On the Reparametrization Symmetry in Physics

and Some of the Key Properties of Physical Systems [1, 2]

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Outline

- Main Points
- Pros and Cons for Reparametrization Invariance.

1. Our Results/Contribution
   - Canonical Form of the 1st-Order Homogeneous Lagrangians.
   - Classical Forces Beyond Electromagnetism and Gravity.
   - Extended Hamiltonian formulation (extended phase-space).
   - Justification of Key Quantum Mechanical phenomenon.
   - Positivity of the mass and the common arrow of time.
2. Summary and References
3. Appendix - Justifying the Reparametrization Invariance
   - Examples of Homogeneous Lagrangians of First Order.
   - Examples of Reparametrization Invariance in Physical Systems.
   - Systems with Constraints ($H \equiv 0$) and the Dirac's formalism.
   - From Lagrangian to Hamiltonian Mechanics
   - From Classical to Quantum Mechanics

Main Points

- The **Principle of Reparametrization Invariance**
  $\Rightarrow$ Lagrangians that are *Homogeneous Functions of First Order*.
  $\Rightarrow$ Forces beyond electromagnetism and gravity.
  The simplest case exhibits a pathological behavior in the limit $v \to 0$.
- The **Extended Hamiltonian formulation** naturally justifies the Superposition principle & the Schrödinger's equation.
- **Evaluate** $[\![t, \boldsymbol{H}]\!]$ to understand the the specific parametrization!
  Amplitude-modulated state in the proper-time parametrization!.
  $\Rightarrow$ State normalizability and the positivity of the rest-energy (mass).

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Pros and Cons for Homogeneous Lagrangians of First Order ($v\partial L/\partial v = L$) [1].

Cons: trouble-making properties

(1) There are constraints among the Euler-Lagrange equations since $\det\left(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}\right) = 0$.

(2) The Legendre transformation $(T(M) \leftrightarrow T^*(M))$, which exchanges velocity and momentum coordinates $(x, v) \leftrightarrow (x, p)$, is problematic!

(3) There is a problem with the canonical quantization approach since the Hamiltonian function is identically ZERO ($h \equiv 0$).

Pros: good properties!

(1) The action $S = \int L(x, \frac{dx}{d\tau})d\tau$ is reparametrization invariant.[1)]

(2) For any Lagrangian $L(t, x^i, \frac{dx^i}{dt})$ one can construct a reparametrization-invariant Lagrangian by enlarging the space to the extended space-time $(x^0 = ct)$.[2)]

(3) Parameterization-independent path-integral quantization is possible!

(4) The reparametrization invariance may help in dealing with singularities.

(5) It is easily generalized to extended objects (p-branes).

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[2] The Euler-Lagrange equations for the two Lagrangians are equivalent as long as $v^0 = dt/d\tau$ is well behaved and τ is a reasonable "time"-parametrization choice.

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Canonical Form of the First-Order Homogeneous Lagrangians ($v\partial L/\partial v = L$) [1].

Theorem (one-to-one correspondence b/w interaction field and its source)

For the canonical form of the first-order homogeneous Lagrangian:

$$L(\vec{x},\vec{v}) = \sum_{n=1}^{\infty} \sqrt[n]{S^{(n)}(\vec{v},\ldots,\vec{v})} = A_\alpha v^\alpha + \sqrt{g_{\alpha\beta}v^\alpha v^\beta} + \ldots + \sqrt[m]{S^{(m)}(\vec{v},\ldots,\vec{v})}.$$

There is a one-to-one correspondence between the interaction field $S^{(m)}$: $S^{(m)}_{\mu_1\ldots\mu_m} v^{\mu_1}\ldots v^{\mu_m}$ and its source type $\sim v^{\mu_1}\ldots v^{\mu_m}$.

Corollary

Classical Forces Beyond Electromagnetism and Gravity (m>2).

Proposition

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Pure $S^{(n)}$ systems ($n > 2$) exhibit pathological behavior in the Newtonian limit [1].

Consider the simplest pure $S_n(v)$ Lagrangian Systems:

$$S^{(n)}(r, w, u) = \psi(r)w^n + \phi(r)u^n.$$

It is for a static, curvilinear, Newton-like space-time with $w = dx^0/d\tau$, $u = dr/d\tau$, and in coordinate-time parametrization $v = dr/dt = c\, u/w$.
The corresponding equations of motion for $L = S^{(n)}(r, w, u)$ are:

$$\frac{du}{d\tau} = -\frac{u^2\phi'(r)}{(n-1)\phi(r)} + \frac{1}{u^{n-2}}\frac{w^n\psi'(r)}{n(n-1)\phi(r)}, \quad \frac{dw}{d\tau} = -\frac{wu\psi'(r)}{(n-1)\psi(r)}.$$

Dividing by w^2 and by using $u/w = v/c$, the first equation will become:

$$\frac{(n-1)}{w^2}\frac{du}{d\tau} = -\frac{v^2\phi'(r)}{c^2\phi(r)} + \frac{c^{n-2}}{v^{n-2}}\frac{\psi'(r)}{n\phi(r)}.$$

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From Classical to Relativistic Quantum Mechanics [2, 5].

Extended Hamiltonian formalism.

On the extended phase-space ($h \equiv 0$), then the Hilbert space $\mathcal{H}$ is $\hat{H} \equiv 0$.

- Natural superposition of states for the Hilbert space: $\mathcal{H} = \{\hat{\boldsymbol{H}}\psi \equiv 0\}$
- Extended Poisson bracket where $\mu = 0, 1, \ldots, n$ and $\eta_{\mu\nu}$ is the Lorentz-invariant tensor with signature $1, -1, \ldots, -1$ such that $[\![q_i, p_j]\!] = \delta_{i,j}$ and $[\![q_0, p_0]\!] = -1$[a]:

$$\{f, g\} \to [\![f, g]\!] = -\eta_{\mu\nu}\left(\frac{\partial f}{\partial q_\mu}\frac{\partial g}{\partial p_\nu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q_\nu}\right)$$

- Evolution of operators: $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \to \frac{df}{d\lambda} = [\![f, \boldsymbol{H}]\!]$.
- Then the Extended Hamiltonian $\boldsymbol{H}$ is: $\boldsymbol{H} = H(q_i, p_i) - cp_0$.
- Schrödinger's equation from $\hat{\boldsymbol{H}}\psi \equiv 0$ and the form of $\hat{p}_0 = +i\hbar\partial_{q_0}$.

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The Schrödinger's equation and the principle of superposition [2].

Superposition Principle!

When defining the Hilbert space via $\mathcal{H} = \{\boldsymbol{\hat{H}}\psi \equiv 0\}$, by promoting the extended phase-space Hamiltonian constraint ($h \equiv 0$) into an operator such that $\hat{H} \equiv 0$, then states in the Hilbert space satisfying $\boldsymbol{\hat{H}}\psi \equiv 0$ are guaranteed to obey the Superposition Principle!

Justifying the Schrödinger's equation

The Extended Hamiltonian $\boldsymbol{H}$ has the form $\boldsymbol{H} = H(q_i, p_i) - cp_0$, while the states within the Hilbert space satisfy $\boldsymbol{\hat{H}}\psi \equiv 0$.
Thus, upon canonical quantization one can identify $\hat{p}_0 = i\hbar\partial_{q_0}$ and arrives at the usual Schrödinger's equation $i\hbar\partial_t\psi = H(q_i, p_i)\psi$.

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The Meaning of λ and the Role of $H \equiv 0$ in the Extended Phase-space [2].

The simplest possible case of *only one space-time coordinate q:*

$$v\frac{\partial L(q,v)}{\partial v} = L(q,v) \Rightarrow L(q,v) = \phi(q)v.$$

Therefore, $p = \frac{\partial L(q,v)}{\partial v} = \phi(q)$ and therefore $H = pv - L \equiv 0$.
For $q = t$ the relationship $p_0 = \phi(t)$ suggests two extensions for $\boldsymbol{H}$:

$$\boldsymbol{H} = \phi(t) - p_0, \text{ and } \boldsymbol{H} = 1 - \frac{\mathrm{p}_0}{\phi(\mathrm{t})}.$$

The first one corresponds to coordinate-time parametrization and the second to the proper-time parametrization! Consider $dt/d\lambda = [\![t, \boldsymbol{H}]\!]$, which gives 1, therefore coordinate-time parametrization; while $1/\phi(t)$ corresponds to the proper-time parametrization $d\lambda = \phi(t)dt = d\tau$.

Evaluating $dt/d\lambda = [\![t, \boldsymbol{H}]\!]$ provides information about the parametrization implied by the choice of the extended Hamiltonian.

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Wave-function's form in coordinate and proper-time parametrization [2]!

Conserved energy and onset of quantum-fluctuations scale.

$p = \hat{p}_0 = \phi(t)$ should be related to the energy of the system. A process can be considered classical with conserved energy if the fowling quantity is independent of the observational time interval Δt:

$$E = E(\Delta t) \coloneqq \langle \phi \rangle_{\Delta t} = \frac{1}{\Delta t} \int_0^{\Delta t} \phi(t) dt = \frac{1}{\Delta t} \int_0^{\Delta \tau} d\tau = \frac{\Delta \tau}{\Delta t}$$

Plane-waves in Coordinate parametrization (λ = t).

$$H = \phi(t) - p_0 \to \hat{H}\psi(t) = 0 \Rightarrow i\hbar \frac{\partial \psi}{\partial t} = \phi(t)\psi$$

where the wave function solutions $\psi(t)$ are given by: $\psi(t) = \frac{1}{\mathcal{N}} \exp\left[-\frac{i}{\hbar}\left(\int_0^{t<\delta} \phi(t)dt + E\int_\delta^{t\gg\delta} dt\right)\right]$

Modulated plane-wave in the proper-time parametrization ($\lambda = \tau$).

$$H = 1 - \frac{p_0}{\phi(t)} \to \hat{H} = 1 - \frac{1}{2}\left(\frac{1}{\phi(t)}p_0 + p_0\frac{1}{\phi(t)}\right); \Rightarrow i\hbar\frac{\partial \psi}{\partial t} = \left[\phi(t) + \frac{i\hbar}{2}\left(\frac{\partial \ln \phi(t)}{\partial t}\right)\right]\psi(t)$$

The corresponding Schrödinger like equation now will have an extra term. This factor will not disappear for $\Delta t \gg \delta$ when the energy $p_0 = E$ is conserved and now the wave function will be:

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Positive Mass and the Normalizability of the Wave-Function [2]!

Inner Product and Statistically Conserved Energy.

$$\langle\Psi|\Phi\rangle_{\Delta} = \frac{1}{\Delta}\int_{t_0}^{t_0+\Delta}\Psi^*\Phi dt\,;\; E = E(\Delta t) := \langle\phi\rangle_{\Delta t} = \frac{1}{\Delta t}\int_0^{\Delta t}\phi(t)dt = \frac{1}{\Delta t}\int_0^{\Delta\tau}d\tau = \frac{\Delta\tau}{\Delta t}$$

Positivity of the norm requires positivity of the rest energy

$E = p_0 > 0$ since $\phi(t) \to p_0$.

$$\|\psi\|^2 = \langle\psi|\psi\rangle_{\Delta} = \frac{1}{\mathcal{N}^2}\frac{1}{\Delta}\int_0^{\Delta\gg\delta}\phi(t)dt \underset{\Delta\to\infty}{\longrightarrow} \frac{p_0}{\mathcal{N}^2} > 0.$$

In the rest frame this should correspond to the rest mass of the particle.

Common Arrow of Time - commonly studied processes / $g(\vec{v},\vec{v}) > 0$ & $m(0) > 0$.

$$m(0)d\tau = m(v_{space})\sqrt{g(\vec{v},\vec{v})}d\lambda \Rightarrow dt/d\tau > 0.$$

Positive Mass and the Normalizability of the Wave-Function [2]!

Inner Product and Statistically Conserved Energy.

$$\langle\Psi|\Phi\rangle_{\Delta}=\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta}\Psi^{*}\Phi dt;\; E=E(\Delta t)\coloneqq\langle\phi\rangle_{\Delta t}=\frac{1}{\Delta t}\int_{0}^{\Delta t}\phi(t)dt=\frac{1}{\Delta t}\int_{0}^{\Delta\tau}d\tau=\frac{\Delta\tau}{\Delta t}$$

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Summary

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  $\Rightarrow$ Lagrangians that are *Homogeneous Functions of First Order*.
  $\Rightarrow$ Forces beyond electromagnetism and gravity.
  The simplest case exhibits a pathological behavior in the limit $v \to 0$.
- The **Extended Hamiltonian formulation** naturally justifies the Superposition principle & the Schrödinger's equation.
- **Evaluate** $[\![t, \boldsymbol{H}]\!]$ to understand the the specific parametrization! *Amplitude-modulated state in the proper-time parametrization!*.
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Example (Relativistic Particle Lagrangian L_1)

$$S_1 = \int d\lambda L_1(x,v) = \int d\lambda \sqrt{g_{\mu\nu} v^\mu v^\nu} \to S_1 = \int d\tau.$$

Example (Quadratic Lagrangian L_2)

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Equivalent Euler-Lagrange equations!

The equations obtained from S_1 and S_2 are equivalent, and are also equivalent to the geodesic equation:

$$\frac{d}{d\tau}\vec{v} = D_{\vec{v}}\vec{v} = v^\beta \nabla_\beta \vec{v} = 0, \qquad v^\beta \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} v^\gamma \right) = 0.$$

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Example (E-dimensional Extended Objects)

Solving for $\phi : E \to M$, where $\dim E = E = p + 1$ with $p = 0, 1, 2, \ldots$ is the "trajectory" of a p-brane, submerged in space M with coordinate charts $(x : M \to \mathbb{R}^m)$. Then the action for the embedding ϕ is:

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\begin{aligned}
S[\phi] = \int_E \phi^*(\Omega) &= \int_E A_\Gamma(\vec{\phi})\omega^\Gamma dz \to \int_E L(\vec{\phi}, \omega)\, dz,\ \omega^\Gamma = \frac{\Omega^\Gamma}{dz} = Y^\Gamma, \\
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Example (1-brane/strings, $\dim E = 2, p = 1,\ \omega^\Gamma \to Y^{\alpha\beta}$)

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L\left(x^\alpha, \partial_i x^\beta\right) = \sqrt{Y^{\alpha\beta} Y_{\alpha\beta}},\ Y^{\alpha\beta} = \frac{\partial(x^\alpha, x^\beta)}{\partial(\tau, \sigma)} = \det\begin{pmatrix} \partial_\tau x^\alpha & \partial_\sigma x^\alpha \\ \partial_\tau x^\beta & \partial_\sigma x^\beta \end{pmatrix} = \partial_\tau x^\alpha \partial_\sigma x^\beta - \partial_\sigma x^\alpha \partial_\tau x^\beta.
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Systems with Constraints and the Dirac's formalism approach [3, 4].

Start with $H = vp - L$ and the primary constraints $\phi_i \approx 0$, proceed to:

- Finding all the "secondary" constraints ϕ_j, and closing the algebra of the constraints $\{\phi_i, \phi_j\} = c_{ij}^k \phi_k$.
- Extending the Hamiltonian

$$H_+ = H + u^k \phi_k + v^a V_a^k \phi_k.$$

- If $H \equiv 0$ then one can use some of the constraints as Hamiltonian.
- Extending the Poisson bracket via the second-class $(\{\tilde{\phi}_a, \tilde{\phi}_b\} \not\approx 0)$ constraints by using $M_{ab} = \{\tilde{\phi}_a, \tilde{\phi}_b\}$ to define the Dirac bracket:

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From Lagrangian to Hamiltonian Mechanics [2].

Lagrangian formulation in $T(M)$

(1) Euler-Lagrange equations from an action principle over $T(M)$ with $\{q(t) : \delta A = 0\}$ using an action $\mathcal{A} = \int L(t, q, \dot{q})dt$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}.$$

(2) Velocity $v = \dot{q}$ and momentum p:

$$p := \frac{\partial L}{\partial v},$$

(3) Hamiltonian function $H(t, q, v)$:

$$H = pv - L(q, v)$$

for *homogeneous function of order* n: $v\frac{\partial f(v)}{\partial v} = n f(v)$, one has $H = (n-1)L$.

(4) Rate of change of $f(t, q, v)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\left(\frac{dq}{dt}\right) + \frac{\partial f}{\partial v}\left(\frac{dv}{dt}\right) + \frac{\partial f}{\partial t}$$

Hamiltonian formulation in $T^*(M)$

(1) Using the canonical *Poisson bracket* $\{,\}$:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = -\{g, f\}$$

(2) The Euler-Lagrange equations can be written as *Hamilton's equations* using $H(t, q, p)$:

$$\frac{dp}{dt} = \{p, H\} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \{q, H\} = \frac{\partial H}{\partial p} \qquad (1)$$

(3) The rate of change of an observable over the phase-space $f(t, q, p)$ is given by:

$$\frac{df}{dt} - \frac{\partial f}{\partial t} = \frac{\partial f}{\partial q}\{q, H\} + \frac{\partial f}{\partial p}\{p, H\} = \{f, H\}$$

(4) When applied to the Hamiltonian function H:

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From Classical to Quantum Mechanics [2]

Canonical Quantization:

- Observables are replaced by operators $A \to \hat{A}$ over a Hilbert space $\mathcal{H}$, while the Poisson bracket is replaced by a commutator: $\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$ along with $\{q, p\} = 1 \Rightarrow [\hat{q}, \hat{p}] = i\hbar$.
- In particular, if the Hilbert space $\mathcal{H}$ is taken to be the space of square integrable functions over the configuration space $\mathcal{H} = \mathcal{L}_2[\psi(q) : \hat{q}\psi(q) = q\psi(q), \int ||\psi(q)||^2 = 1]$ than the momentum operator $\hat{p}$ becomes: $\hat{p} = -i\hbar\frac{\partial}{\partial q}$ consistent with the notion that momentum is a generator of translations along the coordinate q.
- The evolution of operators is now given by the *Heisenberg equation** which is very similar to what one had in the Hamiltonian formalism (17):

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \to \frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t}$$

Commonly used structures in relativistic QM & QFT:

- Measuring energy using the Hamiltonian: $E = H(q, p) \to E_\psi = <\psi|\hat{H}|\psi>$.
- Energy and the time translation operator: $p_0 \to \hat{p}_0 = i\hbar\frac{\partial}{\partial t}$.
- Constructing Hilbert space basis states: $c\hat{p}_0\psi_E(t, q) = E\psi_E(t, q)$.
- Useful H-space: $\mathcal{H} = \mathcal{L}_2\{\psi(q) = \sum\!\!\!\!\!\!\int c(E)\phi_E(q)d\mu_E : H\phi_E(q) = E\phi_E(q), \int d\mu_E||\psi(E)||^2 = 1\}$.