

Relation between the wave functions associated to the infinite dimensional representations of the Lorentz group and the wave functions associated to the infinite dimensional representations of the translation group:

$$\Psi_{n_r,\mu,\ell,m}(r,\eta,\theta,\phi) = \sum_{n_t,n_x,n_y,n_z} \int d^4\alpha_\nu\, e^{-\frac{1}{2}\sum_\nu |\alpha_\nu|^2} C^{n_t,n_x,n_y,n_z}_{n_r,\mu,\ell,m} \frac{\alpha_t^{*\,n_t}\alpha_x^{*\,n_x}\alpha_y^{*\,n_y}\alpha_z^{*\,n_z}}{\pi^4\sqrt{n_t!n_x!n_y!n_z!}} \Psi_{\alpha_t,\alpha_x,\alpha_y,\alpha_z}(t,x,y,z)\,.$$

X

B. The quantum triangle

$$e_1 + e_2 = e_3 \quad \Rightarrow \quad \zeta_\mu + \lambda_\mu = \omega_\mu \,,$$

$$b := e_1 \wedge e_2 = e_1 \wedge e_3 = e_3 \wedge e_2 \,.$$

The steps to extracting the representation of the quantum bivector are:

- We take the skew part of the tensor product of the direct sum of all the expansors E^n (the edges e_i): the wedge product condition
- Each of these decomposes into a tower of copies of the bisector representation $R\left(0,\mu\right)$ and of the right parity.
- We then project onto the **balanced** part (simple bivector) ie. only the copies of $R\left(0,\mu\right)$ where now μ is any positive real number.
- Thus we get copies of the direct integral of all the $R\left(0,\mu\right)$ for each combination of two indices n, and two μ_i skew symmetrized with respect to the pair of indices.

- The quantization of Minkowski space $L^2[M^4]$ is realised as the Hilbert space associated to the translation group.

$$b := e_1 \wedge e_2 = e_1 \wedge e_3 = e_3 \wedge e_2 .$$

➡ Consider the Hilbert spaces associated to two vectors $e_1, e_2 \in M^4$ and then take the tensor product of the associated Hilbert spaces

$$L^2[e_1, e_2] := L^2[e_1] \otimes L^2[e_2] ,$$

- Denote $T_q^A[e_1, e_2]$ its skew symmetric part.

➡ The anti-symmetric condition ensures that the elements of $T_q^A[e_1, e_2]$ represents the bi-vectors obtained as the wedge product of e_1, e_2, and thus are normal of a triangle.

$$e_1 + e_2 = e_3 \quad \Rightarrow \quad \zeta_\mu + \lambda_\mu = \omega_\mu ,$$

- closure of the edge vectors of the triangle:

➡The Hilbert space of a bi-vector is the space $T_q^A[e_2, e_2]$ such that $L^2[e_2, e_2]$ is invariant under the switching operator

$$\sigma_q : L^2[e_1, e_2] \to L^2[e_1, e_3] .$$

This ensures that, given a (quantum) triangle, its description is not affected by the choice of the two edge vectors used to construct the bi-vector (normal to the triangle).

- Associate to such edges the position and momentum (or translation) operators
➡they are expressed as a combination of the ladder operators of the harmonic oscillator

$$a_1^\dagger := a_t\zeta_t + a_x^\dagger\zeta_x + a_y^\dagger\zeta_y + a_z^\dagger\zeta_z\,, \qquad a_2^\dagger := a_t\lambda_t + a_x^\dagger\lambda_x + a_y^\dagger\lambda_y + a_z^\dagger\lambda_z\,,$$

$$a_1 := a_t^\dagger\zeta_t + a_x\zeta_x + a_y\zeta_y + a_z\zeta_z\,, \qquad a_2 := a_t^\dagger\lambda_t + a_x\lambda_x + a_y\lambda_y + a_z\lambda_z\,.$$

a: generators of translations on Minlowski space,
$a^\dagger$: can be seen as the quantization of their dual momenta (position operators ξ on Minkowski space).

- The wedge product of the two edge vectors $e_1 \wedge e_2$ can be associated to an operator acting on the Hilbert space $L^2[\zeta, \lambda]$

$$b_{e_1\wedge e_2} := -ia_1^\dagger \wedge a_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1)$$

$b_{e_1\wedge e_2}$
=
quantization of the simple bivector $e_1 \wedge e_2$.

- **The wave function of the quantum bivector** $b_{e_1\wedge e_2} := -ia_1^\dagger \wedge a_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1)$ $\quad M^4 \wedge M^4 \cong so^*(1,3)$

- Use the operator associated to the bi-vector to expand the wave function in the Fourier decomposition:

$$f(\lambda_1, \lambda_2) := -i\int d\alpha d\alpha' \langle \alpha_\nu \,|\, a^\dagger(\lambda_1)a(\lambda_2) - a^\dagger(\lambda_2)a(\lambda_1) \,|\, \alpha'_\nu\rangle f_{\alpha_\nu,\alpha'_\nu}$$

role of plane wave in the Fourier transform.

$\langle \alpha_\nu \,|\, a^\dagger(\lambda_1)a(\lambda_2) - a^\dagger(\lambda_2)a(\lambda_1) \,|\, \alpha'_\nu\rangle$