Residual Diffeomorphisms in Kantowski-Sachs Spacetime

Ian W. Bornhoeft, Rafael Guolo Dias, Jonathan Engle

Florida Atlantic University

LOOPS'22

Summary

Background

Kantowski-Sachs as a black hole interior model

- Kantowski-Sachs: homogeneous model with spatial section topology $\approx S^2 \times \mathbb{R}$.
- Metric:

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2 \tag{1}$$

- Relates to Schwarzschild interior metric by:

$$|p_c| = \tau^2, \qquad p_b^2 = L_0^2 \left(\frac{2m}{\tau} - 1\right) \tau^2, \qquad N^2 = \left(\frac{2m}{\tau} - 1\right)^{-1}. \tag{2}$$

Kantowski-Sachs as a black hole interior model

- Kantowski-Sachs: homogeneous model with spatial section topology $\approx S^2 \times \mathbb{R}$.

- Metric:

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2 \tag{1}$$

- Relates to Schwarzschild interior metric by:

$$|p_c| = \tau^2, \qquad p_b^2 = L_0^2 \left(\frac{2m}{\tau} - 1\right)\tau^2, \qquad N^2 = \left(\frac{2m}{\tau} - 1\right)^{-1}. \tag{2}$$

Kantowski-Sachs as a black hole interior model

- Kantowski-Sachs: homogeneous model with spatial section topology $\approx S^2 \times \mathbb{R}$.

- Metric:

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2 \tag{1}$$

- Relates to Schwarzschild interior metric by:

$$|p_c| = \tau^2, \qquad p_b^2 = L_0^2 \left(\frac{2m}{\tau} - 1 \right) \tau^2, \qquad N^2 = \left(\frac{2m}{\tau} - 1 \right)^{-1}. \tag{2}$$

Kantowski-Sachs in Ashtekar-Barbero variables

- Ashtekar-Barbero variables:

$$\begin{aligned} A_a^1 &= -b\sin\theta\partial_a\phi \quad , \quad E_1^a = -\frac{p_b}{L_0}\phi^a \\ A_a^2 &= b\partial_a\theta \quad , \quad E_2^a = \frac{p_b}{L_0}\sin\theta\theta^a \\ A_a^3 &= \frac{c}{L_0}\partial_a x + \cos\theta\partial_a\phi \quad , \quad E_3^a = p_c\sin\theta x^a \end{aligned} \tag{3}$$

- Symplectic Structure

$$\{b, p_b\} = G\gamma \quad , \quad \{c, p_c\} = 2G\gamma \tag{4}$$

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{5}$$

Kantowski-Sachs in Ashtekar-Barbero variables

- Ashtekar-Barbero variables:

$$A_a^1 = -b\sin\theta\partial_a\phi \quad , \quad E_1^a = -\frac{p_b}{L_0}\phi^a$$
$$A_a^2 = b\partial_a\theta \quad , \quad E_2^a = \frac{p_b}{L_0}\sin\theta\theta^a \tag{3}$$
$$A_a^3 = \frac{c}{L_0}\partial_a x + \cos\theta\partial_a\phi \quad , \quad E_3^a = p_c\sin\theta x^a$$

- Symplectic Structure

$$\{b, p_b\} = G\gamma \quad , \quad \{c, p_c\} = 2G\gamma \tag{4}$$

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\mathrm{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{5}$$

Kantowski-Sachs in Ashtekar-Barbero variables

- Ashtekar-Barbero variables:

$$\begin{aligned} A_a^1 &= -b\sin\theta\partial_a\phi \quad , \quad E_1^a = -\frac{p_b}{L_0}\phi^a \\ A_a^2 &= b\partial_a\theta \quad , \quad E_2^a = \frac{p_b}{L_0}\sin\theta\theta^a \\ A_a^3 &= \frac{c}{L_0}\partial_a x + \cos\theta\partial_a\phi \quad , \quad E_3^a = p_c\sin\theta x^a \end{aligned} \tag{3}$$

- Symplectic Structure

$$\{b, p_b\} = G\gamma \quad , \quad \{c, p_c\} = 2G\gamma \tag{4}$$

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\operatorname{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{5}$$

Residual Diffeomorphisms

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action.
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p}_b + \frac{\partial E_i{}^a}{\partial p_c}\dot{p}_c$ for some $\dot{p}_b$ and $\dot{p}_c$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^{i}{}_{a} = \dot{A}^{i}{}_{a} = \frac{\partial A^{i}{}_{a}}{\partial b}\dot{b} + \frac{\partial A^{i}{}_{a}}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_{i}{}^{a} = \dot{E}_{i}{}^{a} = \frac{\partial E_{i}{}^{a}}{\partial p_b}\dot{p_b} + \frac{\partial E_{i}{}^{a}}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \qquad (6)$$

Residual Diffeomorphisms

- Residual diffeomorphisms: a group of transformations preserving the form of (A, E).
- First we must find the most general vector field v generating a flow preserving the Kantowski-Sachs form for A_a
  - That is, we must solve $\mathcal{L}_{\vec{v}} A^i{}_a = \dot{A}^i{}_a = \frac{\partial A^i{}_a}{\partial b}\dot{b} + \frac{\partial A^i{}_a}{\partial c}\dot{c}$ for some $\dot{b}$ and $\dot{c}$.
  - $\left\{\vec{\phi}, \vec{x}, x\vec{x}\right\}$ forms a basis for the most general $\vec{v}$.
  - The subgroup generated by $\left\{\vec{\phi}, \vec{x}\right\}$ has trivial action
  - The subgroup with non-trivial action is generated by $\{x\vec{x}\}$.
  - $\mathcal{L}_{\vec{v}} E_i{}^a = \dot{E}_i{}^a = \frac{\partial E_i{}^a}{\partial p_b}\dot{p_b} + \frac{\partial E_i{}^a}{\partial p_c}\dot{p_c}$ for some $\dot{p_b}$ and $\dot{p_c}$ must also be solved.
- The resulting flow equations requires:

$$\dot{b} = 0 \quad , \quad \dot{p_b} = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p_c} = 0 \tag{6}$$

Classical Covariance

Covariance Equation

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2} \frac{\mathrm{sgn}(p_c)}{\sqrt{|p_c|}} \left[(b^2 + \gamma^2)p_b + 2bcp_c\right] \tag{7}$$

- This Hamiltonian Constraint is of density weight one so, under the residual diffeomorphism flow,

$$\dot{H}_{c\ell} = H_{c\ell}. \tag{8}$$

- This is the equation we will then attempt to quantize.
- Unfortunately, the flows in (6) are not canonical, so the left hand side of (8) cannot be quantized in the usual way.

Covariance Equation

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{7}$$

- This Hamiltonian Constraint is of density weight one so, under the residual diffeomorphism flow,

$$\dot{H}_{c\ell} = H_{c\ell}. \tag{8}$$

- This is the equation we will then attempt to quantize.
- Unfortunately, the flows in (6) are not canonical, so the left hand side of (8) cannot be quantized in the usual way.

Covariance Equation

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{7}$$

- This Hamiltonian Constraint is of density weight one so, under the residual diffeomorphism flow,

$$\dot{H}_{c\ell} = H_{c\ell}. \tag{8}$$

- This is the equation we will then attempt to quantize.
- Unfortunately, the flows in (6) are not canonical, so the left hand side of (8) cannot be quantized in the usual way.

Covariance Equation

- Hamiltonian Constraint (for an arbitrary lapse N)

$$H_{c\ell} = -\frac{8\pi N}{\gamma^2}\frac{\text{sgn}(p_c)}{\sqrt{|p_c|}}\left[(b^2+\gamma^2)p_b + 2bcp_c\right] \tag{7}$$

- This Hamiltonian Constraint is of density weight one so, under the residual diffeomorphism flow,

$$\dot{H}_{c\ell} = H_{c\ell}. \tag{8}$$

- This is the equation we will then attempt to quantize.
- Unfortunately, the flows in (6) are not canonical, so the left hand side of (8) cannot be quantized in the usual way.

Maxwell Field in Kantowski-Sachs Spacetime

Maxwell Field

- A similar approach can be taken with a general Maxwell Field in Kantowski-Sachs.

$$\begin{aligned} A_a &= \alpha\left(\phi,\theta,x\right)\ \partial_a\phi + \beta\left(\phi,\theta,x\right)\ \partial_a\theta + \gamma\left(\phi,\theta,x\right)\ \partial_a x \\ E^a &= \chi\left(\phi,\theta,x\right)\ \phi^a + \psi\left(\phi,\theta,x\right)\ \theta^a + \omega\left(\phi,\theta,x\right)\ x^a \end{aligned} \tag{9}$$

- Our basis of the Killing vector fields for the Kantowski-Sachs symmetry group is:

$$\begin{aligned} \vec{R}_0 &:= x^a \\ \vec{R}_1 &:= \cot\theta\ \cos\phi\,\phi^a + \sin\phi\,\theta^a \\ \vec{R}_2 &:= -\cot\theta\ \sin\phi\,\phi^a + \cos\phi\,\theta^a \\ \vec{R}_3 &:= \phi^a \end{aligned} \tag{10}$$

- Our residual diffeomorphisms then become:

$$\pounds_{\vec{R}}\,A_a = 0 = \pounds_{\vec{R}}\,E^a \tag{11}$$

- It turns out, the most general forms of A_a and E^a are:

$$\begin{aligned} A_a &= \gamma\,\partial_a x \\ E^a &= \omega\,\sin\theta\,x^a. \end{aligned} \tag{12}$$

where γ and ω are constants.

- Therefore:

$${}^4A_a = V\left(\tau\right)\ \partial_a\tau + A_a = V\left(\tau\right)\ \partial_a\tau + \gamma\left(\tau\right)\ \partial_a x. \tag{13}$$

Maxwell Field

- A similar approach can be taken with a general Maxwell Field in Kantowski-Sachs.

$$\begin{aligned} A_a &= \alpha\left(\phi,\theta,x\right)\ \partial_a\phi + \beta\left(\phi,\theta,x\right)\ \partial_a\theta + \gamma\left(\phi,\theta,x\right)\ \partial_a x \\ E^a &= \chi\left(\phi,\theta,x\right)\ \phi^a + \psi\left(\phi,\theta,x\right)\ \theta^a + \omega\left(\phi,\theta,x\right)\ x^a \end{aligned} \tag{9}$$

- Our basis of the Killing vector fields for the Kantowski-Sachs symmetry group is:

$$\begin{aligned} \vec{R}_0 &:= x^a \\ \vec{R}_1 &:= \cot\theta\ \cos\phi\,\phi^a + \sin\phi\,\theta^a \\ \vec{R}_2 &:= -\cot\theta\ \sin\phi\,\phi^a + \cos\phi\,\theta^a \\ \vec{R}_3 &:= \phi^a \end{aligned} \tag{10}$$

- Our residual diffeomorphisms then become:

$$\pounds_{\vec{R}}\, A_a = 0 = \pounds_{\vec{R}}\, E^a \tag{11}$$

- It turns out, the most general forms of A_a and E^a are:

$$\begin{aligned} A_a &= \gamma\,\partial_a x \\ E^a &= \omega\ \sin\theta\, x^a. \end{aligned} \tag{12}$$

where γ and ω are constants.

- Therefore:

$${}^4A_a = V\left(\tau\right)\ \partial_a\tau + A_a = V\left(\tau\right)\ \partial_a\tau + \gamma\left(\tau\right)\ \partial_a x. \tag{13}$$

Maxwell Field

- A similar approach can be taken with a general Maxwell Field in Kantowski-Sachs.

$$\begin{aligned} A_a &= \alpha(\phi,\theta,x)\ \partial_a\phi + \beta(\phi,\theta,x)\ \partial_a\theta + \gamma(\phi,\theta,x)\ \partial_a x \\ E^a &= \chi(\phi,\theta,x)\ \phi^a + \psi(\phi,\theta,x)\ \theta^a + \omega(\phi,\theta,x)\ x^a \end{aligned} \tag{9}$$

- Our basis of the Killing vector fields for the Kantowski-Sachs symmetry group is:

$$\begin{aligned} \vec{R}_0 &:= x^a \\ \vec{R}_1 &:= \cot\theta\ \cos\phi\,\phi^a + \sin\phi\,\theta^a \\ \vec{R}_2 &:= -\cot\theta\ \sin\phi\,\phi^a + \cos\phi\,\theta^a \\ \vec{R}_3 &:= \phi^a \end{aligned} \tag{10}$$

- Our residual diffeomorphisms then become:

$$\pounds_{\vec{R}}\, A_a = 0 = \pounds_{\vec{R}}\, E^a \tag{11}$$

- It turns out, the most general forms of A_a and E^a are:

$$\begin{aligned} A_a &= \gamma\,\partial_a x \\ E^a &= \omega\ \sin\theta\, x^a. \end{aligned} \tag{12}$$

where γ and ω are constants.

- Therefore:

$${}^4A_a = V(\tau)\ \partial_a\tau + A_a = V(\tau)\ \partial_a\tau + \gamma(\tau)\ \partial_a x. \tag{13}$$

Maxwell Field

- A similar approach can be taken with a general Maxwell Field in Kantowski-Sachs.

$$\begin{aligned} A_a &= \alpha\left(\phi,\theta,x\right)\ \partial_a\phi + \beta\left(\phi,\theta,x\right)\ \partial_a\theta + \gamma\left(\phi,\theta,x\right)\ \partial_a x \\ E^a &= \chi\left(\phi,\theta,x\right)\ \phi^a + \psi\left(\phi,\theta,x\right)\ \theta^a + \omega\left(\phi,\theta,x\right)\ x^a \end{aligned} \tag{9}$$

- Our basis of the Killing vector fields for the Kantowski-Sachs symmetry group is:

$$\begin{aligned} \vec{R}_0 &:= x^a \\ \vec{R}_1 &:= \cot\theta\ \cos\phi\,\phi^a + \sin\phi\,\theta^a \\ \vec{R}_2 &:= -\cot\theta\ \sin\phi\,\phi^a + \cos\phi\,\theta^a \\ \vec{R}_3 &:= \phi^a \end{aligned} \tag{10}$$

- Our residual diffeomorphisms then become:

$$\pounds_{\vec{R}}\, A_a = 0 = \pounds_{\vec{R}}\, E^a \tag{11}$$

- It turns out, the most general forms of A_a and E^a are:

$$\begin{aligned} A_a &= \gamma\,\partial_a x \\ E^a &= \omega\ \sin\theta\, x^a. \end{aligned} \tag{12}$$

where γ and ω are constants.

- Therefore:

$${}^4A_a = V\left(\tau\right)\ \partial_a\tau + A_a = V\left(\tau\right)\ \partial_a\tau + \gamma\left(\tau\right)\ \partial_a x. \tag{13}$$

Maxwell Field

- A similar approach can be taken with a general Maxwell Field in Kantowski-Sachs.

$$A_a = \alpha(\phi,\theta,x)\ \partial_a\phi + \beta(\phi,\theta,x)\ \partial_a\theta + \gamma(\phi,\theta,x)\ \partial_a x$$
$$E^a = \chi(\phi,\theta,x)\ \phi^a + \psi(\phi,\theta,x)\ \theta^a + \omega(\phi,\theta,x)\ x^a \tag{9}$$

- Our basis of the Killing vector fields for the Kantowski-Sachs symmetry group is:

$$\vec{R}_0 := x^a$$
$$\vec{R}_1 := \cot\theta\ \cos\phi\,\phi^a + \sin\phi\,\theta^a$$
$$\vec{R}_2 := -\cot\theta\ \sin\phi\,\phi^a + \cos\phi\,\theta^a \tag{10}$$
$$\vec{R}_3 := \phi^a$$

- Our residual diffeomorphisms then become:

$$\pounds_{\vec{R}}\, A_a = 0 = \pounds_{\vec{R}}\, E^a \tag{11}$$

- It turns out, the most general forms of A_a and E^a are:

$$A_a = \gamma\,\partial_a x$$
$$E^a = \omega\ \sin\theta\, x^a. \tag{12}$$

where γ and ω are constants.
- Therefore:

$${}^4A_a = V(\tau)\ \partial_a\tau + A_a = V(\tau)\ \partial_a\tau + \gamma(\tau)\ \partial_a x. \tag{13}$$

Comparison to the Reissner-Nordström Solution

- The Reissner-Nordstrom metric is given by:

$$g_{ab}^{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t + \left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r \\ + r^2\sin^2\theta\,\partial_a\phi\,\partial_b\phi + r^2\,\partial_a\theta\,\partial_b\theta \tag{14}$$

$${}^4A_a = \frac{Q}{r}\partial_a t \tag{15}$$

- To compare to the Reissner-Nordström solution, we need to determine the corresponding metric:

$$h^{ab} = \delta^{ij} e^a_i e^b_j = \frac{E^a_i E^{bi}}{|\det\{h_{ab}\}|} \\ = \frac{1}{|\tilde{p}_c|}\csc^2\theta\,\phi^a\phi^b + \frac{1}{|\tilde{p}_c|}\theta^a\theta^b + \frac{|\tilde{p}_c|}{\tilde{p}_b^2}\,x^a x^b. \tag{16}$$

  - This leads to:

$$g_{ab} = -N^2\partial_a\tau\,\partial_b\tau + |\tilde{p}_c|\sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta + \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a x\,\partial_b x. \tag{17}$$

Comparison to the Reissner-Nordström Solution

- The Reissner-Nordstrom metric is given by:

$$g^{RN}_{ab} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t + \left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r + r^2\sin^2\theta\,\partial_a\phi\,\partial_b\phi + r^2\,\partial_a\theta\,\partial_b\theta \quad (14)$$

$${}^4A_a = \frac{Q}{r}\partial_a t \quad (15)$$

- To compare to the Reissner-Nordström solution, we need to determine the corresponding metric:

$$\begin{aligned} h^{ab} &= \delta^{ij} e^a_{\ i} e^b_{\ j} = \frac{E^a_{\ i} E^{bi}}{|\det\{h_{ab}\}|} \\ &= \frac{1}{|\tilde{p}_c|}\csc^2\theta\,\phi^a\phi^b + \frac{1}{|\tilde{p}_c|}\theta^a\theta^b + \frac{|\tilde{p}_c|}{\tilde{p}_b^2}\,x^a x^b. \end{aligned} \quad (16)$$

- This leads to:

$$g_{ab} = -N^2\partial_a\tau\,\partial_b\tau + |\tilde{p}_c|\sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta + \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a x\,\partial_b x. \quad (17)$$

Comparison to the Reissner-Nordström Solution

- The Reissner-Nordstrom metric is given by:

$$g_{ab}^{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t + \left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r + r^2\sin^2\theta\,\partial_a\phi\,\partial_b\phi + r^2\,\partial_a\theta\,\partial_b\theta \tag{14}$$

$$^4A_a = \frac{Q}{r}\partial_a t \tag{15}$$

- To compare to the Reissner-Nordström solution, we need to determine the corresponding metric:

$$h^{ab} = \delta^{ij} e^a_{\ i} e^b_{\ j} = \frac{E^a_i E^{bi}}{|\det\{h_{ab}\}|} = \frac{1}{|\tilde{p}_c|}\csc^2\theta\,\phi^a\phi^b + \frac{1}{|\tilde{p}_c|}\theta^a\theta^b + \frac{|\tilde{p}_c|}{\tilde{p}_b^2}\,x^a x^b. \tag{16}$$

  - This leads to:

$$g_{ab} = -N^2\partial_a\tau\,\partial_b\tau + |\tilde{p}_c|\sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta + \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a x\,\partial_b x. \tag{17}$$

Comparison to the Reissner-Nordström Solution Continued

- Identifying (x, τ) with (t, r):

$$g_{ab} = \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a t\,\partial_b t - N^2 \partial_a r\,\partial_b r + |\tilde{p}_c| \sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta \tag{18}$$

$$^4A_a = V(r)\;\partial_a r + \gamma(r)\;\partial_a t \tag{19}$$

- Comparing the ^4A_a:

$$V(r) = 0 \;\wedge\; \gamma(r) = \frac{Q}{r} \tag{20}$$

- Comparing the rest:

$$\begin{aligned} |\tilde{p}_c| &= r^2 \\ \tilde{p}_b^2 &= -r^2\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right) \\ N^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1} \end{aligned} \tag{21}$$

Comparison to the Reissner-Nordström Solution Continued

- Identifying (x, τ) with (t, r):

$$g_{ab} = \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a t\,\partial_b t - N^2 \partial_a r\,\partial_b r + |\tilde{p}_c| \sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta \tag{18}$$

$$^4A_a = V(r)\;\partial_a r + \gamma(r)\;\partial_a t \tag{19}$$

- Comparing the ^4A_a:

$$V(r) = 0 \,\wedge\, \gamma(r) = \frac{Q}{r} \tag{20}$$

- Comparing the rest:

$$|\tilde{p}_c| = r^2$$
$$\tilde{p}_b^2 = -r^2\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right) \tag{21}$$
$$N^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1}$$

Comparison to the Reissner-Nordström Solution Continued

- Identifying (x, τ) with (t, r):

$$g_{ab} = \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\,\partial_a t\,\partial_b t - N^2 \partial_a r\,\partial_b r + |\tilde{p}_c|\sin^2\theta\,\partial_a\phi\,\partial_b\phi + |\tilde{p}_c|\,\partial_a\theta\,\partial_b\theta \tag{18}$$

$$^4A_a = V(r)\;\partial_a r + \gamma(r)\;\partial_a t \tag{19}$$

- Comparing the ^4A_a:

$$V(r) = 0 \wedge \gamma(r) = \frac{Q}{r} \tag{20}$$

- Comparing the rest:

$$\begin{aligned} |\tilde{p}_c| &= r^2 \\ \tilde{p}_b^2 &= -r^2\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right) \\ N^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}\right)^{-1} \end{aligned} \tag{21}$$

Comparison to the Reissner-Nordström Solution Continued

- The flow defined by $x\vec{x}$ (now identified as $t\vec{t}$) also preserves the form of the electromagnetic variables A and E.
- This flow results in:

$$\begin{aligned} g_{ab}(\lambda) = &-e^{(\alpha+2)\lambda}\left(1-\frac{2M}{r}+\frac{\left[\frac{e^{\alpha\lambda}}{\alpha}-4\pi\epsilon_0 r\left(r-2M\right)\right]}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t \\ &+e^{\alpha\lambda}\left(1-\frac{2M}{r}+\frac{\left[\frac{e^{\alpha\lambda}}{\alpha}-4\pi\epsilon_0 r\left(r-2M\right)\right]}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r \\ &+r^2\left(\partial_a\theta\,\partial_b\theta+\sin^2\theta\partial_a\phi\,\partial_b\phi\right) \end{aligned} \tag{22}$$

Where

$$\alpha = \left(Q^2-4\pi\epsilon_0 r\left(r-2M\right)\right)^{-1} \tag{23}$$

- I haven't finished the rest of this yet, as we have been focused on quantizing the gravitational part.

Comparison to the Reissner-Nordström Solution Continued

- The flow defined by $x\vec{x}$ (now identified as $t\vec{t}$) also preserves the form of the electromagnetic variables A and E.
- This flow results in:

$$g_{ab}(\lambda) = -e^{(\alpha+2)\lambda}\left(1 - \frac{2M}{r} + \frac{\left[\frac{e^{\alpha\lambda}}{\alpha} - 4\pi\epsilon_0 r\left(r - 2M\right)\right]}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t$$
$$+ e^{\alpha\lambda}\left(1 - \frac{2M}{r} + \frac{\left[\frac{e^{\alpha\lambda}}{\alpha} - 4\pi\epsilon_0 r\left(r - 2M\right)\right]}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r$$
$$+ r^2\left(\partial_a\theta\,\partial_b\theta + \sin^2\theta\partial_a\phi\,\partial_b\phi\right) \tag{22}$$

Where

$$\alpha = \left(Q^2 - 4\pi\epsilon_0 r\left(r - 2M\right)\right)^{-1} \tag{23}$$

- I haven't finished the rest of this yet, as we have been focused on quantizing the gravitational part.

Comparison to the Reissner-Nordström Solution Continued

- The flow defined by $x\vec{x}$ (now identified as $t\vec{t}$) also preserves the form of the electromagnetic variables A and E.
- This flow results in:

$$\begin{aligned} g_{ab}(\lambda) = &-e^{(\alpha+2)\lambda}\left(1-\frac{2M}{r}+\frac{\left[\frac{e^{\alpha\lambda}}{\alpha}-4\pi\epsilon_0 r\left(r-2M\right)\right]}{4\pi\epsilon_0 r^2}\right)\partial_a t\,\partial_b t \\ &+e^{\alpha\lambda}\left(1-\frac{2M}{r}+\frac{\left[\frac{e^{\alpha\lambda}}{\alpha}-4\pi\epsilon_0 r\left(r-2M\right)\right]}{4\pi\epsilon_0 r^2}\right)^{-1}\partial_a r\,\partial_b r \\ &+r^2\left(\partial_a\theta\,\partial_b\theta+\sin^2\theta\partial_a\phi\,\partial_b\phi\right) \end{aligned} \tag{22}$$

Where

$$\alpha = \left(Q^2-4\pi\epsilon_0 r\left(r-2M\right)\right)^{-1} \tag{23}$$

- I haven't finished the rest of this yet, as we have been focused on quantizing the gravitational part.

Next Steps

- The next step is to quantize the non-canonical covariance equation for the gravitational field, which Rafael Dias will go into in more detail in his presentation.
- Usually in quantum theory one only defines unitary flows corresponding to <u>canonical</u> transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{24}$$

- However, as Rafael will explain, the flows in (6) are <u>non-canonical</u>, so (24) cannot be directly applied.
- Fortunately, we were able to quantize things by generalizing (24).

Next Steps

- The next step is to quantize the non-canonical covariance equation for the gravitational field, which Rafael Dias will go into in more detail in his presentation.
- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{24}$$

- However, as Rafael will explain, the flows in (6) are non-canonical, so (24) cannot be directly applied.
- Fortunately, we were able to quantize things by generalizing (24).

Next Steps

- The next step is to quantize the non-canonical covariance equation for the gravitational field, which Rafael Dias will go into in more detail in his presentation.
- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{24}$$

- However, as Rafael will explain, the flows in (6) are non-canonical, so (24) cannot be directly applied.
- Fortunately, we were able to quantize things by generalizing (24).

Next Steps

- The next step is to quantize the non-canonical covariance equation for the gravitational field, which Rafael Dias will go into in more detail in his presentation.
- Usually in quantum theory one only defines unitary flows corresponding to canonical transformations:

$$\dot{F} = \{\Lambda, F\} \qquad \Rightarrow \qquad \dot{\hat{F}} = \frac{1}{i\hbar}\left[\hat{F}, \hat{\Lambda}\right] \qquad \Rightarrow \qquad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}}\hat{F}(0)e^{-\frac{t}{i\hbar}\hat{\Lambda}} \tag{24}$$

- However, as Rafael will explain, the flows in (6) are non-canonical, so (24) cannot be directly applied.
- Fortunately, we were able to quantize things by generalizing (24).

Discussion

Questions?

References

Ashtekar, A. & Bojowald, M. Quantum geometry and the Schwarzschild singularity. *Classical and Quantum Gravity* **23,** 391–411. https://doi.org/10.1088%2F0264-9381%2F23%2F2%2F008 (Dec. 2005).

Ashtekar, A., Olmedo, J. & Singh, P. Quantum extension of the Kruskal spacetime. *Physical Review D* **98.** https://doi.org/10.1103%2Fphysrevd.98.126003 (Dec. 2018).

Corichi, A. & Singh, P. Loop quantization of the Schwarzschild interior revisited. *Classical and Quantum Gravity* **33,** 055006. https://doi.org/10.1088%2F0264-9381%2F33%2F5%2F055006 (Feb. 2016).

Engle, J. & Vilensky, I. Deriving loop quantum cosmology dynamics from diffeomorphism invariance. *Physical Review D* **98.** https://doi.org/10.1103%2Fphysrevd.98.023505 (July 2018).

Engle, J. & Vilensky, I. Uniqueness of minimal loop quantum cosmology dynamics. *Physical Review D* **100.** https://doi.org/10.1103%2Fphysrevd.100.121901 (Dec. 2019).

Thank You!