Diffeomorphism Covariant Dynamics in Quantum Kantowski-Sachs

Rafael Guolo Dias, Ian Bornhoeft, Jonathan Engle

Florida Atlantic University

LOOPS'22

Summary

Introduction

General idea and motivation

- We offer a fresh perspective of finding a Hamiltonian constraint operator, by quantizing its required properties, instead of following the standard procedure.
    - Covariance under residual diffeomorphisms
- Good results for isotropic model (Engle-Vilensky, 2019).
- We aim to generalize the procedure for a Kantowski-Sachs framework (interior of a black hole).

General idea and motivation

- We offer a fresh perspective of finding a Hamiltonian constraint operator, by quantizing its required properties, instead of following the standard procedure.
  - Covariance under residual diffeomorphisms
- Good results for isotropic model (Engle-Vilensky, 2019).
- We aim to generalize the procedure for a Kantowski-Sachs framework (interior of a black hole).

General idea and motivation

- We offer a fresh perspective of finding a Hamiltonian constraint operator, by quantizing its required properties, instead of following the standard procedure.
  - Covariance under residual diffeomorphisms

- Good results for isotropic model (Engle-Vilensky, 2019).

- We aim to generalize the procedure for a Kantowski-Sachs framework (interior of a black hole).

General idea and motivation

- We offer a fresh perspective of finding a Hamiltonian constraint operator, by quantizing its required properties, instead of following the standard procedure.
  - Covariance under residual diffeomorphisms
- Good results for isotropic model (Engle-Vilensky, 2019).
- We aim to generalize the procedure for a Kantowski-Sachs framework (interior of a black hole).

Kantowski-Sachs as a black hole interior model

- Kantowski-Sachs: homogeneous model with spatial section of topology $\approx S^2 \times \mathbb{R}$.
- Metric:

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2 \tag{1}$$

- Relates to Schwarzschild interior metric by

$$|p_c| = \tau^2, \qquad p_b^2 = L_0^2 \left(\frac{2m}{\tau} - 1 \right) \tau^2, \qquad N^2 = \left(\frac{2m}{\tau} - 1 \right)^{-1}. \tag{2}$$

Kantowski-Sachs in Ashtekar-Barbero variables

- Ashtekar-Barbero variables:

$$
\begin{aligned}
A_a^1 &= -b\sin\theta\partial_a\phi \quad , \quad E_1^a = -\frac{p_b}{L_0}\phi^a \\
A_a^2 &= b\partial_a\theta \quad , \quad E_2^a = \frac{p_b}{L_0}\sin\theta\theta^a \\
A_a^3 &= \frac{c}{L_0}\partial_a x + \cos\theta\partial_a\phi \quad , \quad E_3^a = p_c\sin\theta x^a
\end{aligned} \tag{3}
$$

- Symplectic Structure

$$
\{b, p_b\} = G\gamma \quad , \quad \{c, p_c\} = 2G\gamma \tag{4}
$$

- Hamiltonian Constraint (for an arbitrary lapse N)

$$
H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\mathrm{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{5}
$$

Diffeomorphism Covariance

Classical diffeomorphism covariance and path to quantization

- Residual diffeomorphisms: group of transformations preserving the form of (A, E).
  - subgroup with non-trivial action generated by $\{x\vec{x}\}$.
  - Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{6}$$

- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{7}$$

- However, the flows in (6) are non-canonical, so (7) cannot be directly applied;

- Nevertheless, they can be cast in a related form

$$\begin{aligned} \dot{F} &= \omega_1\{\Lambda_1, F\}(b, p_b) + \omega_2\{\Lambda_2, F\}(c, p_c) \\ &= p_b\left\{\frac{b}{\gamma G}, F\right\} - c\left\{\frac{p_c}{2\gamma G}, F\right\} \end{aligned} \tag{8}$$

Classical diffeomorphism covariance and path to quantization

- Residual diffeomorphisms: group of transformations preserving the form of (A, E).
  - subgroup with non-trivial action generated by $\{x\vec{x}\}$.
  - Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{6}$$

- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{7}$$

- However, the flows in (6) are non-canonical, so (7) cannot be directly applied;
- Nevertheless, they can be cast in a related form

$$\begin{aligned}\dot{F} &= \omega_1\{\Lambda_1, F\}(b, p_b) + \omega_2\{\Lambda_2, F\}(c, p_c)\\ &= p_b\left\{\frac{b}{\gamma G}, F\right\} - c\left\{\frac{p_c}{2\gamma G}, F\right\}\end{aligned} \tag{8}$$

Classical diffeomorphism covariance and path to quantization

- Residual diffeomorphisms: group of transformations preserving the form of (A, E).
  - subgroup with non-trivial action generated by $\{x\vec{x}\}$.
  - Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{6}$$

- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{7}$$

- However, the flows in (6) are non-canonical, so (7) cannot be directly applied;

- Nevertheless, they can be cast in a related form

$$\begin{aligned}\dot{F} &= \omega_1\{\Lambda_1, F\}(b, p_b) + \omega_2\{\Lambda_2, F\}(c, p_c) \\ &= p_b\left\{\frac{b}{\gamma G}, F\right\} - c\left\{\frac{p_c}{2\gamma G}, F\right\}\end{aligned} \tag{8}$$

Classical diffeomorphism covariance and path to quantization

- Residual diffeomorphisms: group of transformations preserving the form of (A, E).
  - subgroup with non-trivial action generated by $\{x\vec{x}\}$.
  - Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{6}$$

- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{7}$$

- However, the flows in (6) are non-canonical, so (7) cannot be directly applied;
- Nevertheless, they can be cast in a related form

$$\begin{aligned}\dot{F} &= \omega_1\{\Lambda_1, F\}(b, p_b) + \omega_2\{\Lambda_2, F\}(c, p_c)\\ &= p_b\left\{\frac{b}{\gamma G}, F\right\} - c\left\{\frac{p_c}{2\gamma G}, F\right\}\end{aligned} \tag{8}$$

Obtaining Hamiltonian operator

- Because the Hamiltonian is of density weight one, one shows that, under the residual diffeomorphism flow, $\dot{H}_{c\ell} = H_{c\ell}$.

- We seek to require the quantum Hamiltonian to follow a quantization of this condition (*quantum covariance*).

- Turn our quantities into operators, Poisson brackets into commutators and choose the Weyl ordering for quantizing the products:

$$\hat{H} = \frac{1}{2i\hbar\gamma G}\left\{\hat{p}_b\left[\hat{b},\hat{H}\right] + \left[\hat{b},\hat{H}\right]\hat{p}_b\right\} + \frac{1}{4i\hbar\gamma G}\left\{\hat{c}\left[\hat{p}_c,\hat{H}\right] + \left[\hat{p}_c,\hat{H}\right]\hat{c}\right\}. \quad (9)$$

- Since $\hat{b}$ and $\hat{c}$ are ill-defined in LQG, first find the general solution for the matrix elements in the $|p_b, p_c\rangle$ basis of the Schrodinger representation, with later imposition of preservation of the Bohr-Hilbert space.

Obtaining Hamiltonian operator

- Because the Hamiltonian is of density weight one, one shows that, under the residual diffeomorphism flow, $\dot{H}_{c\ell} = H_{c\ell}$.

- We seek to require the quantum Hamiltonian to follow a quantization of this condition (*quantum covariance*).

- Turn our quantities into operators, Poisson brackets into commutators and choose the Weyl ordering for quantizing the products:

$$\hat{H} = \frac{1}{2i\hbar\gamma G}\left\{\hat{p}_b\left[\hat{b},\hat{H}\right]+\left[\hat{b},\hat{H}\right]\hat{p}_b\right\}+\frac{1}{4i\hbar\gamma G}\left\{\hat{c}\left[\hat{p}_c,\hat{H}\right]+\left[\hat{p}_c,\hat{H}\right]\hat{c}\right\}. \quad (9)$$

- Since $\hat{b}$ and $\hat{c}$ are ill-defined in LQG, first find the general solution for the matrix elements in the $|p_b, p_c\rangle$ basis of the Schrodinger representation, with later imposition of preservation of the Bohr-Hilbert space.

Obtaining Hamiltonian operator

- Because the Hamiltonian is of density weight one, one shows that, under the residual diffeomorphism flow, $\dot{H}_{c\ell} = H_{c\ell}$.
- We seek to require the quantum Hamiltonian to follow a quantization of this condition (*quantum covariance*).
- Turn our quantities into operators, Poisson brackets into commutators and choose the Weyl ordering for quantizing the products:

$$\hat{H} = \frac{1}{2i\hbar\gamma G}\left\{\hat{p}_b\left[\hat{b},\hat{H}\right] + \left[\hat{b},\hat{H}\right]\hat{p}_b\right\} + \frac{1}{4i\hbar\gamma G}\left\{\hat{c}\left[\hat{p}_c,\hat{H}\right] + \left[\hat{p}_c,\hat{H}\right]\hat{c}\right\}. \tag{9}$$

- Since $\hat{b}$ and $\hat{c}$ are ill-defined in LQG, first find the general solution for the matrix elements in the $|p_b, p_c\rangle$ basis of the Schrodinger representation, with later imposition of preservation of the Bohr-Hilbert space.

Obtaining Hamiltonian operator

- Because the Hamiltonian is of density weight one, one shows that, under the residual diffeomorphism flow, $\dot{H}_{c\ell} = H_{c\ell}$.
- We seek to require the quantum Hamiltonian to follow a quantization of this condition (*quantum covariance*).
- Turn our quantities into operators, Poisson brackets into commutators and choose the Weyl ordering for quantizing the products:

$$\hat{H} = \frac{1}{2i\hbar\gamma G}\left\{\hat{p}_b\left[\hat{b},\hat{H}\right] + \left[\hat{b},\hat{H}\right]\hat{p}_b\right\} + \frac{1}{4i\hbar\gamma G}\left\{\hat{c}\left[\hat{p}_c,\hat{H}\right] + \left[\hat{p}_c,\hat{H}\right]\hat{c}\right\}. \quad (9)$$

- Since $\hat{b}$ and $\hat{c}$ are ill-defined in LQG, first find the general solution for the matrix elements in the $|p_b, p_c\rangle$ basis of the Schrodinger representation, with later imposition of preservation of the Bohr-Hilbert space.

Obtaining Hamiltonian operator

- The general solution for the matrix elements $\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle$ is

$$\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle = C_{\operatorname{sgn}(p_b''+p_b')} \left[p_b'' - p_b', p_c'' + p_c', (p_b''+p_b')(p_c''-p_c')\right] (p_b''+p_b')^2 \quad (10)$$

- Use completeness of momentum basis to obtain the action of the Hamiltonian on a general state $|p_b', p_c'\rangle$:

$$\hat{H}|p_b', p_c'\rangle = \int |p_b'', p_c''\rangle\langle p_b'', p_c''|\hat{H}|p_b', p_c'\rangle dp_b'' dp_c'' \quad (11)$$

- But p_b'', p_c'' relates to p_b', p_c' by shifts

$$\begin{aligned} p_b'' &= p_b' + (p_b'' - p_b') := p_b' + \gamma \ell_p^2 A \\ p_c'' &= p_c' + \frac{\frac{1}{2}(p_b' + p_b'')(p_c'' - p_c')}{p_b' + \frac{1}{2}(p_b'' - p_b')} := p_c' + \frac{4\gamma \ell_p^4 B}{p_b' - \frac{1}{2}\gamma \ell_p^2 A} \end{aligned} \quad (12)$$

- By changing variables, we rewrite (11) in terms of the shifts (12) and an unconstrained parameter function $\alpha : \mathbb{R}^3 \to \mathbb{C}$

$$\hat{H}|p_b', p_c'\rangle = \int e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} \hat{p}_b\ \alpha(A, B, \hat{p}_c, \operatorname{sgn} p_b) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} |p_b', p_c'\rangle\ dA dB \quad (13)$$

Obtaining Hamiltonian operator

- The general solution for the matrix elements $\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle$ is

$$\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle = C_{\mathrm{sgn}(p_b''+p_b')} \left[p_b'' - p_b', p_c'' + p_c', (p_b'' + p_b')(p_c'' - p_c') \right] (p_b'' + p_b')^2 \quad (10)$$

- Use completeness of momentum basis to obtain the action of the Hamiltonian on a general state $|p_b', p_c'\rangle$:

$$\hat{H}|p_b', p_c'\rangle = \int |p_b'', p_c''\rangle \langle p_b'', p_c''|\hat{H}|p_b', p_c'\rangle dp_b'' dp_c'' \quad (11)$$

- But p_b'', p_c'' relates to p_b', p_c' by shifts

$$p_b'' = p_b' + (p_b'' - p_b') := p_b' + \gamma \ell_p^2 A$$
$$p_c'' = p_c' + \frac{\frac{1}{2}(p_b' + p_b'')(p_c'' - p_c')}{p_b' + \frac{1}{2}(p_b'' - p_b')} := p_c' + \frac{4\gamma \ell_p^4 B}{p_b' - \frac{1}{2}\gamma \ell_p^2 A} \quad (12)$$

- By changing variables, we rewrite (11) in terms of the shifts (12) and an unconstrained parameter function $\alpha : \mathbb{R}^3 \to \mathbb{C}$

$$\hat{H}|p_b', p_c'\rangle = \int e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}\ \hat{p}_b\ \alpha(A, B, \hat{p}_c, \mathrm{sgn}\, p_b) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} |p_b', p_c'\rangle\ dA dB \quad (13)$$

Obtaining Hamiltonian operator

- The general solution for the matrix elements $\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle$ is

$$\left\langle p_b'', p_c'' \left| \hat{H} \right| p_b', p_c' \right\rangle = C_{\mathrm{sgn}(p_b''+p_b')} \left[p_b'' - p_b', p_c'' + p_c', (p_b'' + p_b')(p_c'' - p_c') \right] (p_b'' + p_b')^2 \quad (10)$$

- Use completeness of momentum basis to obtain the action of the Hamiltonian on a general state $|p_b', p_c'\rangle$:

$$\hat{H}|p_b', p_c'\rangle = \int |p_b'', p_c''\rangle\langle p_b'', p_c''|\hat{H}|p_b', p_c'\rangle dp_b'' dp_c'' \quad (11)$$

- But p_b'', p_c'' relates to p_b', p_c' by shifts

$$\begin{aligned} p_b'' &= p_b' + (p_b'' - p_b') := p_b' + \gamma \ell_p^2 A \\ p_c'' &= p_c' + \frac{\frac{1}{2}(p_b' + p_b'')(p_c'' - p_c')}{p_b' + \frac{1}{2}(p_b'' - p_b')} := p_c' + \frac{4\gamma \ell_p^4 B}{p_b' - \frac{1}{2}\gamma \ell_p^2 A} \end{aligned} \quad (12)$$

- By changing variables, we rewrite (11) in terms of the shifts (12) and an unconstrained parameter function $\alpha : \mathbb{R}^3 \to \mathbb{C}$

$$\hat{H}|p_b', p_c'\rangle = \int e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} \; \hat{p}_b \; \alpha(A, B, \hat{p}_c, \mathrm{sgn}\, p_b) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} |p_b', p_c'\rangle \; dA dB \quad (13)$$

Obtaining Hamiltonian operator

- The general solution for the matrix elements $\left\langle p_b'', p_c'' \middle| \hat{H} \middle| p_b', p_c' \right\rangle$ is

$$\left\langle p_b'', p_c'' \middle| \hat{H} \middle| p_b', p_c' \right\rangle = C_{\mathrm{sgn}(p_b''+p_b')} \left[p_b'' - p_b', p_c'' + p_c', (p_b'' + p_b')(p_c'' - p_c')\right] (p_b'' + p_b')^2 \quad (10)$$

- Use completeness of momentum basis to obtain the action of the Hamiltonian on a general state $|p_b', p_c'\rangle$:

$$\hat{H}|p_b', p_c'\rangle = \int |p_b'', p_c''\rangle\langle p_b'', p_c''|\hat{H}|p_b', p_c'\rangle dp_b'' dp_c'' \quad (11)$$

- But p_b'', p_c'' relates to p_b', p_c' by shifts

$$\begin{aligned} p_b'' &= p_b' + (p_b'' - p_b') := p_b' + \gamma \ell_p^2 A \\ p_c'' &= p_c' + \frac{\frac{1}{2}(p_b' + p_b'')(p_c'' - p_c')}{p_b' + \frac{1}{2}(p_b'' - p_b')} := p_c' + \frac{4\gamma \ell_p^4 B}{p_b' - \frac{1}{2}\gamma \ell_p^2 A} \end{aligned} \quad (12)$$

- By changing variables, we rewrite (11) in terms of the shifts (12) and an unconstrained parameter function $\alpha : \mathbb{R}^3 \to \mathbb{C}$

$$\hat{H}|p_b', p_c'\rangle = \int e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} \ \hat{p}_b \ \alpha(A, B, \hat{p}_c, \mathrm{sgn}\, p_b) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} |p_b', p_c'\rangle \ dA dB \quad (13)$$

Hamiltonian Operator

- Prescription ordering for quantization: for a general function $f(p_b, p_c)$,

$$\widehat{f(p_b, p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} f(\hat{p}_b, \hat{p}_c) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} \tag{14}$$

- and thus

$$\hat{H} = \int \widehat{p_b\ a(A, B, p_c) e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}}\ dAdB. \tag{15}$$

- *Preservation of Bohr-Hilbert space*: for any p'_b, p'_c there must be at most countable p''_b, p''_c such that the matrix elements $\left\langle p''_b, p''_c \middle| \hat{H} \middle| p'_b, p'_c \right\rangle$ are non-zero:
    - Require $\alpha(A, B, p_c, \operatorname{sgn} p_b) = \sum_n \alpha_n(p_c, \operatorname{sgn} p_b)\delta[A - A_n(p_c)]\delta[B - B_n(p_c)]$, then

$$\hat{H} = \sum_n \widehat{p_b \alpha_n(p_c, \operatorname{sgn} p_b) e^{i\left(A_n(p_c)b+B_n(p_c)\ell_p r\frac{c}{p_b}\right)}}. \tag{16}$$

- *Classical analogue* of operator: preimage under quantization map:

$$H = \sum_n p_b \alpha_n(p_c, \operatorname{sgn} p_b) e^{i\left(A_n(p_c)b+B_n(p_c)\frac{c}{p_b}\right)} \tag{17}$$

Hamiltonian Operator

- Prescription ordering for quantization: for a general function $f(p_b, p_c)$,

$$\widehat{f(p_b,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}}e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}f(\hat{p}_b,\hat{p}_c)e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}e^{\frac{iA}{2}\hat{b}} \tag{14}$$

- and thus

$$\hat{H} = \int \widehat{p_b\ a(A,B,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}}\ dAdB. \tag{15}$$

- *Preservation of Bohr-Hilbert space*: for any p_b', p_c' there must be at most countable p_b'', p_c'' such that the matrix elements $\left\langle p_b'', p_c'' \middle| \hat{H} \middle| p_b', p_c' \right\rangle$ are non-zero:
  - Require $\alpha(A, B, p_c, \operatorname{sgn} p_b) = \sum_n \alpha_n(p_c, \operatorname{sgn} p_b)\delta[A - A_n(p_c)]\delta[B - B_n(p_c)]$, then

$$\hat{H} = \sum_n \widehat{p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\ell_p r\frac{c}{p_b}\right)}}. \tag{16}$$

- *Classical analogue* of operator: preimage under quantization map:

$$H = \sum_n p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\frac{c}{p_b}\right)} \tag{17}$$

Hamiltonian Operator

- Prescription ordering for quantization: for a general function $f(p_b, p_c)$,

$$\widehat{f(p_b,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}}e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}f(\hat{p}_b,\hat{p}_c)e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}e^{\frac{iA}{2}\hat{b}} \tag{14}$$

- and thus

$$\hat{H} = \int \widehat{p_b\ a(A,B,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}}\ dAdB. \tag{15}$$

- *Preservation of Bohr-Hilbert space*: for any p_b', p_c' there must be at most countable p_b'', p_c'' such that the matrix elements $\left\langle p_b'', p_c'' \middle| \hat{H} \middle| p_b', p_c' \right\rangle$ are non-zero:
  - Require $\alpha(A, B, p_c, \operatorname{sgn} p_b) = \sum_n \alpha_n(p_c, \operatorname{sgn} p_b)\delta[A - A_n(p_c)]\delta[B - B_n(p_c)]$, then

$$\hat{H} = \sum_n \widehat{p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\ell_p r\frac{c}{p_b}\right)}}. \tag{16}$$

- *Classical analogue* of operator: preimage under quantization map:

$$H = \sum_n p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\frac{c}{p_b}\right)} \tag{17}$$

Hamiltonian Operator

- Prescription ordering for quantization: for a general function $f(p_b, p_c)$,

$$\widehat{f(p_b,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}}e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}f(\hat{p}_b,\hat{p}_c)e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}}e^{\frac{iA}{2}\hat{b}} \tag{14}$$

- and thus

$$\hat{H} = \int \widehat{p_b\ a(A,B,p_c)e^{i\left(Ab+B\ell_p^2\frac{c}{p_b}\right)}}\ dAdB. \tag{15}$$

- *Preservation of Bohr-Hilbert space*: for any p'_b, p'_c there must be at most countable p''_b, p''_c such that the matrix elements $\left\langle p''_b, p''_c \middle| \hat{H} \middle| p'_b, p'_c \right\rangle$ are non-zero:
  - Require $\alpha(A,B,p_c,\operatorname{sgn} p_b) = \sum_n \alpha_n(p_c,\operatorname{sgn} p_b)\delta[A - A_n(p_c)]\delta[B - B_n(p_c)]$, then

$$\hat{H} = \sum_n \widehat{p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\ell_p r\frac{c}{p_b}\right)}}. \tag{16}$$

- *Classical analogue* of operator: preimage under quantization map:

$$H = \sum_n p_b\alpha_n(p_c,\operatorname{sgn} p_b)e^{i\left(A_n(p_c)b+B_n(p_c)\frac{c}{p_b}\right)} \tag{17}$$

Hamiltonian Operator

- Prescription ordering for quantization: for a general function $f(p_b, p_c)$,

$$\widehat{f(p_b, p_c) e^{i\left(Ab + B\ell_p^2 \frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} f(\hat{p}_b, \hat{p}_c) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}} \tag{14}$$

- and thus

$$\hat{H} = \int \widehat{p_b \; a(A, B, p_c) e^{i\left(Ab + B\ell_p^2 \frac{c}{p_b}\right)}} \, dA dB. \tag{15}$$

- *Preservation of Bohr-Hilbert space*: for any p_b', p_c' there must be at most countable p_b'', p_c'' such that the matrix elements $\left\langle p_b'', p_c'' \middle| \hat{H} \middle| p_b', p_c' \right\rangle$ are non-zero:
  - Require $\alpha(A, B, p_c, \operatorname{sgn} p_b) = \sum_n \alpha_n(p_c, \operatorname{sgn} p_b) \delta[A - A_n(p_c)] \delta[B - B_n(p_c)]$, then

$$\hat{H} = \sum_n \widehat{p_b \alpha_n(p_c, \operatorname{sgn} p_b) e^{i\left(A_n(p_c) b + B_n(p_c) \ell_p r \frac{c}{p_b}\right)}}. \tag{16}$$

- *Classical analogue* of operator: preimage under quantization map:

$$H = \sum_n p_b \alpha_n(p_c, \operatorname{sgn} p_b) e^{i\left(A_n(p_c) b + B_n(p_c) \frac{c}{p_b}\right)} \tag{17}$$

Required Symmetries

- As a consequence of the ordering prescription for quantization, symmetries can be easily checked directly in the classical analogue.
- Adjust the format of the sum elements to make manifest the required symmetries:
  1. *Hermiticity*: $\hat{H} = \hat{H}^\dagger = \hat{\bar{H}} \;\Rightarrow\; \bar{H} = H$,
  2. *b-Parity*: $\Pi_b : (b, p_b) \mapsto (-b, -p_b)$
     - Equivalent to an internal gauge rotation of π around the 3-axis
     - $\hat{\Pi}_b \hat{H} \hat{\Pi}_b = \widehat{\Pi_b^* H} \;\Rightarrow\; \Pi_b^* H = -H$ (*covariant*)
  3. *c-Parity*: $\Pi_c : (c, p_c) \mapsto (-c, -p_c)$
     - antipodal map $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$ + internal parity under 3-axis
     - $\Pi_c^* H = -H$ (*covariant*)
- The general form for H is then

$$H = p_b \,\mathrm{sgn}(p_c) a_0(|p_c|) + p_b \sum_{n=1}^{N} \left\{ a_n(p_c) \cos\left[A_n(p_c) b + B_n(p_c)\frac{c}{p_b}\right] - a_n(-p_c)\cos\left[A_n(-p_c) b - B_n(-p_c)\frac{c}{p_b}\right]\right\} + |p_b| \sum_{n=1}^{N} \left\{ b_n(p_c) \sin\left[A_n(p_c) b + B_n(p_c)\frac{c}{p_b}\right] - b_n(-p_c)\sin\left[A_n(-p_c) b - B_n(-p_c)\frac{c}{p_b}\right]\right\} \tag{18}$$

for $a_n, b_n, A_n, B_n \in \mathbb{R}$.

Required Symmetries

- As a consequence of the ordering prescription for quantization, symmetries can be easily checked directly in the classical analogue.
- Adjust the format of the sum elements to make manifest the required symmetries:

1. *Hermiticity*: $\hat{H} = \hat{H}^\dagger = \hat{\bar{H}} \;\Rightarrow\; \bar{H} = H$,
2. *b-Parity*: $\Pi_b : (b, p_b) \mapsto (-b, -p_b)$
   - Equivalent to an internal gauge rotation of π around the 3-axis
   - $\hat{\Pi}_b \hat{H} \hat{\Pi}_b = \widehat{\Pi_b^* H} \;\Rightarrow\; \Pi_b^* H = -H$ (*covariant*)
3. *c-Parity*: $\Pi_c : (c, p_c) \mapsto (-c, -p_c)$
   - antipodal map $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$ + internal parity under 3-axis
   - $\Pi_c^* H = -H$ (*covariant*)

- The general form for H is then

$$H = p_b \,\mathrm{sgn}(p_c) a_0(|p_c|) + p_b \sum_{n=1}^{N} \left\{ a_n(p_c) \cos\left[A_n(p_c) b + B_n(p_c)\frac{c}{p_b}\right] - a_n(-p_c)\cos\left[A_n(-p_c) b - B_n(-p_c)\frac{c}{p_b}\right]\right\} + |p_b| \sum_{n=1}^{N} \left\{ b_n(p_c) \sin\left[A_n(p_c) b + B_n(p_c)\frac{c}{p_b}\right] - b_n(-p_c)\sin\left[A_n(-p_c) b - B_n(-p_c)\frac{c}{p_b}\right]\right\} \quad (18)$$

for $a_n, b_n, A_n, B_n \in \mathbb{R}$.

Required Symmetries

- As a consequence of the ordering prescription for quantization, symmetries can be easily checked directly in the classical analogue.
- Adjust the format of the sum elements to make manifest the required symmetries:
  1. *Hermiticity*: $\hat{H} = \hat{H}^{\dagger} = \hat{\bar{H}} \;\Rightarrow\; \bar{H} = H$,
  2. *b-Parity*: $\Pi_b : (b, p_b) \mapsto (-b, -p_b)$
     - Equivalent to an internal gauge rotation of π around the 3-axis
     - $\hat{\Pi}_b \hat{H} \hat{\Pi}_b = \widehat{\Pi_b^* H} \;\Rightarrow\; \Pi_b^* H = -H$ (*covariant*)
  3. *c-Parity*: $\Pi_c : (c, p_c) \mapsto (-c, -p_c)$
     - antipodal map $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$ + internal parity under 3-axis
     - $\Pi_c^* H = -H$ (*covariant*)
- The general form for H is then

$$H = p_b \,\mathrm{sgn}(p_c) a_0(|p_c|) + p_b \sum_{n=1}^{N} \left\{ a_n(p_c) \cos\left[A_n(p_c) b + B_n(p_c) \frac{c}{p_b} \right] - a_n(-p_c) \cos\left[A_n(-p_c) b - B_n(-p_c) \frac{c}{p_b} \right] \right\} + |p_b| \sum_{n=1}^{N} \left\{ b_n(p_c) \sin\left[A_n(p_c) b + B_n(p_c) \frac{c}{p_b} \right] - b_n(-p_c) \sin\left[A_n(-p_c) b - B_n(-p_c) \frac{c}{p_b} \right] \right\} \tag{18}$$

for $a_n, b_n, A_n, B_n \in \mathbb{R}$.

Classical asymptotic behavior

- We expect classical behavior for limits of low curvature $\Rightarrow b, c \to 0$.
- Expand $\cos[...]$ and $\sin[...]$ in powers of b, c.
- Match with terms of same order in $H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right]$

$$\mathcal{O}(1): \quad -8\pi N\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = a_0(|p_c|) + \sum_{n=1}^{N}\left[a_n(p_c) + a_n(-p_c)\right] \tag{19}$$

$$\mathcal{O}(b): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)A_n(p_c) + b_n(-p_c)A_n(-p_c)\right] \tag{20}$$

$$\mathcal{O}(c): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)B_n(p_c) + b_n(-p_c)B_n(-p_c)\right] \tag{21}$$

$$\mathcal{O}(bc): \quad -\frac{16\pi N}{\gamma^2}\sqrt{|p_c|} = \sum_{n=1}^{N}\left[a_n(-p_c)A_n(-p_c)B_n(-p_c) - a_n(p_c)A_n(p_c)B_n(p_c)\right] \tag{22}$$

$$\mathcal{O}(b^2): \quad \frac{16\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = \sum_{n=1}^{N}\left[a_n(p_c)A_n^2(p_c) + a_n(-p_c)A_n^2(-p_c)\right] \tag{23}$$

$$\mathcal{O}(c^2): \quad 0 = \sum_{n=1}^{N}\left[a_n(p_c)B_n^2(p_c) + a_n(-p_c)B_n^2(-p_c)\right] \tag{24}$$

Classical asymptotic behavior

- We expect classical behavior for limits of low curvature $\Rightarrow b, c \to 0$.
- Expand $\cos[...]$ and $\sin[...]$ in powers of b, c.
- Match with terms of same order in $H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right]$

$$\mathcal{O}(1): \quad -8\pi N\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = a_0(|p_c|) + \sum_{n=1}^{N}\left[a_n(p_c) + a_n(-p_c)\right] \tag{19}$$

$$\mathcal{O}(b): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)A_n(p_c) + b_n(-p_c)A_n(-p_c)\right] \tag{20}$$

$$\mathcal{O}(c): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)B_n(p_c) + b_n(-p_c)B_n(-p_c)\right] \tag{21}$$

$$\mathcal{O}(bc): \quad -\frac{16\pi N}{\gamma^2}\sqrt{|p_c|} = \sum_{n=1}^{N}\left[a_n(-p_c)A_n(-p_c)B_n(-p_c) - a_n(p_c)A_n(p_c)B_n(p_c)\right] \tag{22}$$

$$\mathcal{O}(b^2): \quad \frac{16\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = \sum_{n=1}^{N}\left[a_n(p_c)A_n^2(p_c) + a_n(-p_c)A_n^2(-p_c)\right] \tag{23}$$

$$\mathcal{O}(c^2): \quad 0 = \sum_{n=1}^{N}\left[a_n(p_c)B_n^2(p_c) + a_n(-p_c)B_n^2(-p_c)\right] \tag{24}$$

Classical asymptotic behavior

- We expect classical behavior for limits of low curvature $\Rightarrow b, c \to 0$.
- Expand $\cos[...]$ and $\sin[...]$ in powers of b, c.
- Match with terms of same order in $H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right]$

$$\mathcal{O}(1): \quad -8\pi N\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = a_0(|p_c|) + \sum_{n=1}^{N}\left[a_n(p_c) + a_n(-p_c)\right] \tag{19}$$

$$\mathcal{O}(b): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)A_n(p_c) + b_n(-p_c)A_n(-p_c)\right] \tag{20}$$

$$\mathcal{O}(c): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)B_n(p_c) + b_n(-p_c)B_n(-p_c)\right] \tag{21}$$

$$\mathcal{O}(bc): \quad -\frac{16\pi N}{\gamma^2}\sqrt{|p_c|} = \sum_{n=1}^{N}\left[a_n(-p_c)A_n(-p_c)B_n(-p_c) - a_n(p_c)A_n(p_c)B_n(p_c)\right] \tag{22}$$

$$\mathcal{O}(b^2): \quad \frac{16\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = \sum_{n=1}^{N}\left[a_n(p_c)A_n^2(p_c) + a_n(-p_c)A_n^2(-p_c)\right] \tag{23}$$

$$\mathcal{O}(c^2): \quad 0 = \sum_{n=1}^{N}\left[a_n(p_c)B_n^2(p_c) + a_n(-p_c)B_n^2(-p_c)\right] \tag{24}$$

Classical asymptotic behavior

- We expect classical behavior for limits of low curvature $\Rightarrow b, c \to 0$.
- Expand $\cos[...]$ and $\sin[...]$ in powers of b, c.
- Match with terms of same order in $H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right]$

$$\mathcal{O}(1): \quad -8\pi N\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = a_0(|p_c|) + \sum_{n=1}^{N}\left[a_n(p_c) + a_n(-p_c)\right] \tag{19}$$

$$\mathcal{O}(b): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)A_n(p_c) + b_n(-p_c)A_n(-p_c)\right] \tag{20}$$

$$\mathcal{O}(c): \quad 0 = \sum_{n=1}^{N}\left[b_n(p_c)B_n(p_c) + b_n(-p_c)B_n(-p_c)\right] \tag{21}$$

$$\mathcal{O}(bc): \quad -\frac{16\pi N}{\gamma^2}\sqrt{|p_c|} = \sum_{n=1}^{N}\left[a_n(-p_c)A_n(-p_c)B_n(-p_c) - a_n(p_c)A_n(p_c)B_n(p_c)\right] \tag{22}$$

$$\mathcal{O}(b^2): \quad \frac{16\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}} = \sum_{n=1}^{N}\left[a_n(p_c)A_n^2(p_c) + a_n(-p_c)A_n^2(-p_c)\right] \tag{23}$$

$$\mathcal{O}(c^2): \quad 0 = \sum_{n=1}^{N}\left[a_n(p_c)B_n^2(p_c) + a_n(-p_c)B_n^2(-p_c)\right] \tag{24}$$

Discussion

Ongoing work / Next steps

- Cast the equations from asymptotic behavior in a better way to solve them (even/odd parts?).
- Check conditions a minimal number of terms
  - Promising results for isotropic case with $N = 1$, matching previous proposals.
- Compare with other proposals of quantum Hamiltonian for Kantowski-Sachs framework (suggestions?)

References

Ashtekar, A. & Bojowald, M. Quantum geometry and the Schwarzschild singularity. *Classical and Quantum Gravity* **23,** 391–411. https://doi.org/10.1088%2F0264-9381%2F23%2F2%2F008 (Dec. 2005).

Ashtekar, A., Olmedo, J. & Singh, P. Quantum extension of the Kruskal spacetime. *Physical Review D* **98.** https://doi.org/10.1103%2Fphysrevd.98.126003 (Dec. 2018).

Corichi, A. & Singh, P. Loop quantization of the Schwarzschild interior revisited. *Classical and Quantum Gravity* **33,** 055006. https://doi.org/10.1088%2F0264-9381%2F33%2F5%2F055006 (Feb. 2016).

Engle, J. & Vilensky, I. Deriving loop quantum cosmology dynamics from diffeomorphism invariance. *Physical Review D* **98.** https://doi.org/10.1103%2Fphysrevd.98.023505 (July 2018).

Engle, J. & Vilensky, I. Uniqueness of minimal loop quantum cosmology dynamics. *Physical Review D* **100.** https://doi.org/10.1103%2Fphysrevd.100.121901 (Dec. 2019).

Thank You!