Semiclassical states, high order quantum corrections and cosmology.

Tomasz Pawłowski

Institute for Theoretical Physics, University of Wrocław

Loops'22 Conference
ENS de Lyon (France) 18-22.07.2022

work and presentation supported in part by National Center for Science (NCN) project 2020/37/B/ST2/03604

based mostly on D. Brizuela, TP, arXiv:2107.04342

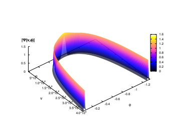
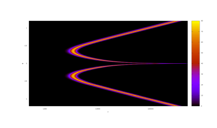
The problem

- Determining the evolution of quantum system on the genuine quantum level: solving partial differential equation(s) or large set of coupled ordinary differential equations

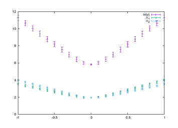
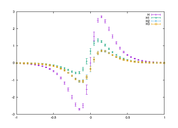
$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$

Example (Loop Quantum Cosmology):

  - Isotropic LQC: 10^5 points in v (equations).

  - Anisotropic homogeneous LQC 10^9 equations. Just 3D

- More variables: impractical!!

However ...

- Often interest in specific systems: behaving classically (small variances)
  - Restriction to semiclassical states very useful.
- Usually just a couple of DOF (specific observables) is of interest.
  - No need to track fine details of the wave function shape.
- Desired: a method of casting the quantum evolution that:
  - Involves just a couple of relevant degrees of freedom.
  - Allows for control of the effects of neglected DOF with arbitrary precision.
- The solution: semiclassical effective dynamics based on Hamburger moments.

Semiclassical effective dynamics

- In quantum optics known for a long time, for cosmology rediscovered by Bojowald, Skirzewski 2006.
- General construction: define a set of observables that
  - Encode all the information about the quantum state.
  - Few of them contain relevant information (expectation values and variances of interesting observables) while
  - the rest can be treated as quantum corrections.
  - Set of equations of motion represents the quantum dynamics fully.
- Closeness and completeness requires a Poisson algebra of observables.

Hamburger moments

- Example: a 1D quantum system with Heisenberg algebra $\{\hat{x}, \hat{p}, \mathbb{I}\}$ and the hamiltonian $\hat{H}$

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{I} \quad i\hbar\partial_t\Psi(x) = [\hat{H}\Psi](x).$$

- Observables: analogs of statistical moments

$$G^{mn} ='' \langle : (\hat{x} - \langle\hat{x}\rangle\mathbb{I})^m(\hat{p} - \langle\hat{p}\rangle\mathbb{I})^n :\rangle''$$
$$:= \sum_{i,j=0}^{m,n} \binom{m}{i}\binom{n}{j}(-1)^{(m+n)-(i+j)}\langle : \hat{x}^i\hat{p}^j :\rangle\langle\hat{x}\rangle^{m-i}\langle\hat{p}\rangle^{n-j}$$

- Poisson algebra structure uniquely defined by requirement

$$\{\langle : \hat{x}^m\hat{p}^n :\rangle, \langle : \hat{x}^{m'}\hat{p}^{n'} :\rangle\} = (i\hbar)^{-1}\langle [: \hat{x}^m\hat{p}^n :, : \hat{x}^{m'}\hat{p}^{n'} :]\rangle$$

  - $\{G^{ab}, G^{cd}\}$ polynomial and at most 2nd order in G
  - couples G^{mn} of order (m+n) up to a+b+c+d-1.
  - In practice: found automatically by metalanguage procedures
- Semiclassicality: all the $F^{mn} := \langle : \hat{x}^{m'}\hat{p}^{n'} :\rangle$ must be finite.
  - In practice: $\sum_{m,n\in\mathbb{N}} |G^{mn}| < \infty$.

The dynamics

- Taylor-like expansion of the Hamiltonian

$$H(\hat{x},\hat{p}) = H(\langle\hat{x}\rangle\mathbb{I} + (\hat{x} - \langle\hat{x}\rangle\mathbb{I}), \langle\hat{p}\rangle\mathbb{I} + (\hat{p} - \langle\hat{p}\rangle\mathbb{I}))$$

  - In Weyl ordering ($x := \langle\hat{x}\rangle$, $p := \langle\hat{p}\rangle$):

$$H(x,p,\{G^{mn}\}) := \langle\hat{H}(\hat{x},\hat{p})\rangle = \sum_{mn} \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial^m x \partial^n p} H(x,p) G^{mn}$$

  - In practice the ordering of operators in F^{mn} is adapted to the factor ordering in $\hat{H}$.
- Any observable can be decomposed the same way.
- The equations of motion: Hamilton-Jacobi equations for G^{mn}, x, p,

$$\dot{G}^{mn} = \{H(\{G^{mn}\}), G^{mn}\}$$

  - Countable set of equations.
  - Polynomial and at most quadratic in G^{mn} for $m+n \geq 2$. Nonlinear!
  - Can be cut-off at arbitrary order $m+n$.

Loop quantized system (LQC)

- LQC isotropic geometry example
  - Classical variables: volume V and momentum b

$$\{V, b\} = 4\pi\gamma\sqrt{\Delta}G = a \propto \hbar^{1/2}$$

    γ - Barbero-Immirzi par; Δ - LQC area gap
  - Generators: volume V and holonomy component $N = e^{ib}$ (promoted to operators)
    - Hamburger moments $\tilde{G}^{ab}$ algebra complex!!
  - Basic observables: V, $c = (N + N^{-1})/2$, $s = (N - N^{-1})/(2i)$

$$\{V, c\} = -as, \ \{V, s\} = ac, \ \{s, c\} = 0$$

  - Absolute moments: $F^{abc} := \langle : \hat{V}^a \hat{s}^b \hat{c}^c : \rangle \to G^{abc}$
  - Necessary step of tranformation $G^{abc} \leftrightarrow \tilde{G}^{ab}$
    - Performed via metalanguage

Simple FRLW inflationary universe

- The model: flat isotropic FRLW universe with matter:
  - Dust providing the internal clock (negligible mass to not affect the dynamics).
  - Massive scalar field: quadratic inflationary potential.
- The metric:

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\,{}^o q \qquad {}^o q = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- Degrees of freedom:
  - geometry: canonical pair $(V = a^3, \pi_V)$ s.t. $\{V, \pi_V\} = 1$,
  - matter: canonical pair $\{\phi, p_\phi\}$,
  - time: dust field potential T.
- The dynamics generated by the (deparametrized) Hamiltonian

$$H(V, \pi_V, \phi, p_\phi) = -6\pi G V \pi_V^2 + \frac{p_\phi^2}{2V} + \frac{m^2}{2} V \phi^2$$

  - Polynomial with exception of $1/V$ term!

The quantization

- Quantization: Wheeler-deWitt one (following the geometrodynamics program)
  - Both dynamical DOF quantized using Schrödinger representation:
    - variables (V, π_V, ϕ, p_ϕ) promoted to operators,
    - Hilbert space: $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}V) \otimes L^2(\mathbb{R}, \mathrm{d}\phi)$.
  - The quantum dynamics generated by the Schrödinger equation

$$i\hbar\partial_T \Psi_T(V,\phi) = \left[-6\pi G \hat{\pi_V} \hat{V} \hat{\pi_V} + \frac{1}{2}\hat{V}^{-1}\hat{p}_\phi^2 + \frac{m^2}{2}\hat{V}\hat{\phi}^2\right]\Psi_T(V,\phi)$$

- The semiclassical description:
  - expectation values V, π_V, ϕ, p_ϕ,
  - the moments

$$G^{abcd} := \langle : (\hat{V} - \langle\hat{V}\rangle\mathbb{I})^a(\hat{\pi_V} - \langle\hat{\pi_V}\rangle\mathbb{I})^b : \\ : (\hat{\phi} - \langle\hat{\phi}\rangle\mathbb{I})^c(\hat{p_\phi} - \langle\hat{p_\phi}\rangle\mathbb{I})^d :\rangle$$

where $:\hat{x}\hat{p} ::= (1/2)(\hat{x}\hat{p} + \hat{p}\hat{x})$.

Framework for numerical analysis

D. Brizuela, TP, 2021

- The system:
  - Generated sets of EOM for cutoffs n from 2 to 6.
    - The respective numbers of EOM: $14, 34, 69, 125, 209$.
  - Initial data: evaluated moments for Gaussian (and couple other) shapes in basic variables.
  - Inflaton mass: $\sim 10^{-6} m_{\mathrm{Pl}}$.
  - Initial data specified in (high density) kinetic dominated region before inflation.
  - Since V is the infrared regulator we "densitized" the moments

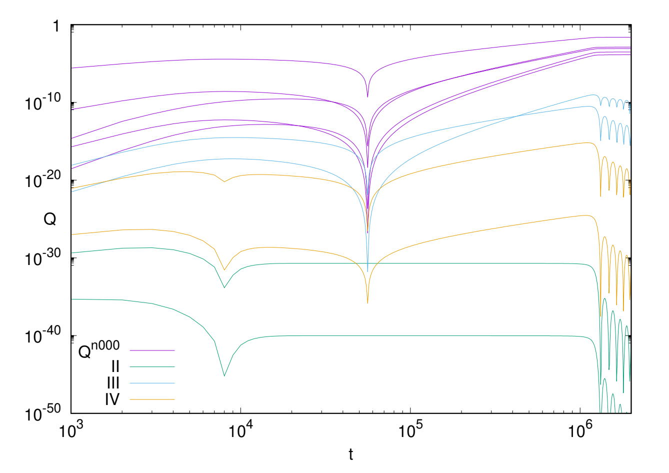
$$G^{abcd} \to Q^{abcd} := G^{abcd}/V^{a+d}, p_\phi \to \sigma := p_\phi/V$$

- The evolution range: well past inflation end.
- The data processing:
  - Compared the trajectories of expectation values and higher moments for various order of cutoff.
  - Convergence: determined convergence of the trajectories as n increases.

Regulator removal limit

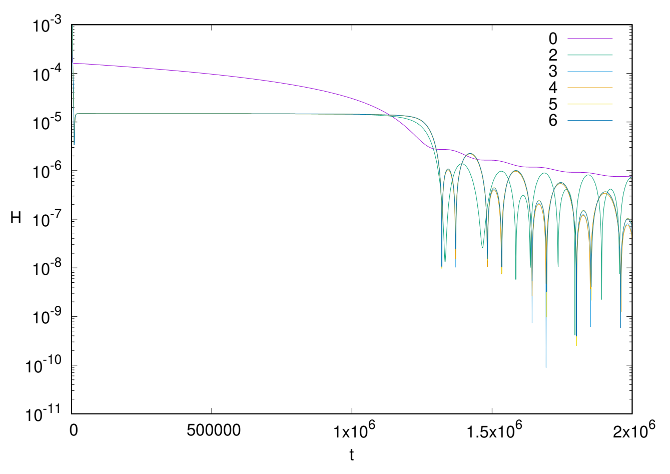
- Infrared regulator:
  - Action/Hamiltonian/momentum variables defined by integrating densities over (infinite) Cauchy slice.
  - For finiteness integration restricted to finite comoving region.
  - Crucial requirement: model has to admit nontrivial limit as chosen region is expanded to encompass whole slice!
  - Addressed only partially: Corichi, Montoya, Singh, ...
- Crucial properties of EOM:
  - At 2nd order cutoff V decouples. System manifestly invariant wrt. fiducial cell choice.
  - For higher order V appears only through terms $(\hbar/V)^n$.
- Consequences:
  - System admits well defined limit $V \to \infty$.
  - Taking that limit is mathematically equivalent to setting $\hbar \to 0$: system behaves like an ensemble of statistical ones.

Densitized central moments



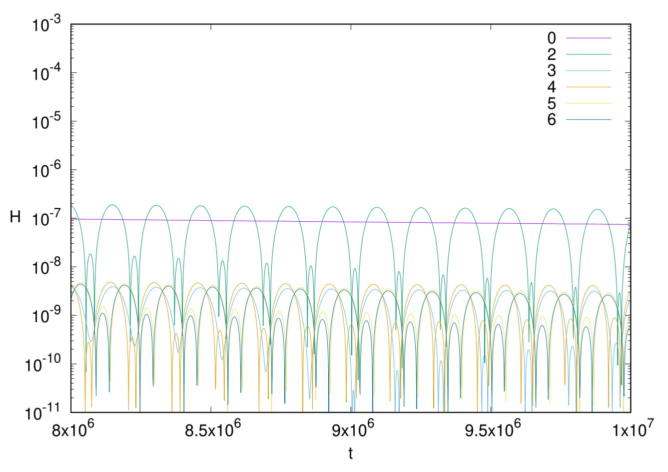
Rysunek: Evolution of sample of the moments for order 6.

Hubble parameter corrections



Rysunek: Quantum corrections to H_r.

Hubble parameter corrections (2)



Rysunek: Quantum corrections to H_r: late reheating.

Distinguished dynamical results

- Significant dispersion in volume
  - Moments Q^{n000} start rapidly increasing mid-inflation, becoming constant upon inflation end.
  - Approximate analysis of 2nd order EOM shows that transition point is distinguished by a certain function of expectation values.
- Counterintuitive behavior past inflation (classicalization)
  - After exiting the inflation the quantum corrections to locally measurable observables decrease as the cutoff order increases!
    - Holds for all evolved data, though variances in conjugate variables differ by at most 1 level of magnitude.

General insights

- General observations inferred:
  - Method robustness: described semiclassical framework is a technically viable and powerful tool of controlling the dynamical evolution of quantum systems up to high order (of quantum corrections).
  - Consistency check: it allows to probe the models consistency in the infrared regulator removal limit in a precise and robust manner.
  - Surprises: in certain domains the 1st subleading correction can actually be least accurate and including higher order corrections may prove necessary to accurately describe systems past 0th order approximations.
- Point to remember:
  - What is the real result is the limit to which the trajectories converge with increasing order!

Thank you for your attention!

Research and presentation supported in part by National Center for Science (NCN), Poland under grant no. 2020/37/B/ST2/03604.