Prescriptions and analytic control over quantum dynamics in loop quantum cosmology

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Outline

Classical theory

Phase Space

We are going to consider simplest setting: a model of a flat FLRW universe admitting the massless scalar field. Ashtekar-Barbero variables takes form:

$$A_i^a = cV_0^{-\frac{1}{3}}\delta_i^a \qquad E_a^i = pV_0^{-\frac{2}{3}}\delta_a^i$$

with Poisson Bracket:

$$\{c, p\} = \frac{8\pi}{3}G\gamma$$

Constraint

Due to symmetries and partial gauge fixing the Gauss and Diffeonorphism one are already satisfied. The only remaining is Hamiltonian one:

$$C = C_{gr}^{(E)} - 2(1+\gamma^2)C_{gr}^{(L)} + 8\pi G p^{-\frac{3}{2}} p_\phi^2$$

$$C_{gr}^{(E)} = \int_{\mathcal{V}} d^3x \epsilon_{ijk} \frac{1}{\sqrt{det(E)}} E^{ai}E^{bj}F_{ab}^k \qquad C_{gr}^{(L)} = \int_{\mathcal{V}} d^3x \frac{1}{\sqrt{det(E)}} E^{ai}E^{bj}K_{[a}^j K_{b]}^i$$

Quantization

On the kinematical level

Kinematical Hilbert space is a tensor product of $\mathcal{H}^{\phi} = L^2(\phi, d\phi)$ (the basic operators $\hat{\phi}$ and $\hat{p}_{\phi}$) and $\mathcal{H}^{gr} = L^2(\bar{\mathbb{R}}, d\mu)$ with elementary operators:

$$\hat{V}\,|v\rangle = \alpha|v|\,|v\rangle \qquad \hat{N}\,|v\rangle = |v+1\rangle$$

$$\alpha = 2\pi G\hbar\gamma\sqrt{\Delta} \qquad \hat{N} := e^{i\frac{\bar{\mu}c}{2}}$$

Of the constraint

Due to ambiguities of the regularization procedure, quantization of the gravity part of the constraint is not unique. So far three distinct examples of such have been proposed in the literature.

LQC and Strict Thiemann regularization

Constraint in LQC

- on classical level:

$$E^{ai}E^{bj}K^j_{[a}K^i_{b]} = \frac{1}{2\gamma^2}\epsilon_{ijk}E^{ai}E^{bj}(F^k_{ab}-\Omega^k_{ab})$$

  for flat model $\Omega^k_{ab}=0$, then the Lorentzian part can be subsumed into the Euclidean part.
- $\hat{\Theta}=\sqrt{\hat{V}}\hat{C}_{gr}\sqrt{\hat{V}}$ acts on the state in volume representation as a second order difference operator and
- is a self-adjoint operator

Constraint in Strict Thiemann regularization

- by using Thiemann's algorithm K^i_a can be obtained by a Poisson bracket which in cosmological settings (in improved dynamics scheme) reduce to:

$$K^i_a = -\frac{2}{3\kappa\gamma^3\bar{\mu}}h_i^{(\bar{\mu})}\{(h_i^{(\bar{\mu})})^{-1},\{C^E,V\}\}$$

- $\hat{\Theta}$ acts on the state in volume representation as a fourth order difference operator and
- admits multiple self-adjoint extensions

Regularization with $K_a^i = \frac{1}{\gamma} A_a^i$

In volume representation $\hat{\Theta}$ act on the state as as 4th order difference operator. If we consider its eigenvalue problem (corresponding to an eigenvalue of matter part operator ω), this equation can be rewritten as recurrence relation with unstable solutions.

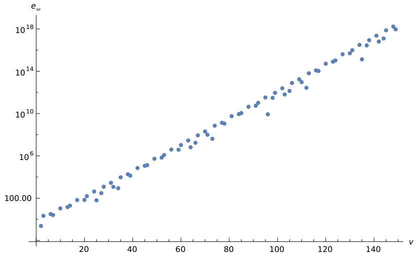

Figure: Values of the eigenfunction e_ω to Θ -in logarytmic scale, for $\omega = 10$

Regularization with $K_a^i = \frac{1}{\gamma} A_a^i$

But by defining a type of a Fourier transformation of state in volume base to momentum b base and by introducing a coordinate $x(b)$: scalar product in takes form:

$$\langle\psi_1|\psi_2\rangle = -\frac{2i}{\pi}\int_{-\infty}^{\infty} dx\bar{\psi}_1(x)\partial_x\psi_2(x)$$

operator Θ will simplified to:

$$\hat{\Theta} = -12\pi G\hbar^2\partial_x^2$$

The operator $\hat{V}$

$$\hat{V} = -i\frac{\alpha}{\sqrt{\gamma^2+1}}(\gamma^2\tanh^2(x)+1)\cosh(x)\partial_x$$

General prescription to probe dynamics

With observation that equation of the constraint can be formulated as:

$$\partial_\phi^2 \Psi(x,\phi) = \beta^2 \partial_x^2 \Psi(x,\phi)$$

Solutions, are find by using standard quantum mechanical methods to be:

$$\Psi(x,\phi) = \int_0^\infty dk \frac{\psi(k)}{\sqrt{k}} e^{ikx} e^{i\beta k\phi}$$

and $\hat{V}$ could be rewritten:

$$\hat{V} = -iAf(x)\partial_x$$

we obtained general formula to probe Loop Quantum Cosmology system where constant A and β along side with function f is define via model we considering.

In th energy k representation expectation value become:

$$\langle \hat{V} \rangle = A \sum_{l=0}^{\infty} \frac{1}{l!} f^{(l)}(\beta\phi) \langle \sqrt{k} (i\partial_k)^l \sqrt{k} \rangle$$

Expectation value of the product of operators

The expectation value of the product of operators, in general, is not equal to products of expectation values, to describe the system completely after quantization we need moments defined as:

$$G^{a,b} := \langle (\hat{k} - \langle k \rangle)^a (i\hat{\partial}_k - \langle i\partial_k \rangle)^b \rangle_{Weyl}$$
$$= \sum_{n=0}^{a} \sum_{n=m}^{b} (-1)^{a+b-n-m} \binom{a}{n} \binom{b}{m} \langle k \rangle^{a-n} \langle i\partial_k \rangle^{b-m} \langle \hat{k}^n i\hat{\partial}_k{}^m \rangle_{Weyl}$$

After inverting this relation:

$$\langle \hat{k}^n i\hat{\partial}_k{}^m \rangle_{Weyl} = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \langle k \rangle^{n-i} \langle i\partial_k \rangle^{m-j} G^{i,j}$$

Remark

After quantization of product of classical variables ordering of corresponding operators is not unique. Because of that, relation of general arbitrary ordering x and Weyl ordering is required.

Procedure

Rewriting product of the operator in expectation value of Volume as a Weyl ordering

$$\sqrt{k}(i\partial_k)^l\sqrt{k} =: k(i\partial_k)^l : +\frac{1}{2}[\sqrt{k},[(i\partial_k)^l,\sqrt{k}]]$$

Double commutator

Because in general we are interested in large energy of the universe, expectation value of k should be in non negative power to make reasonable contribution and what we obtain from double commutator is at best proportional to k^{-1}. Thus for $\langle\hat{V}\rangle$ this contribution is negligible. But for the standard deviation it is not. And what we considerate was only its leading term:

$$\frac{1}{2}\langle[\sqrt{k},[(i\partial_k)^l,\sqrt{k}]]\rangle = \frac{l(l-1)}{16}\frac{1}{k_0}\sum_{n=0}^{\infty}\sum_{a=0}^{n}\frac{(-1)^{n+a}}{k_0^{n-a}}\binom{n}{a}\langle k^{n-a}(i\partial_k)^{l-2}\rangle_s$$

Expectation Value expressed in moments

In the next step to obtain expectation value in desired form we use inverse relation for central moments and change order of summation.

Result

Expectation value of volume operator in terms of central moments

$$\langle \hat{V} \rangle = A \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(\beta\phi - \langle i\partial_k \rangle)\Big(\langle k \rangle \, G^{0,i} + G^{1,i} \Big) + o(k^{-a-b})$$

Second power of standard deviation:

$$\begin{aligned}
\sigma^2(V) =& A^2 \sum_{i=0}^{\infty} \Bigg(\langle k \rangle^2 \Big(\sum_{a=0}^{i} \frac{1}{(i-a)!a!} f^{(i-a)} f^{(a)} G^{0,i} - \sum_{i'=0}^{\infty} \frac{1}{i!} \frac{1}{i'!} f^{(i)} f^{(i')} G^{0,i} G^{0,i'} \Big) \\
&+ 2 \langle k \rangle \Big(\sum_{a=0}^{i} \frac{1}{(i-a)!a!} f^{(i-a)} f^{(a)} G^{1,i} - \sum_{i'=0}^{\infty} \frac{1}{i!} \frac{1}{i'!} f^{(i)} f^{(i')} G^{0,i} G^{1,i'} \Big) \\
&+ \Big(\sum_{a=0}^{i} \frac{1}{(i-a)!a!} f^{(i-a)} f^{(a)} G^{2,i} - \sum_{i'=0}^{\infty} \frac{1}{i!} \frac{1}{i'!} f^{(i)} f^{(i')} G^{1,i} G^{1,i'} \Big) \\
&+ \Big(\frac{1}{8} \sum_{a=0}^{i+2} \frac{(i+2)(i+1)}{(i+2-a)!a!} f^{(i+2-a)} f^{(a)} G^{0,i} - \frac{1}{4} \sum_{i'=0}^{\infty} \frac{1}{i!} \frac{1}{i'!} f^{(i)} f^{(i'+2)} G^{0,i} G^{0,i'} \Big) \Bigg)
\end{aligned}$$

Application

Extracting Physics from constant A

To this point we boldly refer to $\langle k \rangle$ as energy of the universe completely forgetting about its physical dimension.

$$p_g = \hbar \beta k \qquad \Pi_g = \frac{i}{\beta}\partial_k \qquad \tilde{G}^{a,b} = \frac{1}{\hbar^a}\beta^{b-a}G^{a,b}$$

Application

To check validity of this results in the lowest order of central moments we should obtain classical trajectories which should be consistent with predictions of effective dynamics. From what we checked for mLQC-II:

$$\langle \hat{V} \rangle = \frac{\gamma\sqrt{\pi G \Delta}}{\sqrt{3(\gamma^2+1)}} p_g \left(\gamma^2 \tanh^2(\beta\phi + \beta\Pi_g) + 1\right) \cosh(\beta\phi + \beta\Pi_g)$$

$$+ \frac{\gamma\sqrt{\pi G \Delta}}{\sqrt{3(\gamma^2+1)}} \sum_{i=1}^{\infty} \frac{\beta^{i-1}}{i!} \partial^i_{\beta\phi}\left(\left(\gamma^2 \tanh^2(\beta\phi + \beta\Pi_g) + 1\right) \cosh(\beta\phi + \beta\Pi_g)\right)\left(p_g \tilde{G}^{0,i} + \tilde{G}^{1,i}\right)$$

first line recreates effective dynamics and second line is higher order quantum corrections.

Summary

In our work:

- we investigate properties of operator Θ in so called mLQC-II and probe its dynamics,
- then we generalized this prescription for the variety of Loop quantum Cosmology systems, example mainstream LQC:

$$\langle \hat{V} \rangle = \frac{\sqrt{\pi G \Delta}}{\sqrt{3}} p_g \cosh(\beta\phi + \beta\Pi_g) + \frac{\sqrt{\pi G \Delta}}{\sqrt{3}} \sum_{i=1}^{\infty} \frac{\beta^{i-1}}{i!} \partial^i_{\beta\phi} \cosh(\beta\phi + \beta\Pi_g) \left(p_g \tilde{G}^{0,i} + \tilde{G}^{1,i} \right)$$

and even Geometrodynamics:

$$\langle \hat{V} \rangle = \frac{\gamma\sqrt{\pi G \Delta}}{\sqrt{3}} p_g \exp(\beta\phi + \beta\Pi_g) + \frac{\gamma\sqrt{\pi G \Delta}}{\sqrt{3}} \sum_{i=1}^{\infty} \frac{\beta^{i-1}}{i!} \exp(\beta\phi + \beta\Pi_g) \left(p_g \tilde{G}^{0,i} + \tilde{G}^{1,i} \right)$$

Thank you for your attention