The Physical Relevance of the Fiducial Cell in Loop Quantum Cosmology

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based on

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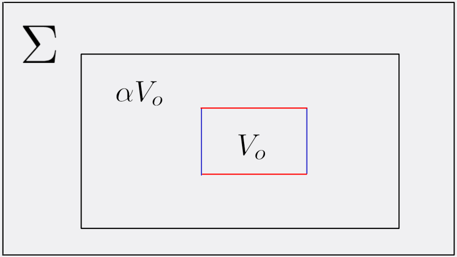
Fiducial Cell in Cosmology



Regularise divergencies

1. Restrict to $V_o \subset \Sigma$
2. Impose periodic boundary conditions ($V_o \simeq_{top} \mathbb{T}^3$)
3. What depends on V_o?

Observation: V_o is the scale on which homogeneity is effectively imposed

Can we send $V_o \to \infty$? Is V_o just a regulator? Physical consequences?

What is the role of the fiducial cell for the quantum theory and fluctuations?

Variables and Observables

Full theory counterparts [Bodendorfer '16]

$$v(x) = \sqrt{q}(x) \quad , \quad b(x) = -\frac{2q_{ab}P^{ab}(x)}{3\sqrt{q}}$$

Observables for $V, V' \subset V_o$

$$\text{vol}(V) = \int_V \mathrm{d}^3x\,\sqrt{q} = V \cdot v \quad , \quad b(V) = \frac{1}{\text{vol}(V)} \int_V \mathrm{d}^3x\,\sqrt{q}\,b(x) = b$$

Poisson bracket: $\{b(V')\,,\,\text{vol}(V)\} = \frac{V}{V_o}$

Key observations

1. $\{\cdot\ ,\ \cdot\} \propto \frac{1}{V_o}$, is a property of the Poisson bracket itself
2. $V_o \mapsto \alpha V_o$ is a non-canonical trafo
   $\rightarrow V_o$ labels a family of canonically inequivalent theories!
3. However: **The classical theory is on-shell independent of V_o!**

Quantisation of the V_o-family of theories

Weyl Canoical Commutation Relations

$$\widehat{e^{-i\xi v}}\,\widehat{e^{-i\mu b}} = \widehat{e^{-i\mu b}}\,\widehat{e^{-i\xi v}}\, e^{-\frac{i\mu\xi}{V_o}}$$

$\Rightarrow$ operator representations have to contain V_o!

$$\hat{v}\,|\nu\rangle = \frac{\eta^\gamma}{V_o^\gamma}\nu\,|\nu\rangle \quad , \quad \widehat{e^{-i\lambda\mu b}}\,|\nu\rangle = |\nu - \frac{\lambda\mu}{\eta^\gamma V_o^\delta}\rangle$$

$\eta = \kappa^{3/2}$, $\gamma + \delta = 1$: freedom in putting V_o into v or b

Transformation behaviour ((i) labels qu. of theory with $V_o^{(i)}$)

$$\hat{H}_{true}\Big|_{V_o^{(i)}}\psi_E^{(i)} = E\psi_E^{(i)} \quad , \quad \psi_E^{(1)}(\nu) = \psi_E^{(2)}\left(\left(\frac{V_o^{(1)}}{V_o^{(2)}}\right)^\delta \nu\right)$$

Quantum dynamics is V_o-independent after this identification!

$$\left\langle \hat{v}\big|^n_{V_o^{(1)}} \right\rangle_{\psi^{(1)}} = \left(\frac{V_o^{(2)}}{V_o^{(1)}}\right)^n \left\langle \hat{v}\big|^n_{V_o^{(2)}} \right\rangle_{\psi^{(2)}}$$

$$\left\langle \widehat{e^{-i\lambda\mu b}}\Big|^n_{V_o^{(1)}} \right\rangle_{\psi^{(1)}} = \left\langle \widehat{e^{-i\lambda\mu b}}\Big|^n_{V_o^{(2)}} \right\rangle_{\psi^{(2)}}$$

Fiducial Cell and Uncertainty Relation

Recall $\widehat{\text{vol}(V_o)} = V_o\hat{v}$ [Ashtekar, Bojowald, Lewandowski '03]

$$\left\langle \left(V_o^{(1)}\hat{v}\Big|_{V_o^{(1)}} \right)^n \right\rangle_{\psi^{(1)}} = \left\langle \left(V_o^{(2)}\hat{v}\Big|_{V_o^{(2)}} \right)^n \right\rangle_{\psi^{(2)}}$$

- Assigned volume to both cells is the same $\rightarrow$ diff. invariance
- Rescaling V_o doesn't affect the physical size of the fiducial cell

Remove the regulator: Assign larger volume to V_o

Find family of states s.t. $\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \rightarrow \infty$

A reasonable observable is $V \subset V_o$

$$\left\langle \widehat{\text{vol}(V)} \right\rangle_\psi = \frac{V}{V_o} \left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \xrightarrow{\frac{V}{V_o}\rightarrow 0\,,\ \left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \rightarrow \infty} \text{finite}$$

e.g. size of observable universe today

No fluctuations left (at any point in ϕ-evolution!):

$$\Delta_\psi \widehat{\text{vol}(V)} \Delta_\psi \frac{\widehat{\sin(\lambda b)}}{\lambda} \geq \frac{V}{2\,V_o} \left| \widehat{\cos(\lambda b)} \right| \rightarrow 0$$

Fluctuations know about the size of homogeneity!

Conclusions

- Classical and quantum dynamics can be made independent of V_o
- Quantum fluctuations depend on the physical size $\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi$
- Full homogeneity is obtained by choosing ψ s.t. $\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \to \infty$
- Quantum fluctuation of any finite volume become arbitrarily small

Future Directions

- **What is the physical scale of homogeneity?**
- Better understanding of symmetry reduction from a full theory perspective [Mele, Münch to be published]
- What exact approximations were done? What is their validity? What role plays renormalisation? [Bodendorfer, Han, Haneder 21; Bodendorfer, Wuhrer 20; Bodendorfer, Haneder 19]

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Thank you for your attention!

Setup

Homogeneous/isotropic cosmology + massless scalar field

Metric:

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + q_{ab}\mathrm{d}x^a\mathrm{d}x^b = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2$$

Action:

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_M = \int \mathrm{d}t\, \mathcal{L} + \text{boundary terms}$$

$$\mathcal{S}_{EH} = \frac{1}{2\kappa}\int_M \mathrm{d}^4x\,\sqrt{g}R \quad , \quad \mathcal{S}_M = -\frac{1}{2}\int_M \mathrm{d}x^4\,\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$\mathcal{L} = \int_\Sigma \mathrm{d}^3x\,\mathscr{L} = \int_\Sigma \mathrm{d}^3x\left(-\frac{3}{\kappa}\frac{a\,\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N}\right)$$

Assuming $M \simeq \mathbb{R}\times\Sigma$, Σ non-compact
$\implies \int_\Sigma \mathrm{d}x^3 \to \infty$

Further Scaling Behaviour

Hamiltonian

$$H_T = V_o \cdot N\mathcal{H} \quad , \quad \mathcal{H} = -\frac{3\kappa}{4} v b^2 + \frac{p_\phi^2}{2a^3} \approx 0$$

For two different fiducial cells $V_o^{(1)}$, $V_o^{(2)}$:

$$H_T^{(1)} = V_o^{(1)} \cdot N\mathcal{H} = \frac{V_o^{(1)}}{V_o^{(2)}} H_T^{(2)}$$

Poisson structure

For $V, V' \subset V_o^{(1)}, V_o^{(2)}$
vol(V) , $b(V)$ are independent of $V_o^{(1)}$ and $V_o^{(2)}$

$$\{b(V'), \mathrm{vol}(V)\}_{V_o^{(1)}} = \frac{V}{V_o^{(1)}} \quad , \quad \{b(V'), \mathrm{vol}(V)\}_{V_o^{(2)}} = \frac{V}{V_o^{(2)}}$$

Poisson structure itself is V_o-dependent:

$$\{\,\cdot\,,\,\cdot\,\}_{V_o^{(1)}} = \frac{V_o^{(2)}}{V_o^{(1)}} \{\,\cdot\,,\,\cdot\,\}_{V_o^{(2)}}$$

V_o-dependence: Summary + Dynamics

The choice of V_o

1. transforms the Poisson structure inversely $\propto 1/V_o$
2. enters linearly the Hamiltonian

- V_o labels a family of symmetry reduced theories (no canonical transformation)
- There is a well defined map between different V_o's

Dynamics

Choose $V_o^{(1)}$ and $V_o^{(2)}$ + observable $\mathcal{O}$ defined in both

$$\dot{\mathcal{O}} = \left\{\mathcal{O}\,,\, H_T^{(1)}\right\}_{(1)} = \frac{V_o^{(2)}}{V_o^{(1)}} \left\{\mathcal{O}\,,\, \frac{V_o^{(1)}}{V_o^{(2)}} \cdot H_T^{(2)}\right\}_{(2)} = \left\{\mathcal{O}\,,\, H_T^{(2)}\right\}_{(2)}$$

$\Rightarrow$ Dynamics is independent of V_o
On-shell the regulator can be removed $V_o \to \infty$

Quantisation

Quantisation of the V_o-family of theories

Setup

- ϕ-clock de-parametrisation:

$$p_\phi(V_o) = \sqrt{\frac{3\kappa}{2}} V_o v b =: H_{true}$$

- Polymer quantisation in b
- Hilbert spaces (different for each V_o)

$$\mathscr{H}_{LQC} = L^2\left(\mathbb{R}_{Bohr}, \mathrm{d}\mu_{\mathsf{Bohr}}\right) \quad , \quad |\psi\rangle = \sum_{\nu\in\mathbb{R}} \psi(\nu)\,|\nu\rangle$$

$$\langle \nu \mid \nu' \rangle = \delta_{\nu,\nu'} \quad , \quad \langle b \mid \nu \rangle = e^{i\lambda b \nu} \quad , \quad \psi(\nu) = \langle \nu \mid \psi \rangle \ ,$$

$|\nu\rangle$: Eigenstate of $\hat{v}$

$|b\rangle$: Eigenstate of $\widehat{e^{i\nu\lambda b}}$

Quantum Dynamics

$$i\frac{\partial}{\partial\phi}|\psi\rangle = \hat{H}_{true}|\psi\rangle \qquad , \qquad |\psi;\phi\rangle = e^{i\phi\hat{H}_{true}}|\psi\rangle$$

Hamiltonian

Regularisation of the Hamiltonian [Martín-Benito, Maruguán, Olmedo '09]:

$$\hat{H}_{true}\psi(\nu) = \frac{i}{4}\sqrt{\frac{3\kappa}{2}}\cdot\Big(s_+(n)\sqrt{|n||n+1|}\psi\left(\theta\cdot(n+1)\right) - s_-(n)\sqrt{|n||n-1|}\psi\left(\theta\cdot(n-1)\right)\Big)$$

$$\nu = \theta n \quad , \quad n\in\mathbb{R} \quad , \quad \theta = \frac{\lambda}{\eta^\gamma V_o^\delta} \quad , \quad s_\pm(n) = \mathsf{sign}(n\pm 1) + \mathsf{sign}(n)$$

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Eigenstates depend only on $n = \nu/\theta$: Look for $\Psi_E : \mathbb{R} \to \mathbb{C}$ with

$$-\frac{i}{2}\sqrt{\frac{3\kappa}{2}}\cdot\Big(s_+(n)\sqrt{|n||n+1|}\Psi_E(n+1) - s_-(n)\sqrt{|n||n-1|}\Psi_E(n-1)\Big) = E\Psi_E(n)\ .$$

It follows: $\psi_E(\nu) = \Psi_E\left(\frac{\nu}{\theta}\right)$