Unitarity and clock dependence in quantum cosmology

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What is the problem of time (POT)?

Diffeomorphism invariance $\implies$ GR is a constrained system:

$$\mathcal{S}_{EH} = \int \mathrm{d}t\mathrm{d}^3x \left[p^{ab}\dot{h}_{ab} - N\mathcal{H}^g_\perp - N^a\mathcal{H}^g_a\right]$$

N and N^a (lapse and shift functions) are Lagrange multipliers. Hence,

$$\mathcal{H}^g_\perp = 0, \quad \mathcal{H}^g_a = 0$$

When we quantise we find

$$\hat{\mathcal{H}}^g_\perp \Psi = 0, \quad \hat{\mathcal{H}}^g_a \Psi = 0$$

There is no external time variable. How do we make observables evolve?

The model we studied

Ingredients

- flat FLRW: $\mathrm{d}s^2 = -N(\tau)^2\mathrm{d}\tau^2 + a(\tau)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$
- a cosmological constant Λ from unimodular gravity
- free massless scalar field ϕ

$$\mathcal{S}_{PUM} = V_0 \int_{\mathbb{R}} \mathrm{d}\tau \left\{ \frac{3\dot{a}^2 a}{N\kappa} - Na^3\frac{\Lambda}{\kappa} + \Lambda\dot{T} + \frac{a^3}{2N}\dot{\phi}^2 \right\}$$

Change of variables:

$$v \propto a^3,\ \lambda \propto \Lambda,\ t \propto T, \varphi \propto \phi,$$

where $\{t, \lambda\} = 1$. The cosmological constant is a constant of motion.

$$\mathcal{H} = \tilde{N}\left[-\pi_v^2 + \frac{\pi_\varphi^2}{v^2} + \lambda\right] \overset{\text{constraint}}{\Longrightarrow} \mathcal{C} = -\pi_v^2 + \frac{\pi_\varphi^2}{v^2} + \lambda = 0$$

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Wheeler–DeWitt equation

$$\hat{\mathcal{C}}\Psi = 0 \implies \left(\hbar^2 \frac{\partial^2}{\partial v^2} + \frac{\hbar^2}{v}\frac{\partial}{\partial v} - \frac{\hbar^2}{v^2}\frac{\partial^2}{\partial \varphi^2} - i\hbar\frac{\partial}{\partial t}\right)\Psi(v,\varphi,t) = 0 \tag{WdW}$$

POT: how to extract time evolution from this equation?

Answer: A $\rightarrow$ choose an internal variable to play the rôle of time parameter (relational quantisation), and B $\rightarrow$ Dirac quantisation (ask me about that in the breaks :D)

Answer A: choose a clock

t and φ are good clocks classically, hence both can a priori be good clocks for the quantum theory

The t-clock theory

$$i\hbar\partial_t\Psi = -\underbrace{\left(\hbar^2\partial_v^2 + \frac{\hbar^2}{v}\partial_v - \frac{\hbar^2}{v^2}\partial_\varphi^2\right)}_{\hat{\mathcal{H}}}\Psi$$

Schr. eq. $\implies$ Schr. inner product

$$\langle\Psi|\Phi\rangle_t = \int \mathrm{d}\varphi\mathrm{d}v\ v\bar{\Psi}\Phi$$

The φ-clock theory

Multiply (WdW) by v^2

$$\hbar^2\partial_\varphi^2\Psi = \underbrace{\left(\hbar^2(v\partial_v)^2 - iv^2\hbar\partial_t\right)}_{\hat{\mathcal{G}}}\Psi$$

K.G. eq. $\implies$ K.G. inner product

$$\langle\Psi|\Phi\rangle_\varphi = i\int \mathrm{d}t\frac{\mathrm{d}v}{v}\left(\bar{\Psi}\partial_\varphi\Phi - \Phi\partial_\varphi\bar{\Psi}\right)$$

choose a clock (continued)

$\hat{\mathcal{H}}$ and $\hat{\mathcal{G}}$ are not self-adjoint $\implies$ boundary conditions have to be imposed to preserve unitarity

t-clock theory	φ-clock theory
$\int \mathrm{d}\varphi \left[v\bar{\Psi}\partial_v\Phi - v\Phi\partial_v\bar{\Psi}\right]_{v=0} = 0$	$\int \mathrm{d}t \left[v\bar{\Psi}\partial_v\Phi - v\Phi\partial_v\bar{\Psi}\right]^{v=\infty} = 0$

Remark:

- In the t-theory, the boundary condition is at $v = 0$ (classical singuarity)
- In the φ-theory, the boundary condition is at $v = \infty$

Singularity resolution

expectation values $= \langle \Psi_{sc} | v(t) | \Psi_{sc} \rangle = \langle v(t) \rangle_{\Psi_{sc}}$ and $\langle \Psi_{sc} | v(\varphi) | \Psi_{sc} \rangle = \langle v(\varphi) \rangle_{\Psi_{sc}}$

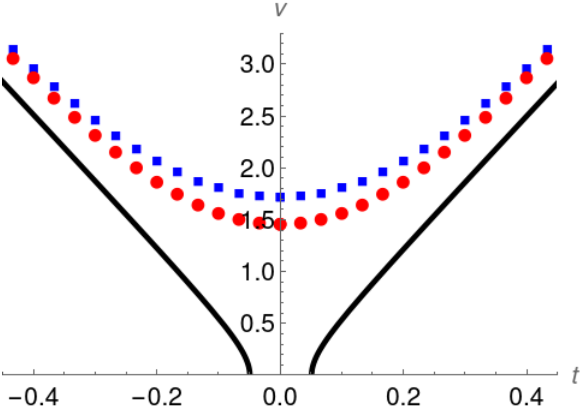

Figure: $\langle v(t) \rangle_{\Psi_{sc}}$ for different semiclassical states

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Quantum recollapse

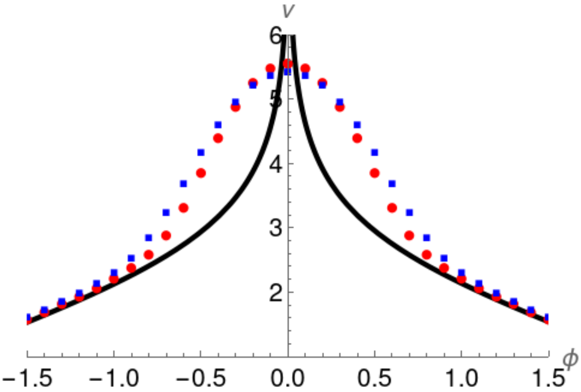

Figure: $\langle v(\varphi) \rangle_{\Psi_{sc}}$ for different semiclassical states

A few things to take from this talk

- The problem of time has many nuances
- Quantum cosmology is a good testing ground
- Unitarity requirements are what lead to different theories and,
- They are present in different quantisation schemes
- Singularity resolution is not a feature of the theory, it is a feature of the clock $\Longrightarrow$ clock choices should be given more attention when studying these models
- Path integral quantisation?

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Thank you!

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Classical solutions

For $\lambda > 0$ the classical solutions are

$$v(t) = \sqrt{-\frac{\pi_\varphi^2}{\lambda} + 4\lambda(t - t_0)^2}, \qquad \varphi(t) = \frac{1}{2}\log\left|\frac{\pi_\varphi - 2\lambda(t - t_0)}{\pi_\varphi + 2\lambda(t - t_0)}\right| + \varphi_0$$

$$v(\varphi) = \frac{|\pi_\varphi|}{\sqrt{\lambda}|\sinh(\varphi - \varphi_0)|}, \qquad t(\varphi) = -\frac{\pi_\varphi}{2\lambda}\coth(\varphi - \varphi_0) + t_0$$

$$t(v) = t_0 - \mathrm{sgn}(\pi_v)\frac{1}{2}\sqrt{\frac{v^2}{\lambda} + \frac{\pi_\varphi^2}{\lambda^2}}, \quad \varphi(v) = \varphi_0 + \log\left|\frac{\pi}{\sqrt{\lambda}v} + \sqrt{\frac{\pi_\varphi^2}{\lambda v^2} + 1}\right|$$

The big bang/big crunch singularity is at $t_{\text{sing}} = \frac{|\pi_\varphi|}{2\lambda}$, $\log\frac{v_{\text{sing}}}{v_0} = -\infty$, $\varphi_{\text{sing}} = \pm\infty$
Spatial infinity is at $t_\infty = \pm\infty$, $\log\frac{v_\infty}{v_0} = \infty$, $\varphi_\infty = \varphi_0$

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Answer B: Dirac quantisation

Freedom to multiply the WdW by a phase space function

$$\hat{\mathcal{C}}_1\Psi = 0, \implies \left(\hbar^2\frac{\partial^2}{\partial v^2} + \frac{\hbar^2}{v}\frac{\partial}{\partial v} - \frac{\hbar^2}{v^2}\frac{\partial^2}{\partial\varphi^2} - i\hbar\frac{\partial}{\partial t}\right)\Psi = 0 \quad \text{(WdW1)}$$

$$\hat{\mathcal{C}}_2\Psi = 0, \implies \left(\hbar^2\left(v\frac{\partial}{\partial v}\right)^2 - \hbar^2\frac{\partial^2}{\partial\varphi^2} - i\hbar v^2\frac{\partial}{\partial t}\right)\Psi = 0 \quad \text{(WdW2)}$$

where $\hat{\mathcal{C}}_2 = v^2\hat{\mathcal{C}}_1$. This motivates different kinematical inner products

$$\langle\Psi|\Phi\rangle_{\text{kin}_1} = \int \mathrm{d}t\mathrm{d}\varphi\mathrm{d}v\ v\bar{\Psi}\Phi, \quad \langle\Psi|\Phi\rangle_{\text{kin}_2} = \int \mathrm{d}t\mathrm{d}\varphi\frac{\mathrm{d}v}{v}\ \bar{\Psi}\Phi$$

Demanding self-adjointness of $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ reduces to demanding self-adjointness of of $\hat{\mathcal{H}}$ and $\hat{\mathcal{G}}$

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Dirac quantisation extra

Demanding $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ to be self-adjoint with respect to their respective inner products corresponds to self-adjointness of the one dimensional operators

$$\hat{\mathcal{D}}_1 = \hbar^2 \left(-\frac{\partial^2}{\partial v^2} - \frac{k^2 + \frac{1}{4}}{v^2} \right), \quad \hat{\mathcal{D}}_2 = -\hbar^2 \left(v \frac{\partial}{\partial v} \right)^2 - \lambda v^2$$

Recall,

$$\hat{\mathcal{H}} = \left(\hbar^2 \frac{\partial^2}{\partial v^2} + \frac{\hbar^2}{v} \frac{\partial}{\partial v} - \frac{\hbar^2}{v^2} \frac{\partial^2}{\partial \varphi^2} \right), \quad \hat{\mathcal{G}} = \left(\hbar^2 \left(v \frac{\partial}{\partial v} \right)^2 - i v^2 \hbar \frac{\partial}{\partial t} \right)$$

Despite being two dimensional, study $\hat{\mathcal{H}}$ and $\hat{\mathcal{G}}$ reduces to analysing $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$

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Allowed wave functions in the different theories

$$\Psi_t \sim \int \frac{\mathrm{d}\lambda}{2\pi\hbar} \frac{\mathrm{d}k}{2\pi} \; e^{ik\varphi} e^{i\lambda \frac{t}{\hbar}} \alpha(k,\lambda)\, \mathrm{Re}\left[e^{i\vartheta(k) - i|k| \log \sqrt{\frac{\lambda}{\lambda_0}}} J_{i|k|}\left(\frac{\sqrt{\lambda}}{\hbar} v \right) \right]$$

$$\Psi_\varphi \sim \int \frac{\mathrm{d}\lambda}{2\pi\hbar} \frac{\mathrm{d}k}{2\pi} e^{ik\varphi} e^{i\lambda \frac{t}{\hbar}} \alpha(k,\lambda)\, \mathrm{Re}\left[\sqrt{\frac{\sinh\left((|k| - i\kappa_0(\lambda)) \frac{\pi}{2} \right)}{\sinh\left((|k| + i\kappa_0(\lambda)) \frac{\pi}{2} \right)}} J_{i|k|}\left(\frac{\sqrt{\lambda}}{\hbar} v \right) \right]$$