Perspectives on Emergent Space-time

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Paradigm presented in the final form in : AA & M. Vadararajan (2021)
Based on work carried our by a large number of researchers, especially:
Jacobson, Lewandowski, Mason, Newman, Penrose, Plebanski, Smolin, Thiemann, . . .

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Preamble

- In GR, the space-time metric g encodes gravity. No background structures! All fields, are dynamical and physical: No spectators in fundamental physics. But Einstein's equations are technically complicated; highly non-polynomial in g.

  By contrast, the other three basic forces are described by theories of connections A^i and equations are much simpler: low order polynomials in A^i. But they have a spectator: g.

- Can we combine the core-strengths of both? Yes! Idea: Start with a diffeomorphism invariant gauge theory; no background fields, not even g. Write down the simplest equations possible. These are to be regarded as 'fundamental'. Introduce a dictionary that defines the Riemannian structures of GR such that Einstein's equations emerge from the gauge theory ones, exactly.

  $\Rightarrow$ The dictionary is rather intricate, since gauge theory equations are very simple, while Einstein's equations are so complicated! Explore new insights into GR from the gauge theory perspective.

- Of course this idea lies at the very foundations of LQG. But its full force is generally not felt because standard treatments begin with GR and reformulate it as a theory of connections. Regarding space-time geometry as emerging from a 'more fundamental' gauge theory provides qualitatively new insights.

Organization

1. Geometrodynamics in a nutshell.
2. From Gauge Theory to GR (not the AdS/CFT gauge-gravity duality!).
3. Application: Transmutation of ‘Time Evolution’ into ‘Spatial Motions’.
4. Application: Unforeseen connection: Penrose’s ‘Non-linear Graviton’
   and the group of volume preserving diffeomophisms.
4. Application: Geometrization of the Constraint Algebra:
   door to anomaly-free quantization.
6. Outlook: Summary and Bridges to Other Areas

The three applications are just illustrations. In the last part I will point out others, but that will not a full list. These applications provide interesting directions not only for further advances within LQG but also to build bridges to other areas such as Twistor theory, Developments in scattering amplitudes, ‘Double copy’, Mathematics and Philosophy of spacetime.

1 Geometrodynamics in a nutshell

- GR: 4-metric $g_{\mu\nu}$ on ${}^4M = M \times R$. Gravitational Phase space $\mathbf{\Gamma} \ni (q_{ab};\, p^{ab})$:

  q_{ab}: +,+,+ metric; $p^{ab} = -\epsilon\, \sqrt{q}(k^{ab} - kq^{ab})$; k_{ab} : Extrinsic curvature.

Canonically conjugate pair: $\mathbf{\Omega}|_{(q,p)}(\delta_1,\, \delta_2) = \int_M \big(\delta_1 q_{ab}\, \delta_2 p^{ab} \,-\, 1 \leftrightarrow 2\big) \mathrm{d}^3x.$

$\epsilon = 1$ in the Riemannian signature +,+,+,+ and -1 in the Lorentzian signature -,+,+,+.
We will consider both signatures: Generalized Wick transform (Thiemann; AA; Varadarajan) .
For simplicity: (i) No matter sources $G_{\mu\nu} = 0$; and (ii) M: Oriented, compact w/o boundary.
We can drop these restrictions. Main results unaltered; just more fields and surface terms.

- Einstein's 10 equations on g are naturally divided into 2 parts. (i) 4 constraint equations on initial data $(q;\, p)$; have no time-derivatives; and, (ii) 6 evolution equations. Constraints are:

$$C^a := -2D_b\, p^{ab} = 0, \quad \text{and} \quad C := -\tfrac{1}{2}\big(q^{\frac{1}{2}}\, \mathcal{R} + \epsilon\, q^{-\frac{1}{2}}\big(q_{ac}q_{bd} - \tfrac{1}{2} q_{ab}q_{cd}\big)\, p^{ab}\, p^{cd}\big) \;=\; 0.$$

D: covariant derivative operator of q_{ab}; q, its determinant; $\mathcal{R}$, its scalar curvature.
Highly non-polynomial in q. DOF: 6 - 4 = 2

- Fundamental fact: If all fields are dynamical (no background metric) Hamiltonian is a linear combination of constraints: $H_{N,\vec{N}}(q,p) = \int_M \big(N^a C_a + NC\big)\, \mathrm{d}^3x$. Given (N, N^a), each dynamical trajectory in $\mathbf{\Gamma}$ gives $q_{ab}(t)$. By stacking, one assembles a 4-dimensional solution $g_{\mu\nu}$ on ${}^4M$.

Dynamics

Hamiltonian: $H_{N,\vec{N}}(q,p) = \int_M \left(N^a C_a + NC\right) \mathrm{d}^3x\,.$

- The canonical transformation generated by the first term,

$$\dot{q}_{ab} = \mathcal{L}_{\vec{N}}\, q_{ab}; \qquad \dot{p}^{ab} = \mathcal{L}_{\vec{N}}\, p_{ab}$$

has a simple geometric meaning: It generates diffeomorphisms along the shift N^a. This interpretation makes it possible to lift this action to the quantum level.

- That generated by the second term is

(i) Very complicated, with non-polynomial terms:

$$\dot{q}_{ab} = 2N\, q^{-\frac{1}{2}} \left(q_{ac}q_{bd} - \tfrac{1}{2}q_{ab}q_{cd}\right) p^{cd}$$

$$\begin{aligned} \dot{p}^{ab} \;=\;& \epsilon\, q^{\frac{1}{2}} \left(q^{ac}q^{bd} - q^{ab}q^{cd}\right) D_c D_d N - \epsilon\, N\, q^{\frac{1}{2}} \left(q^{ac}q^{bd} - \tfrac{1}{2}q^{ab}q^{cd}\right) \mathcal{R}_{cd} \\ -\;& q^{-\frac{1}{2}}\, N \left(2\delta^a_d \delta^b_n q_{cm} - \delta^a_m \delta^b_n q_{cd} - \tfrac{1}{2} q^{ab}(q_{cm}q_{dn} - \tfrac{1}{2} q_{cd}q_{mn})\right) p^{cd} p^{mn}. \end{aligned}$$

(ii) Also, it does not have a geometric interpretation. This is because the H_N is of the form $G^{\alpha\beta}(q)\, P_\alpha P_\beta + V(q)$ where $V(q) = \int N\, \sqrt{q}\, \mathcal{R} \mathrm{d}^3x$ and the integrand is a non-polynomial functional of q_{ab}. This is why the WDW equation has remained formal for over 50 years.

- It is well-known in the LQG community that the evolution equation in the connection formulation are much simpler. We will see that it also has a geometrical interpretation very similar to that generated by $H_{\vec{N}}$, thereby casting 'time evolution' as 'space-evolution'.

2. Changing Gears: a Gauge Theory

- Let us begin ab-initio and consider the phase space of an SU(2) gauge theory; $(A_a^i,\ E_i^a) \in \mathbf{\Gamma}$. Symplectic Structure: $\mathbf{\Omega}|_{(A,E)}(\delta_1,\,\delta_2) = \int_M \big(\delta_1 A_a^i \delta_2 E_i^a - 1 \leftrightarrow 2\big)$.

- Task of specifying constraints simplifies tremendously: With no space-time metric, very few gauge covariant expressions available!

E_i^a is gauge covariant. While A_a^i is not, it defines a gauge covariant $\mathcal{D}$ (on fields with internal indices $i, j, ...$) and also a gauge covariant field strength $F_{ab}^i = 2\partial_{[a}A_{b]}{}^i + \mathring{\epsilon}^i{}_{jk} A_a^i A_b^j$.
Simplest gauge covariant equations at most quadratic in A, E are $F_{ab}^i = 0$, or $E_i^a E_j^b = 0$ ... will be available; there trivialize the theory.

$$\mathcal{G}_i := \mathcal{D}_a E_i^a = 0, \qquad \mathcal{V}_a := E_i^b F_{ab}{}^i = 0, \qquad \mathcal{S} := \tfrac{1}{2}\mathring{\epsilon}^{ij}{}_k\, E_i^a E_j^b\, F_{ab}{}^k = 0.$$

They constitute a first class system system of constraints.

- No background metric $\Rightarrow$ Hamiltonian is a linear combination of constraints: $H_{\Lambda,\vec{N},N}(A,E) = \int_M \big(\Lambda^i \mathcal{G}_i\ +\ N^a \mathcal{V}_a + N\mathcal{S}\big)\mathrm{d}^3x$. Equations of motion are again at most quadratic in (A, E). As in any gauge theory, the first term generates an internal SU(2) gauge rotations. Second generates gauge covariant Lie derivatives. The third also generates very simple transformation:

$$\dot{A}_a^i = N\, E_j^b F_{ab}{}^k \mathring{\epsilon}_i{}^{jk} \qquad \dot{E}_i^a = \mathcal{D}_a(N\, E_j^a E_k^b)\, \mathring{\epsilon}_i{}^{jk}$$

($\mathcal{D}$ only knows how to act on internal indices. Surprisingly, that suffices!!) Same philosophy underlies the definition of the vertex amplitude in spinfoams.

Dictionary

- A_a^i has 9 components and 3+3+1=7 constraints. So our gauge theory will have 2 local DOF just like GR. (We have a background independence but not a Topological Field Theory.) Could this theory be GR in disguise, in spite of the strikingly complicated form of GR equations?

- As the LQG community knows, the answer is in the affirmative! Let us first consider the Riemannian case and focus on the part of phase space where the three E_i^a are linearly independent at each point of M. Then we regard them as orthonormal triads (with density weight 1), and define a 3-metric q_{ab} as follows:

$$E_j^a E^{bj} =: q\, q^{ab}, \quad \text{where} \quad q \equiv \det(q_{ab}) := \eta_{abc}\, \mathring{\epsilon}^{ijk}\, E_i^a E_j^b E_k^c$$

- Given E_i^a, the conjugate momentum p^{ab} can be recovered from A_a^i or $\mathcal{D}$. The procedure is a bit more intricate. First: E_i^a determines a unique torsion-free $\mathbf{D}_a$ on M that acts on fields $T_{ab\ldots}{}^{ij\ldots}$, with both tensor and internal indices via $\mathbf{D}_a E_i^b = 0$. Gauge theory $\mathcal{D}$ acts only on fields with internal indices. Therefore, $\exists$! field ${\pi_a}^j$ such that

$$(\mathbf{D}_a - \mathcal{D}_a)\lambda_i = \mathring{\epsilon}_{ij}{}^k\, {\pi_a}^j\, \lambda_k\,, \qquad \text{and, we set} \qquad p^{ab} := {\pi_c}^j \left(q^{c(a} E_j^{b)} - q^{ab} E_j^c\right)\,.$$

- (A_a^i, E_i^a) are 'fundamental'. (q_{ab}, p^{ab}) of geometrodynamics are now 'emergent, composite' fields, just as quarks and gluons are fundamental and nuclei composite.

Consequences

• Let us consider the phase space $\mathbf{\Gamma}$ spanned by complex-valued pairs (A_a^i, E_i^a). The dictionary can be naturally extended to two 'real sectors' of the gauge theory $\mathbf{\Gamma}$. It sends:

(R) The sector on which (E_i^a, π_a^i) are real to precisely the $\epsilon = 1$ –i.e., Riemannian– sector of real geometrodynamics; and,

(L) The sector on which (E_i^a, π_a^i) are pure imaginary to precisely the $\epsilon = -1$ –i.e., Lorentzian– sector of real geometrodynamics.

Note: unlike in the standard LQG literature, we have a single symplectic structure on full $\mathbf{\Gamma}$ –no relative factors of i between the R and the L sectors of $\mathbf{\Gamma}$– and we have a single set of constraints that have no ϵ factors in the gauge theory description.

The map $(A_a^i, E_i^a) \to (q_{ab}, p^{ab})$ has the following properties: on both sectors, it

(i) Preserves the symplectic structure, or Poisson brackets;

(ii) Preserves constraints, i.e., maps the constraint surface of the gauge theory phase to that of complexified geometrodynamics;

(iii) Preserves dynamics, i.e. maps dynamical trajectories on the constraint surface of gauge theory to those on the constraint surface of geometrodynamics.

Thus we recover both Riemannian and Lorentzian geometrodynamics. The factors of ϵ arise naturally from the dictionary.

Dictionary: Features & Generalizations

- Interestingly, on the Lorentzian sector the gauge theory equations are symmetric hyperbolic, without any extra work (Reula et al; Shinkai et al) .

- The Dictionary also provides the geometrical meaning of A_a^i. In the 4-d solution $g_{\mu,\nu}$, A_a^i is the self-dual connection: The curvature F_{ab}^i of A_a^I vanishes if and only if $g_{\mu\nu}$ is a self-dual solution.

- Generalization to: M non-compact with asymptotically flat boundary conditions, introduction of scalar spinor, Maxwell and Yang-Mills fields as sources, and inclusion of a cosmological constant were been fully worked out. (AA, Lewandowski, Romano, Tate, Thiemann . . .)

- Riemannian geometry is now emergent: From the gauge theory perspective, the 1st and 2nd fundamental forms on M are composite fields. The gauge theory equations are simple. Those of geometrodynamics are very complicated simply because the expressions of the 'composite fields' (q_{ab}, p^{ab}) in terms of the 'fundamental' ones (A_a^i, E_i^a) are complicated. Analogy: Nuclear Physics $\leftrightarrow$ QCD.

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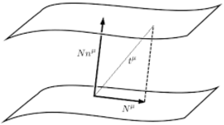
3. Transmutation of ‘Time Evolution’

- Recall: Geometrodynamics had two undesirable features: (i) Eqs complicated; and, (ii) While the canonical transformation generated by $H_{\vec{N}}$ has simple geometric meaning, that by H_N does not. In quantum theory, the action of $\hat{H}_{\vec{N}}$ is geometrical and transparent, that of $\hat{H}_N$ is not, making it difficult to construct an anomaly free constraint algebra.

- While gauge theory equations are simple, the asymmetry between space and time ‘evolutions’ of geometrodynamics has persisted in the LQG literature. Recall: $H_{\vec{N},N} = \int_M \big(N^b(E^a_i F^i_{ab}) + N(\mathring{\epsilon}^{ij}{}_k E^a_i E^b_j F_{ab}{}^k)\big)\, \mathrm{d}^3x$

The Hamiltonian flow generated by the first term has a natural geometrical interpretation on (SU(2) bundles over) M:
Gauge covariant Lie-derivative (GCLD): Ex: $\dot{E}^a_i = \mathfrak{L}_{\vec{N}} E^i_a$.

But that generated by the second term does not:
Ex: $\dot{E}^a_i = \mathring{\epsilon}_i{}^{jk}\, \mathcal{D}_b \big(N E^{[b}_j\, E^{a]}_k\big)$.



- Interestingly, symmetry can be restored using a generalized GCLD along space-like vector fields! Unforeseen bonus of the gauge theory formulation. The generalization opens a window for anomaly-free quantization (Varadarajan’s talk), and also for further work in mathematics. More precisely ...

Gauge Covariant Lie Derivative

- Recall: Operational definition of Lie-derivative à la Penrose-Geroch: $\mathcal{L}_{\vec{N}}$ acts on tensor fields satisfying four axioms: (i) Linearity; (ii) Leibniz rule; (iii) $\mathcal{L}_{\vec{N}} f = \vec{N} \lrcorner \, \mathrm{d}f$, and (iv) commutativity w.r.t. exterior derivative. These conditions yield the familiar formula $\mathcal{L}_{\vec{N}} T_a{}^b = N^c \mathring{D}_c T_a{}^b + T_c{}^b \mathring{D}_a N^c - T_a{}^c \mathring{D}_c N^b$, where $\mathring{D}$ is any torsion-free derivative operator on M. We can take $\mathring{D}$ to be flat: ∂.

- For fields T_i^a with both tensor (manifold) and internal indices: We have the notion of a gauge covariant Lie derivative $\mathfrak{L}_{\vec{N}}$. Fix any torsion-free derivative operator, say ∂ on tensors and extend its action on internal indices using a fixed connection A_a^i as usual: $\mathcal{D}_a T^b{}_i = \partial_a T^b{}_i + \mathring{\epsilon}_{ij}{}^k A_a^j \lambda_k$. Then GC Lie-derivative

$$\mathfrak{L}_{\vec{N}} T^a{}_i = N^c \mathcal{D}_c T^a{}_i - T^c{}_i \, \partial_c N^a$$

is again uniquely selected by the 4 axioms together with a fifth: $\mathfrak{L}_{\vec{N}} \lambda_i = \vec{N} \lrcorner \, \mathcal{D}\lambda_i$.

- But now the Lie bracket also involves a gauge rotation:

$$[\mathfrak{L}_{\vec{N}}, \mathfrak{L}_{\vec{M}}] T_i = \mathfrak{L}_{\vec{V}} T_i + \mathring{\epsilon}_{ij}{}^k \, (N^a M^b F_{ab}{}^j) \, T_k \qquad \text{where} \qquad V^a := \mathfrak{L}_{\vec{N}} M^a \equiv \mathcal{L}_{\vec{N}} M^a$$

Geometrical interpretation: the 'Lie group' generated by the Lie algebra is the group of all automorphisms on $SU(2)$ bundles (i.e. diffeos that preserve the bundle structure). This is well-known. This is the group generated by $H_{\Lambda, \vec{N}}$.

- The challenge is to provide a similar geometrical interpretation of 'time evolution': $\dot{A}_a^i = -\mathring{\epsilon}^{ij}{}_k \, N E_j^b \, F_{ba}{}^k$ and $\dot{E}_i^a = \mathring{\epsilon}_i{}^{jk} \, \mathcal{D}_b \big(N E_j^{[b} E_k^{a]} \big)$.

Generalized Gauge Covariant Lie Derivative

- Idea: The Hamiltonian $H_{\vec{N},N} = \int_M \left(N^b(E^a_i F^i_{ab}) + N(\mathring{\epsilon}^{ij}{}_k E^a_i E^b_j F_{ab}{}^k)\right) \mathrm{d}^3x$ can be recast as:

$$H_{\vec{N},N} = \int_M \left(N^b(E^a_i F^i_{ab}) + N^a_i(\mathring{\epsilon}^{ij}{}_k\, E^b_j F_{ab}{}^k)\right) \mathrm{d}^3x$$

where $N^a_i = NE^a_i$ is called the electric shift. The first term generates GCLD. Suggests: a generalization of GCLD by replacing the vector fields N^a by Lie algebra-valued vector fields N^a_i carrying an internal index. (Note: $H_{\vec{N},N} = \int_M N^{a\,j}_i\, E^a_j F^i_{ab}\, \mathrm{d}^3x$ with $N^{a\,j}_i = N^a \delta^j_i + \mathring{\epsilon}_i{}^{jk} N E^a_k$).

- The same 4 axioms and the requirement of covariance determine the following generalization $\mathbb{L}$ Of the GCLD of $\mathfrak{L}$:

$$\mathbb{L}_{\vec{V}_k}\, T_{ai}{}^{bj} := V^c_k\, \mathcal{D}_c T_{ai}{}^{bj} + T_{ci}{}^{bj}\, \mathcal{D}_a V^c_k - T_{ai}{}^{cj}\, \mathcal{D}_c V^b_k$$

Same as for GCLD but now the gauge covariant $\mathcal{D}$ acts also on the vector field V^c_k since they carry internal indices. Surprise: 'time evolution' equations turn out to be precisely the generalized GCLDs: (AA & Varadarajan)

$$\dot{A}^i_a = \mathring{\epsilon}^{ij}{}_k\, \mathbb{L}_{\vec{N}_j}\, A^k_a \qquad \text{and} \qquad \dot{E}^a_i \approx \tfrac{1}{2}\mathring{\epsilon}_{ij}{}^k \mathbb{L}_{\vec{N}_j}\, E^a_k.$$

Thus, in the gauge theory, all evolutions are given by generalized GCLDs along space-like vectors! Distinction between space and time evolutions arises only when we use the dictionary to pass to geometrodynamics. (Note that the Hamiltonian can also be unified as: $H = \int_M N^{a\,j}_i\, E^a_j F^i_{ab}\, \mathrm{d}^3x$ with $N^a{}_i{}^j = N^a \delta^j_i + \mathring{\epsilon}_i{}^{jk} N E^a_k$).

Ramifications of this Transmutation

• The Lie bracket between vector fields $\vec{M}$, $\vec{N}$ on a manifold M provides us with a Lie algebra whose 'Lie group' is $\mathrm{Diff}(M)$. Given an $SU(2)$ bundle on M, and a connection 1-form on it, we acquire another Lie bracket: $[\mathfrak{L}_{\vec{M}},\, \mathfrak{L}_{\vec{N}}]$. The Lie group it generates is the semi-direct product $\mathrm{SU(2)_{loc}} \ltimes \mathrm{Diff(M)}$ –generally taken to be the kinematical symmetry group in presence of $\mathrm{SU(2)}$ gauge fields.

• We now have a brand new generalized GCLD $\mathbb{L}_{\vec{N}_i}$ associated with vector fields $\vec{N}_i$ with internal indices! Their Lie bracket:

$$[\mathbb{L}_{\vec{N}}, \mathbb{L}_{\vec{M}}]\, T_k \,=\, \mathbb{L}_{\vec{V}_{ij}} T_k \,+\, \underbrace{\mathring{\epsilon}_{k\ell}{}^{m}\left(N^a_i M^b_j F_{ab}{}^{\ell}\right) T_m} \quad \text{where} \quad V^a_{ij} \,=\, \mathbb{L}_{\vec{N}_i}\, M^a_j .$$

again involves an internal rotation in $\mathrm{SU(2)_{loc}}$, just as that for GCLDs. Therefore, again, we have to enlarge the space of vector fields N^a_i by including generators of $\mathrm{SU(2)_{loc}}$.
But, in addition, the bracket between generators of the type N^a_i is a vector field of the type V^a_{ij}. Thus, the Lie algebra will thus have vector fields with arbitrary number of internal indices! The generators span a (huge!) graded vector space. But the commutator bracket on it does satisfy Jacobi identity. So a deep mathematical question is: What is the corresponding 'Lie group'?
Probably the group of automorphisms on a bundle over M with infinite dimensional fibers. Such a rich structure is essential because this Lie-group captures full Einstein dynamics! It is well worth understanding its structure in detail.

4. The Half-flat Sector

For full GR, perhaps the most important result from Twistor Theory is Penrose's "Non-linear Graviton": General anti-self-dual (ASD) complex-valued solutions to Einstein's equations: Encoded in certain deformations of the complex structure of Twistor space.

- Let us begin by collecting all equations in the gauge theory formulation:

  constraints: $\mathcal{G}_i := \mathcal{D}_a E^a_i = 0 \quad \mathcal{V}_a := E^b_i F_{ab}{}^i = 0, \quad \mathcal{S} := \frac{1}{2}\mathring{\epsilon}^{ij}{}_k\, E^a_i E^b_j\, F_{ab}{}^k = 0.$

  evolution equations: $\dot{A}^i_a = \mathring{\epsilon}^{ij}{}_k\, \mathbb{L}_{\vec{N}_j}\, A^k_a$ and $\dot{E}^a_i \approx \frac{1}{2}\mathring{\epsilon}_i{}^{jk}\, \mathbb{L}_{\vec{N}_j}\, E^a_k.$

ASD solutions: self-dual part of curvature vanishes $\Leftrightarrow$ $F^i_{ab} = 0$! Let us go to a gauge in which $A^i_a = 0$. $\Rightarrow$ We are left with just the two Brown equations!

Evolution equations require a lapse N to construct 'the electric shift' $N^a_i := N E^a_i$. Fix a reference volume form $\mathring{\epsilon}_{abc}$ on M and a compatible derivative operator $\mathring{D}$: $\mathring{D}_d \mathring{\epsilon}_{abc} = 0$. Finally, set $N = \mathring{\epsilon}^{abc}\eta_{abc}$.

- Then: multiplying the the two remaining brown equations by N, one obtains:

$$\mathring{D}_a N^a_i = 0 \quad \Leftrightarrow \quad \mathcal{L}_{\vec{N}_i}\mathring{\epsilon}_{abc} = 0, \quad \text{and} \quad \dot{N}^a_i = \frac{1}{2}\mathring{\epsilon}_i{}^{jk}\,\mathcal{L}_{\vec{N}_j}\, N^a_k.$$

This is the entire content of anti-self-dual solutions of Einstein's equations!

Key simplification: If $A^i_a = 0$ initially, it remains zero because time derivative is transmuted to a space derivative by the generalized GCLD $\mathbb{L}_{\vec{N}_j}$.

ASD solutions & Volume Preserving Diffeos

- Thus, ASD solutions to Einstein's equations are in 1-1 correspondence with solutions to $\mathcal{L}_{\vec{N}_i}\mathring{\epsilon}_{abc} = 0,$ and $\dot{N}_i^a = \frac{1}{2}\epsilon_i{}^{jk}\,[N_j, N_k]^a$

Hence, given any linearly independent triplet N_i^a that generates volume preserving diffeomorphisms on M, we can evolve it by the second equation to obtain an ASD solution:

- Define 't-dependent' 3-metrics $q^{ab} := N_i^a\,N^{bi}$ on M and set

$$g^{\alpha\beta} = \hat{q}^{\frac{1}{2}}\,N\left(t^\alpha t^\beta + q^{\alpha\beta}\right) \qquad \text{with} \qquad \hat{q} = \det \hat{q}_{ab}.$$

This is an ASD solution to Einstein;'s equation and every ASD solution can be written in this form locally (AA, Jacobson, Smolin). The simplicity is striking and the interplay with the volume preserving diffeomorphisms (generated by N_i^a) "explains" the hyperKähler structure underlying anti-self-dual solutions (Robinson). It may well be a pointer to other deep mathematical structures.

- The interplay can directly be traced back to our gauge theory formulation of GR; was not seen in earlier literature on self-dual solutions(Twistor theory or Newman and Plebanski work). Furthermore, this characterization holds not only for complex solutions as in earlier work but also for self-dual Riemannian solutions (instantons) that are real.

5. Geometrization of the Constraint Algebra

• The algebra of Diff constraints is simple because the canonical transformation they generate has a simple geometric meaning: Since $H_{\vec{M}}$ acts by Lie-derivatives along M^a we can integrate by parts and obtain:

$$\{H_{\vec{M}},\, H_{\vec{N}}\} = \{H_{\vec{M}},\, -2\int \mathrm{d}^3x\, \big(N^a\, q^{bc}\, D_b p_{ac}\}\big) = -2\int \mathrm{d}^3x\, N^a \mathcal{L}_{\vec{M}} \big(q^{bc} D_b\, p_{ac}\big)$$
$$= 2\int \mathrm{d}^3x\, (\mathcal{L}_{\vec{M}} N^a)\, \big(q^{bc} D_b p_{ac}\big) = H_{\vec{V}}, \quad \text{where} \quad V^a = -\mathcal{L}_{\vec{M}} N^a.$$

By contrast, the Poisson bracket between two scalar constraints takes pages because in geometrodynamics, the action of H_N is messy and non-geometric; we cannot just integrate by parts! Takes > 10 pages! (Thiemann's book).

• In the gauge theory formulation, the action of H_N is geometric: Generalized GCLD! Therefore, the calculation again has geometric interpretation. It only takes 4 steps and the final answer is:

$$[\, H_M\,,\, H_N\,] = H_{\vec{V}} \qquad \text{where} \qquad V^a = -\mathbb{L}_{\vec{M}_i} N^a_i.$$

Suggests that there is something deep about the fact that the generalized GCLD encodes dynamics. This provides a point of departure for Varadarajan's anomaly free implementation of the LQG quantization program. (Of course, the Poisson bracket still involves structure functions because the generalized GCLD $\mathbb{L}_{\vec{N}_j}$ depends on both A^i_a through gauge covariant derivative and E^a_i through its definition $N^a_i = N E^a_i$.)

Constraint Algebra: Derivation

The complete derivation: from (AA & Varadarajan (2021))

$$
\begin{aligned}
&\{H_M,\ H_N\}\\
&= \tfrac{1}{4}\int_\Sigma \mathrm{d}^3x\, M\, \mathring{\epsilon}^{ij}{}_k\, \big(\dot{E}^a_i\, E^b_j\, F_{ab}{}^k + E^a_i\, \dot{E}^b_j\, F_{ab}{}^k + E^a_i\, E^b_j\, \dot{F}_{ab}{}^k\big)\ - M \leftrightarrow N\\
&= \tfrac{1}{4}\int_\Sigma \mathrm{d}^3x\, M\, \mathring{\epsilon}^{ij}{}_k\, \Big[\mathring{\epsilon}_i{}^{mn}\, (\mathbb{L}_{\vec{N}_m}\, E^a_n)\, E^b_j\, F_{ab}{}^k\ -\ E^a_i\, E^b_j\, \mathring{\epsilon}^{km}{}_n\, \mathbb{L}_{\vec{N}m}\, F_{ab}{}^n\Big] - M \leftrightarrow N\\
&= \tfrac{1}{4}\int_\Sigma \mathrm{d}^3x\, M\, \big[(\mathbb{L}_{\vec{N}_j}\, E^a_k - \mathbb{L}_{\vec{N}_k}\, E^a_j)\, E^{b\,j}\, F_{ab}{}^k\ +\ E^a_k E^b_j\, (\mathbb{L}_{\vec{N}j}\, F_{ab}{}^k - \mathbb{L}_{\vec{N}k}\, F_{ab}{}^j)\big] - M \leftrightarrow N\\
&= \tfrac{1}{4}\int_\Sigma \mathrm{d}^3x\, \Big[2M\, \mathbb{L}_{\vec{N}_j} (E^a_k\, F_{ab}{}^k)\, E^{b\,j}\ +\ 2M E^{a\,(k} F_{ab}{}^{j)}\, (\mathbb{L}_{\vec{N}_k}\, E^b_j)\Big] - M \leftrightarrow N\\
&= \tfrac{1}{2}\int_\Sigma \mathrm{d}^3x\, \big(\mathbb{L}_{\vec{N}j}\, M^a_j\ -\ \mathbb{L}_{Mvj}\, \vec{N}^a_j\big)\, E^b_k\, F_{ab}{}^k\\
&= H_{\vec{V}} \qquad \text{with} \qquad V^a = \tfrac{1}{2}\,\big(\mathbb{L}_{\vec{N}j}\, M^b_j\ -\ \mathbb{L}_{\vec{M}j}\, N^b_j\big)\ \equiv\ -\mathbb{L}_{\vec{M}j}\, N^b_j\,.
\end{aligned}
$$

Note: The right side equals but 'geometrizes' the familiar ADM structure function $V^a = \epsilon\, q^{ab}(N\partial_b M - M\partial_b N)$.

6. Summary & Outlook

• The gauge theory equations are very simple: We arrive at them w/o reference to GR, making appeal to diff and gauge covariance. It severely restricts the form of Hamiltonian if there is no background metric! Similar in spirit to spin-foams where it was argued that the vertex amplitude can be arrived at using general covariance requirements. But this emergence of gravity from gauge theory is very different from the AdS/CFT conjecture. The LQG correspondence is more limited in scope. But it is not a conjecture. It is established rigorously and in detail. Other differences: Higher dimensions, supersymmetry, and negative Λ are not needed!

• Equations of GR are complicated because its geometry is emergent and (q_{ab}, p^{ab}) are rather complicated, composite combinations of the elementary building blocks of the gauge theory. After all, Physics of atoms is simple and clean; Chemistry of molecules they form is rich but messy! The perspective lets one cast 'time-evolution' as 'space-evolution' at the fundamental, gauge theory level: Not possible in the metric framework. This geometrization of 'time evolution' has opened the door to an anomaly-free, non-perturbative quantization. It has brought to forefront 'the essential reason' behind integrability of the half flat sector of GR. Insights into the 'infinite dimensional group' defined by generalized GCDs could lead to unforeseen advances in gravitational dynamics and pure mathematics.

Illustrative Directions for Further Work

- Several directions within gravitational physics and LQG.

- Scattering amplitudes: Perturbative amplitudes have been obtained by regarding GR as a double copy of YM theory (S. Bern et al). Can we use our non-perturbative framework –particularly the generalized GCLD– to shed light into the 'origin' of this connection? Can it supply non-perturbative insights?

- Application to geometry? Thurston program on classification of 3-manifolds was recently completed (by Perelman and Hamilton). Einstein metrics play a key important role there. But for 4-manifolds, the 'dream' of trying to understand 4-manifolds via Einstein or other canonical metrics may well be impossible to realize (Gromov). Deep understanding of smooth 4-manifolds has come via the geometry of half-flat gauge fields (Donaldson). But "How far such theories can be carried over to metrics, (the gravitational field), remains to be seen." (Review article, M.T. Anderson). Perhaps the gauge theory framework underlying LQG –together with the 'infinite dimensional group' generated by generalized GCLDs– would be helpful since they bridge the two in a background independent fashion!