The flatness (non-)problem in spin-foams and its significance

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Motivation

The correct classical limit

is **the most basic test** of any quantum theory. **(Bohr's correspondence principle)**

What does it mean to take the classical limit?

1. **Bohr:** Limit within some family of eigenstates in which **quantum numbers become large**.
2. $\hbar \to 0$ (equiv. to (1.), for example, from $L_z = m\hbar$, with L_z constant.)
3. Limit within **some family of coherent states** in which products of labels with dimension of action are **large compared to** $\hbar$, and which **relative uncertainties** of a complete set of observables go to zero.

Transition amplitude has correct classical limit if:

it is **exponentially suppressed** iff labels of initial and final states (=boundary states) are **inconsistent with classical e.o.m.**

Choice of classical limit

- Note that in possibility 3, the most comprehensive one, there are **many compatible choices of limit in the space of coherent state labels**.

- To test recovery of the **full classical equations of motion**, is helpful to additionally require the **flow in the space of labels to be a symmetry of the classical equations of motion**, so that the **classical equations of motion do not become degenerate**.

Explicit example: Classical limit of transition amplitude, for free particle, between coherent states

Coherent states in momentum representation: $\tilde{\psi}_{q',p'}(p) = (2\pi\sigma_p^2)^{-1/4} \exp\left(-\frac{1}{4}\left(\frac{p-p'}{\sigma_p}\right)^2 - \frac{iq'p}{\hbar}\right)$

Transition amplitude:

$$A((q_2,p_2),(q_1,p_1),T) := \langle q_2,p_2|e^{-\frac{i}{\hbar}\hat{H}T}|q_1,p_1\rangle = \langle q_2,p_2|e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}T}|q_1,p_1\rangle$$

$$= (2\pi)^{-1/2}\sigma_p^{-1}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}\left(\frac{p-p_2}{\sigma_p}\right)^2 + \frac{iq_2p}{\hbar} - \frac{iT}{2m\hbar}p^2 - \frac{1}{4}\left(\frac{p-p_1}{\sigma_p}\right)^2 - \frac{iq_1p}{\hbar}\right)dp$$

$$= \alpha^{-1/2}\exp\left(-\frac{1}{8\alpha}\left(\frac{\delta_q}{\sigma_q}\right)^2 - \frac{1}{8}\left(\frac{\delta_p}{\sigma_p}\right)^2 + \frac{ip_{\mathrm{ave}}^2T}{2m\hbar} + \frac{i\delta_q p_{\mathrm{ave}}}{\hbar}\right)$$

where

$$p_{\mathrm{ave}} := (p_1+p_2)/2 \qquad \delta_q := q_2 - \left(q_1 + \frac{p_{\mathrm{ave}}}{m}T\right) \qquad \delta_p := p_2 - p_1 \qquad \alpha := 1 + \frac{i\sigma_p^2 T}{m\hbar}$$

Note $\delta_q = \delta_p = 0$ **iff the boundary data is consistent with the classical e.o.m.**:

$$\boxed{q_2 = \left(q_1 + \frac{p_{\mathrm{ave}}}{m}T\right) \qquad p_2 = p_1 = p_{\mathrm{ave}}}$$

When boundary data is consistent with e.o.m., the amplitude reduces to exponential of action evaluated on history thereby selected, times a coefficient independent of the boundary data.

$$A((q_2, p_2), (q_1, p_1), T) = \alpha^{-1/2} \exp\left(\frac{i p_1^2 T}{2m\hbar}\right) = \alpha^{-1/2} \exp\left(\frac{i}{\hbar} S\right)$$

Classical limit: $\mid q, p\rangle = \mid \lambda q_o, \lambda p_o\rangle, \quad \lambda \to \infty,$ so that $\dfrac{\Delta q}{\langle q \rangle} = \dfrac{\hbar}{2\lambda \sigma_p q_o} \to 0, \quad \dfrac{\Delta p}{\langle p \rangle} = \dfrac{\sigma_p}{\lambda p_o} \to 0$.

Note equations of motion, $q(T) = \left(q(0) + \frac{p(0)}{m} T\right), \; p(T) = p(0)$ are independent of λ, reducing to $q_o(T) = \left(q_o(0) + \frac{p_o(0)}{m} T\right), \; p_o(T) = p_o(0)$.

Classical limit of exact transition amplitude:

$$|A((\lambda q_2, \lambda p_2), (\lambda q_1, \lambda p_1), T)| = |\alpha|^{-1/2} \exp\left(-\frac{\lambda^2}{8|\alpha|}\left(\frac{\delta_q}{\sigma_q}\right)^2 - \frac{\lambda^2}{8}\left(\frac{\delta_p}{\sigma_p}\right)^2\right),$$

so that the transition amplitude is **not** exponentially suppressed **iff** $\delta_q = \delta_p = 0$, that is, **iff the boundary data is consistent with the classical e.o.m.**

Flatness (non-)problem in spin-foams

- To calculate anything in spin-foams, **a cut-off is required**: A choice of **2-complex**, or a choice of a **max number of vertices**.

- Flatness `problem´ has to do with interplay classical limit and limit of removal of cut-off.

- Specifically: With a **fixed cut-off,** amplitude is **exponentially suppressed** in classical limit **unless** boundary data is consistent with a **flat space-time geometry** (roughly speaking).

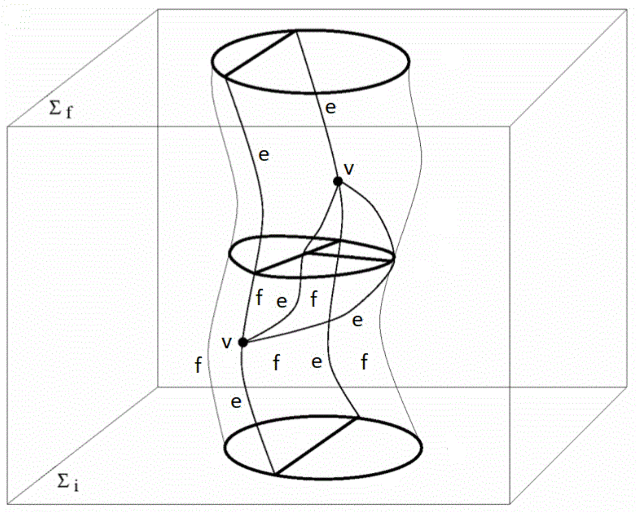
- Of course, this is **a problem with the classical limit of the truncated model, not of the model per se**.

- Nevertheless, it is still very important because it tells us that we **should not take the classical limit on a fixed 2-complex / fixed max number of vertices**. Rather, it suggests **a different prescription** for performing calculations.

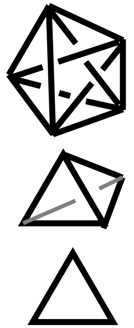
Background

2-complex and dual cell complex / triangulation

2-complex $\mathcal{F} \overset{\text{dual}}{\longleftrightarrow}$ cell complex $\mathcal{F}^*$ (=triangulation Δ in simplicial case).

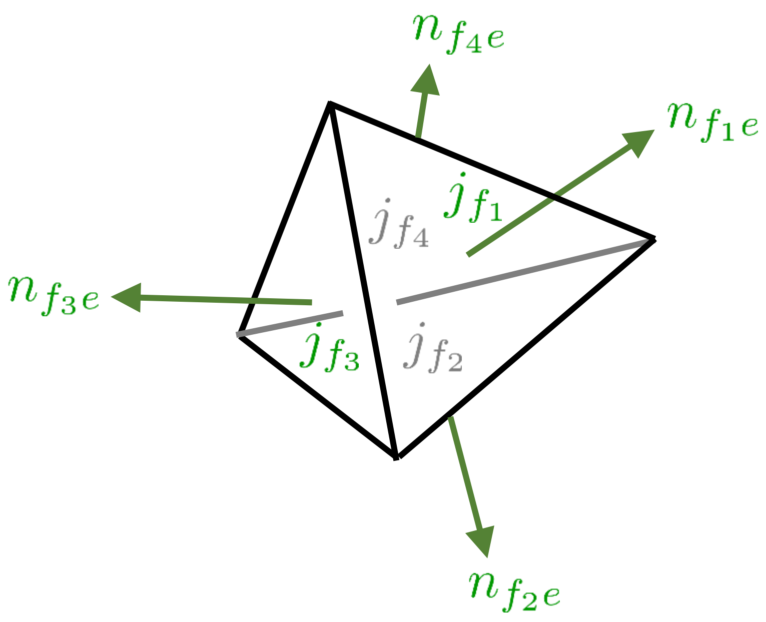


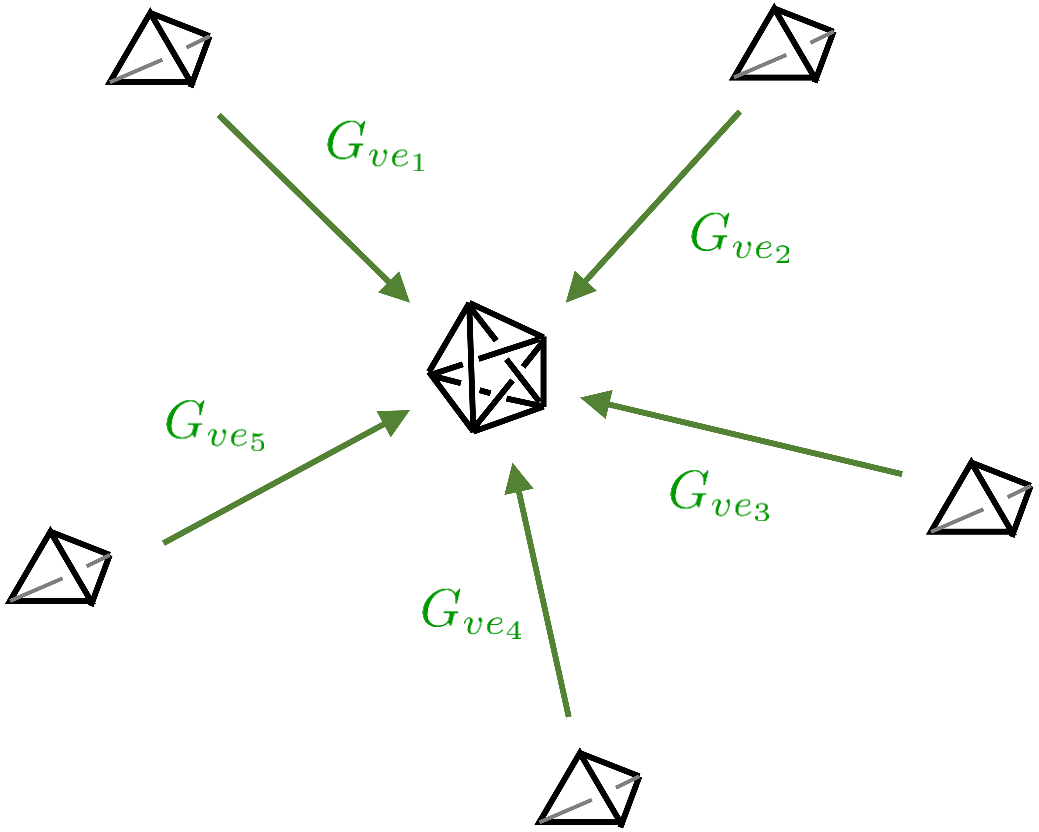
Component of $\mathcal{F}$		Dual to	(Simplicial case)
v	vertex	4-cell	(4-simplex)
e	edge	3-cell	(tetrahedron)
f	face	2-cell	(triangle)



Variables

variable	meaning
$\frac{A_f}{8\pi G\gamma} = j_f \in \frac{\mathbb{Z}}{2}$	area of dual 2-cell (triangle) f^*
$n_{fe} \in S^2$	normal to f^* "in the frame of the 3-cell (tetrahedron) e^*"
$G_{ve} \in Spin(4)$ or $SL(2,\mathbb{C})$	parallel transport from "e^* frame to frame of 4-cell (4-simplex) v^*"
$\Big(h_{vf} \in SU(2)$	Ashtekar-Barbero transport along link $f \cap \partial v^*$ within the boundary of v^* $\Big)$





Transition amplitudes

Eucl. and Lor. EPRL transition amplitudes with boundary states peaked on j_f ´s and n_{fe}´s (Livine-Speziale states) both take the form

$$A\left((j_f, n_{fe})_{e\in f\in\partial\mathcal{F}}\right) = \sum_{\{j_f\}_{f\in \mathrm{int}\mathcal{F}}} \left(\prod_{e\in f\in \mathrm{int}\mathcal{F}} \int d^2 n_{fe}\right)\left(\prod_{f\in\mathcal{F}} A_f(j_f)\right)\left(\prod_{v\in\mathcal{F}} A_v(\{j_f, n_{fe}\}_{v\in e\in f})\right)$$

The Euclidean and Lorentzian cases can be separately further expanded as

$$A^{\mathrm{Eucl}}\left((j_f, n_{fe})_{e\in f\in\partial\mathcal{F}}\right) = \sum_{\{j_f\}_{f\in \mathrm{int}\mathcal{F}}} \left(\prod_{e\in f\in \mathrm{int}\mathcal{F}} \int d^2 n_{fe}\right)\left(\prod_{v\in e\in\mathcal{F}} \int dG_{v\in e}\right)\left(\prod_{f\in\mathcal{F}} A_f(j_f)\right) e^{iS^{\mathrm{Eucl}}(\{j_f, n_{fe}, G_{ev}\})}$$

$$A^{\mathrm{Lor}}\left((j_f, n_{fe})_{e\in f\in\partial\mathcal{F}}\right) = \sum_{\{j_f\}_{f\in \mathrm{int}\mathcal{F}}} \left(\prod_{e\in f\in \mathrm{int}\mathcal{F}} \int d^2 n_{fe}\right)\left(\prod_{v\in e\in\mathcal{F}} \int dG_{v\in e}\right)\left(\prod_{e\in f\in\mathcal{F}} \int_{\mathbb{CP}^1} d^2 z_{fe}\right)$$

$$\left(\prod_{f\in\mathcal{F}} A_f(j_f)\right) e^{iS^{\mathrm{Lor}}(\{j_f, n_{fe}, z_{fe}, G_{ev}\})}$$

where S^{Eucl} and S^{Lor} are both linear in the j_f's':

Can apply stationary phase in large spin limit.

Asymptotic imposition of geometricity.

The data on the 2-complex are **redundant,** so that there are constraints – **geometricity constraints** - that ensure that they define **a consistent piece-wise flat geometry**.

For sufficiently large cell complexes there are degenerate sectors in which geometricity is not satisfied. (**Problem addressed by proper vertex** [Engle, Vilensky, Zipfel (2013-2016)] .)

But for **sufficiently small triangulations**, with appropriate choice of boundary state, these sectors are **exponentially suppressed**, and the contributions to the spin-foam are **exponentially suppressed unless geometricity is satisfied**.

General results

*For **Partition function**: Contribution to sum from asymptotically large **bulk** spins.*

What about contribution from small bulk spins? Physical significance of just **part** of the sum being suppressed is unclear! Contributed to skepticism about significance of the results.

	Models	**Path integral variables**	**Exp. suppr. as $\lambda \to \infty$ unless, for all interior f^*,**
Bonzom (2009)	Eucl. EPRL, Eucl. BC, FK on simplicial 2-complex	$\lambda A_f \in \mathbb{R}^+$, n_{fe}, G_{ve}. N.B. A_f are approximated as continuous!	$\Theta_f = 0$
Perini (2012)	Lor. EPRL on arb. 2-complex.	$\lambda j_f \in \mathbb{N}/2$, n^i_{fe}, G_{ve}, $H_{vf} \equiv h_{vf} e^{\lambda j_f n^i_{fe_-(f,v)} \sigma_i} \in SL(2,\mathbb{C})$. (Holomorphic rep.)	$\gamma\Theta_f = 0 \mod 4\pi$
Han 2013	Lor. EPRL on simplicial 2-complex	$\lambda j_f \in \mathbb{N}/2$, n_{fe}, G_{ve}	$\gamma\Theta_f = 0 \mod 4\pi$

*For **transition amplitude** for a **boundary state** in the limit of large **boundary** spins.* Physical significance is clearer.

	Models	Boundary states	Exp. suppr. as $\lambda \to \infty$ unless	Formulation / Methods
Hellmann and Kaminski (2012, 2013)	Eucl. EPRL, Eucl. BC, FK, on simplicial complex $\mathcal{F}$	Livine-Speziale states $(n_{fe}, \lambda j_f)_{f\in e\in\partial\mathcal{F}}$ (Peaked on intrinsic boundary geom.)	boundary geom. is consistent with an interior simplicial geometry with $$\gamma\Theta_f = 0 \mod 4\pi$$ for all interior f	Holonomy Spin foam formulation / wavefront sets

*Large spin limit yields Flatness also in **simpler** quantum gravity model*

Asante, Dittrich, and Haggard (2021) also find flatness for large spins in a simple path integral for quantum gravity motivated **only by a discrete area spectrum (effective spin foams).**

Suggests that large spin limit yields flatness **quite generally** in path integral quantum gravity when **cut-off to a particular triangulation / 2-complex**.

Testing in a specific case: Δ_3

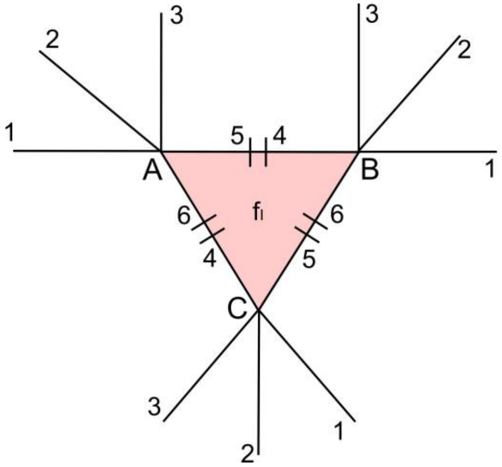
- Δ_3 consists in a **single internal triangle**, with **three 4-simplices in its link** — the simplest triangulation admitting Regge geometries with curvature.
- Equivalently, Δ_3 is the **3-3 Pachner move**.

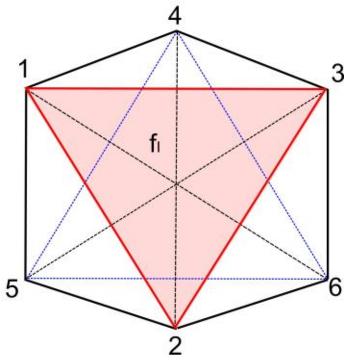
Model: Eucl. EPRL. Boundary states: Livine-Speziale $(n_{fe}, \lambda j_f)_{f \in e \in \partial(\Delta_3^*)}$

Analytical arguments:

	Exp. suppr. as $\lambda \to \infty$ unless boundary geom. extends to a bulk geom. that is Regge-like and	**Assumptions/Methods**
Magliaro and Perini (2011)	No further restriction	**Assumption:** Exp. suppressed bounds for non-critical config.s have coeff.s forming a bounded set as $\lambda \to \infty$. **(False. Set is unbounded in neighborhood of any crit. point.)**
Oliveira (2018)	No further restriction	**Assumption:** Only the magn. of indiv. terms in the sum over spins are relevant for suppression. **(False. Phases and sum also matter.)**
Engle, Kaminski, and Oliveira (2021)	$\gamma\Theta_f = 0 \mod 4\pi$	Uses Poisson resummation of the sum over spins to manifest the peak in the conjugate variable, Θ_f. (Similar to argument in Han 2013.)

Details:





[Oliveira, 2017]

$$A\left((\lambda j_f, n_{fe})_{\partial\Delta}\right) = \sum_{\lambda j_{int} \in \frac{\mathbb{N}}{2} \cap \lambda[j_{int}^{min}, j_{int}^{max}]} \int \prod_{e,v \in \Delta^*} dG_{ev} \prod_{e^* \in int(\Delta)} d^2 n_{fe} \;\; e^{i\lambda S(j_f, n_{fe}, G_{ev})}$$

(using a theorem from Hörmander)

$$\underset{\lambda\to\infty}{\sim} C\lambda^{-54} \sum_{\substack{n = 2\lambda(j_{int}^{min} - j_{int}^{o}) \\ 2\lambda(j_{int} - j_{int}^{o}) =: n \in \mathbb{Z}}}^{2\lambda(j_{int}^{max} - j_{int}^{o})} \exp\left(i\lambda S_{Regge}((j_f, n_{fe})_{\partial\Delta}) + \frac{i\gamma}{2}\Theta n - \frac{\alpha}{\lambda} n^2\right), \text{ where } \mathrm{Re}\alpha > 0$$

$$\underset{\lambda\to\infty}{\sim} C\lambda^{-54} e^{iS_{Regge}((\lambda j_f, n_{fe})_{\partial\Delta})} \sum_{n\in\mathbb{Z}} \underbrace{\exp\left(\frac{i\gamma}{2}\Theta n - \frac{\alpha}{\lambda} n^2\right)}_{=:f(n)}$$

$$\hat{f}(k) := \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{\lambda}x^2 + \frac{i\gamma}{2}\Theta x - 2\pi i k x\right) dx = \sqrt{\frac{\pi\lambda}{\alpha}} e^{-\frac{\lambda}{4\alpha}\left(\frac{\gamma}{2}\Theta - 2\pi k\right)^2}.$$

So, by Poisson resummation, $$A = \tilde{C}\lambda^{-53.5} e^{iS_{Regge}} \sum_{k\in\mathbb{Z}} e^{-\frac{\lambda}{4\alpha}\left(\frac{\gamma}{2}\Theta - 2\pi k\right)^2}.$$

Poisson resummation let's us see the peakedness in Θ_f, the variable *conjugate* to j_f.

Thus, amplitude is exponentially suppressed unless $\gamma\Theta = 4\pi k$ for some $k \in \mathbb{Z}$. That is, unless

$$\boxed{\gamma\Theta = 0 \bmod 4\pi}$$

Matching Han (2013), and Hellmann-Kaminski (2012).

Every analytical argument so far, when sufficiently refined, yields accidental curvature constraint (`flatness´) , in classical limit on a fixed cell complex.

Numerical confirmation for Lorentzian EPRL:

Gozzini (2021), building on prior work Dona, Gozzini, Sarno (2020).

Required spins > 40 to see **flatness / accidental curvature** constraint. Required combining **multiple novel** numerical techniques.

By contrast, **geometricity constraints** can be see already with **spins < 10**. [Dona, Fanizza, Sarno, Speziale (2020)]

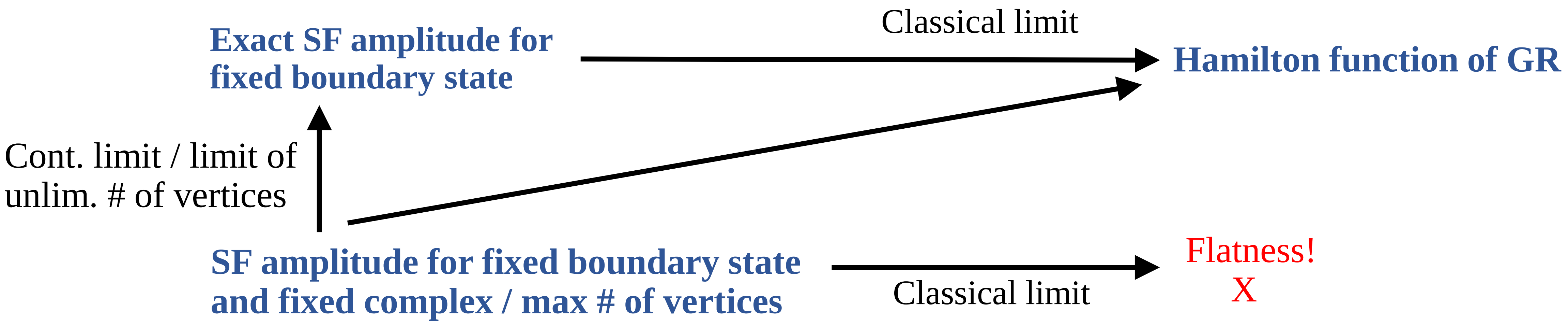
Consensus reached.

Why it is not a problem with the theory

Answer

Because **“the theory”** would include **removal of the cut-off on the number of vertices**, or, equivalently, include actually **taking the refinement limit** of the triangulation.

That is: The flatness problem is not a problem with “the theory”, but **a problem with a truncation of the theory** by a fixed infrared cut-off. As we all learned from basic QFT, infrared cut-offs must be removed to extract physical results.



Toy examples

To provide confidence that it is not a problem with the theory as such, we considered simple **toy examples** in which one can **remove the infrared cut-off explicitly**. In such examples, one has an `accidental' constraint **if one takes the classical limit prior to removing the cut-off**. But if one removes the cut-off first, there is no problem, and one obtains familiar results.

Example 1, Simple Harmonic Oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

(Includes free particle for k=0.)

Boundary states: std. coherent states $\mid q_o, p_o\rangle$, peaked at $q = q_o$, $p = p_o$, with $\Delta q = \sigma$, $\Delta p = \frac{\hbar}{2\sigma}$.

Transition amplitude:

$$W(q_f, p_f; q_i, p_i; t) := \langle q_f, p_f \mid e^{-\frac{i}{\hbar}Ht} \mid q_i, p_i\rangle = \int \frac{dq_n dp_n}{\pi} \prod_{n=0}^{N-1} \langle q_{n+1}, p_{n+1} \mid e^{-\frac{i}{\hbar}H\epsilon} \mid q_n, p_n\rangle$$

$$= \int \frac{dq_n dp_n}{\pi} \prod_{n=0}^{N-1} \left(\langle q_{n+1}, p_{n+1} \mid \mathbb{I} - \frac{i}{\hbar} H\epsilon \mid q_n, p_n\rangle + O(\epsilon^2) \right)$$

$$= \int \frac{dq_n dp_n}{\pi} \prod_{n=0}^{N-1} \langle q_{n+1}, p_{n+1} \mid \mathbb{I} - \frac{i}{\hbar} H\epsilon \mid q_n, p_n\rangle + O(\epsilon)$$

Truncated amplitude:

$$W_N(q_f, p_f; q_i, p_i; t) := \int \frac{dq_n dp_n}{\pi} \prod_{n=0}^{N-1} \langle q_{n+1}, p_{n+1} \mid \mathbb{I} - \frac{i}{\hbar} H\epsilon \mid q_n, p_n\rangle = \langle q_f, p_f \mid \left(\mathbb{I} - \frac{i}{\hbar}H\epsilon\right)^N \mid q_i, p_i\rangle$$

Classical limit: $|\,\lambda q_o, \lambda p_o\rangle, \quad \lambda \to \infty, \quad$ so that $\dfrac{\Delta q}{q} = \dfrac{\sigma}{\lambda q_o} \to 0$ and $\dfrac{\Delta p}{p} = \dfrac{\hbar}{2\hbar\lambda q_o} \to 0$

Note equations of motion, $\quad \dot{q} = \dfrac{p}{m}, \quad \dot{p} = -kq, \quad$ are independent of λ.

Classical limit of *truncated amplitude*:

$$W_N(\lambda q_f, \lambda p_f; \lambda q_i, \lambda p_i; t) = \langle q_f, p_f \,|\, \left(\mathbb{I} - \frac{i}{\hbar} H \epsilon\right)^N \,|\, q_i, p_i\rangle$$

$$= e^{-\lambda^2\left[\frac{\left(q_f - q_i\right)^2}{2\sigma^2} + \frac{\sigma^2\left(p_f - p_i\right)^2}{4\hbar^2}\right]} \sum_{\substack{j=0,J \\ k=0,K}} C_{jk} \left(q_f - i\frac{\sigma^2}{\hbar} p_f\right)^j \left(q_i + i\frac{\sigma^2}{\hbar} p_i\right)^k \lambda^{j+k}$$

is exponentially suppressed ($o(\lambda^{-m})$ for all $m > 0$) unless:

$$\boxed{q_f = q_i \quad \text{and} \quad p_f = p_i}.$$

‘Accidental constraint’ is obtained **if** classical limit is taken **before** $N \to \infty$ is taken.

Example 2: Larmor Precession of charged particle on S^2

Classical analysis:

$$\vec{B} = B\hat{z} \qquad \vec{A} = \frac{1}{2}\vec{B} \times \vec{r} = \frac{B}{2}\left(-y\hat{x} + x\hat{y}\right)$$

$$L = T - U = \frac{1}{2}m\vec{v}^2 + q\vec{v}\cdot\vec{A} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{1}{2}qBR^2\sin^2\theta\dot{\phi}$$

$$\pi_\theta := \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad \pi_\phi := \frac{\partial L}{\partial\dot{\phi}} = mR^2\sin^2\theta\left(\dot{\phi} + \frac{qB}{2m}\right).$$

$$\vec{L} := \vec{r}\times\vec{\pi} := \vec{r}\times\left(\vec{p} + q\vec{A}\right) = \left(-\sin\phi\pi_\theta - \cot\theta\cos\phi\pi_\phi\right)\hat{x} + \left(\cos\phi\pi_\theta - \cot\theta\sin\phi\pi_\phi\right)\hat{y} + \pi_\phi\hat{z}$$

$$\boxed{H} := \pi_\theta\dot{\theta} + \pi_\phi\dot{\phi} - L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = \frac{1}{2}m\vec{v}^2 = \frac{(\vec{r}\times\vec{p})^2}{2mR^2} = \frac{\left(\vec{L} - q\vec{r}\times\vec{p}\right)^2}{2mR^2}$$

$$= \frac{\vec{L}^2}{2mR^2} + \frac{qB}{2m}L_z + \frac{q^2B^2R^2}{8m}\sin^2\theta \quad \boxed{\approx \frac{\vec{L}^2}{2mR^2} + \frac{qB}{2m}L_z}$$

$$\Longrightarrow \quad \dot{\vec{L}} = \{\vec{L}, H\} = \frac{qB}{2m}\{\vec{L}, L_z\} \quad \Longrightarrow \quad \boxed{\omega := \frac{qB}{2m}}$$

Quantum theory and boundary states:

$$\psi(\theta,\phi)\in\mathcal{H} \qquad \langle\psi,\phi\rangle=\int_{S^2}\overline{\psi(\theta,\phi)}\phi(\theta,\phi)\sin\theta d\theta d\phi \qquad \hat{H}=\frac{\hat{\vec{L}}^2}{2mR^2}+\frac{qB}{2m}\hat{L}_z$$

For all $\vec{L}'\in\mathbb{R}^3$ such that $|\vec{L}'|=:\hbar\ell\in\hbar\mathbb{N}$, define $|\vec{L}'\rangle$ by

$$\hat{\vec{L}}^2\mid\vec{L}'\rangle=\hbar^2\ell(\ell+1)\mid\vec{L}'\rangle$$

$$\hat{n}'\cdot\hat{\vec{L}}\mid\vec{L}'\rangle:=(\vec{L}'/|\vec{L}'|)\cdot\hat{\vec{L}}\mid\vec{L}'\rangle=\hbar\ell\mid\vec{L}'\rangle$$

[Livine-Speziale coherent states]

Known properties:

$$\langle\vec{L}'|\hat{L}^i|\vec{L}'\rangle=(L')^i \qquad \Delta:=\langle\hat{\vec{L}}^2\rangle-\langle\hat{\vec{L}}\rangle^2=(\Delta L_x)^2+(\Delta L_y)^2+(\Delta L_z)^2=\hbar^2\ell$$

$$\mathbb{I}=\sum_{\ell=0}^{\infty}(2\ell+1)\int|\ell\hat{n}\rangle\langle\ell\hat{n}|d^2\hat{n}=:\int d\mu(\vec{L})\,|\vec{L}\rangle\langle\vec{L}|$$

Quantum evolution:

$$\exp\left(\frac{i}{\hbar}TH\right)|\vec{L}'\rangle=\text{(phase)}\,|R_z(T\omega)\vec{L}'\rangle$$

matching classical E.O.M.

Transition amplitude, recast as in spin-foams [Rovelli 1998] :

$$W(\vec{L}_f;\vec{L}_i;T) := \langle \vec{L}_f|\exp\left(\frac{i}{\hbar}T\hat{H}\right)|\vec{L}_i\rangle = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N \langle \vec{L}_f|\hat{H}^N|\vec{L}_i\rangle$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N \left(\prod_{n=1}^{N-1}\int d\mu(\overset{N}{\vec{L}_n})\right)\prod_{n=0}^{N-1}\langle \overset{N}{\vec{L}}_{n+1}|\hat{H}|\overset{N}{\vec{L}_n}\rangle$$

	Analogous to
$A(\vec{L}';\vec{L}) := \langle \vec{L}'|\hat{H}|\vec{L}\rangle$	vertex amplitude
N	Number of vertices

Truncation of amplitude to max number M of vertices:

$$W_M(\vec{L}_f;\vec{L}_i;T) := \sum_{N=0}^{M}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N \left(\prod_{n=1}^{N-1}\int d\mu(\overset{N}{\vec{L}_n})\right)\prod_{n=0}^{N-1} A(\overset{N}{\vec{L}}_{n+1};\overset{N}{\vec{L}_n})$$

$$= \sum_{N=0}^{M}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N \langle \vec{L}_f|H^N|\vec{L}_i\rangle$$

Classical limit: $|\,\lambda\vec{L}_o\rangle, \quad \lambda\to\infty, \quad$ so that $\dfrac{\Delta}{\langle\vec{L}^2\rangle} = \dfrac{\hbar^2\lambda\ell_o}{\hbar^2\lambda\ell_o(\lambda\ell_o+1)} = \dfrac{1}{\lambda\ell_o+1}\to 0.$

Note equation of motion for $\vec{L}$, $\vec{L}(T) = R_z(T\omega)\vec{L}(0)$, is independent of λ,
reducing to $\vec{L}_o(T) = R_z(T\omega)\vec{L}_o(0)$.

Classical limit of *truncated* amplitude:

$$W_M(\lambda\vec{L}_f;\lambda\vec{L}_i;T) = \sum_{N=0}^{M}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N\langle\lambda\vec{L}_f|\hat{H}^N|\lambda\vec{L}_i\rangle$$
$$= \sum_{N=0}^{M}\frac{1}{N!}\left(\frac{iT}{\hbar}\right)^N\langle\lambda\vec{L}_f|\left(\frac{\hat{\vec{L}}^2}{2mR^2}+\frac{qB}{2m}\hat{L}_z\right)^N|\lambda\vec{L}_i\rangle$$

Using $\rho_\ell = \bigotimes_{\text{symm}}^{2\ell}\rho_{1/2}$, its not hard to show that

> **Theorem.** *If $\hat{M}$ is any operator on $\mathcal{H}$ polynomial in $\hat{\vec{L}}$, then $\langle\lambda\vec{L}'_o|\hat{M}|\lambda\vec{L}_o\rangle$ is zero or exponentially suppressed as $\vec{L}'_o = \vec{L}_o$.*

Thus, in the classical limit, the **truncated amplitude** is zero or exponentially suppressed unless

$$\boxed{\vec{L}_f = \vec{L}_i}$$

which is **inconsistent** with the **classical** as well as the **exact quantum equations of motion**. It is an accidental constraint arising solely from taking the classical limit before removing the cut-off on the number of vertices.

The difference in the case of spin-foams.

Unlike in the toy examples, in spin-foams **we don't yet know how to define the exact theory without a cut-off to a fixed 2-complex/triangulation/maximum number of vertices**.

It is tempting to be inspired by canonical LQG: There one takes the projective limit of Hilbert spaces on fixed graphs, and operators thereon, to obtain a framework **that includes all graphs simultaneously.**

However, **the calculation of the spin-foam amplitude is fundamentally different**

- In canonical LQG, to calculate

$$\left\langle \Psi_{\gamma'}, \hat{H}(N)\Psi_{\gamma} \right\rangle,$$

it is sufficient to do the calculation on a single graph $\gamma' \cup \hat{H}(\gamma)$.

- But **in Spin-foams**, one calculates the transition amplitude,

$$A\left(\Psi_{\gamma'}, \Psi_{\gamma}\right) = \int \left\langle \Psi_{\gamma'}, e^{i\hat{H}(N)}\Psi_{\gamma} \right\rangle \mathcal{D}N$$

which has **non-zero contributions from an *infinite* number of 2-complexes** connecting γ to γ'.

Proposed solutions

More precisely, the general flatness results find that the spin-foam sum is dominated by configurations satisfying

$$\gamma \Theta_f < j_f^{-1/2} \mod 4\pi \quad \text{for all } f \text{ in the 2-complex} \tag{1}$$

[Han (2013), Dittrich, Asante, Haggard (2021)].

1.) Keep 2-complex/max number of vertices fixed, and don't take the large spin limit of the boundary state.

Instead, **choose boundary spins** so that there exists **at least one compatible bulk geometry**, on some 2-complex included in the sum, **satisfying** (1) [Dittrich, Asante, Haggard (2021); Han, Huang, Liu, Qu (2021)].

2.) Take refinement limit / limit of unbounded number of vertices together with the large spin limit of the boundary state,

such that, **at each stage in the limit,** there exists **at least one bulk geometry** on some 2-complex included in the sum, **satisfying** (1) [Han (2017)].

Outlook

In the immediate future,

while calculations are restricted to relatively simple triangulations, the message is: **Higher spins are not necessarily more semi-classical, and may give unphysical results if taken too high**. Spins should be limited by the inequality (1).

For sufficiently refined triangulations,

in addition to restricting to inequality (1), **the problems of bubble divergences need to be addressed**. Apart from being divergences, they come from critical points in the amplitude unrelated to solutions to Einstein´s equations, and so are arguably unphysical. **The problem of degenerate sectors will also need to be addressed.** The proper vertex – restricting all 4-simplices to a fixed `orientation´ – is **one way to rid of the degenerate sectors** [Engle, Vilensky, Zipfel (2013-2016)] and **might rid of bubble divergences** [Christodoulou, Långvik, Riello, Röken, Rovelli (2012)].

The ideal,

actually taking the refinement limit, or sum over all 2-complexes, is **probably not possible**, though is fun to think about on occasion. Could work on spin-form renormalization help?

Thank you for your attention!

and happy in-person Loops!