“Algebroid” structures for constraint sets and the *BFV* formalism: a disambiguation

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Motivation

Joint works with A.S. Cattaneo [CS]; C. Blohmann and A. Weinstein [BSW]

Canonical structure of (classical) gravity - constraint set.

Clarify structures underlying hypersurface deformation “algebra”.

↳ Lie algebroids vs. L_∞ algebroids.

Discussion of deformation problem.

↳ Natural in L_∞/cohomological BFV setting.

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Hypersurface deformation algebra

Consider spacetime $\Sigma \times \mathbb{R}$ with Σ spacelike.
Geometric phase space: $F^\partial = T^*\mathcal{R}(\Sigma)$ with fields:
h: Riemannian metric;
Π: (densitised) symmetric bivector field.

The momentum and hamiltonian constraints $\mathbf{H}_{\text{mo}}, \mathbf{H}_{\text{ham}}$ give rise to the Hypersurface Deformation "Algebra" (HDA):

$$\{\mathbf{H}_{\text{mo}}(X), \mathbf{H}_{\text{mo}}(Y)\} = \mathbf{H}_{\text{mo}}([X, Y])$$
$$\{\mathbf{H}_{\text{mo}}(X), \mathbf{H}_{\text{ham}}(\phi)\} = \mathbf{H}_{\text{ham}}(L_X\phi)$$
$$\{\mathbf{H}_{\text{ham}}(\phi), \mathbf{H}_{\text{ham}}(\psi)\} = \mathbf{H}_{\text{mo}}(\phi \mathrm{grad}_h\psi - \psi \mathrm{grad}_h\phi)$$

for $\phi, \psi \in C^\infty(\Sigma)$ and $X, Y \in \mathfrak{X}(\Sigma)$.

Defines a Coisotropic "submanifold" $\mathcal{C} \subset T^*\mathcal{R}(\Sigma)$.

Point dependence $\rightsquigarrow$ structure functions. **Lie algebroid?**

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Lie algebroids

What is a Lie algebroid on M? A vector bundle $A \to M$ with

1. a Lie algebra structure on sections $(\Gamma(A), [\![\cdot,\cdot]\!])$;
2. a **bundle morphism** $\rho : A \to TM$, called the *anchor*, inducing a Lie algebra morphism $\rho \colon \Gamma(A) \to \mathfrak{X}(M)$ s.t.:

$$[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + \rho(s_1)(f)s_2 \qquad \forall s_i \in \Gamma(A), f \in C^\infty(M).$$

Note 1: Every Lie algebra action defines a Lie algebroid $A = \mathfrak{g} \times M$ with anchor given by the action.

Note 2: A Lie algebroid anchor is $C^\infty(M)$-linear.

Lie algebra on sections encodes "point dependence".
HDA has point-dependent brackets, but is it enough?

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1st class constraints do not define Lie algebroids

$M = T^*N$, n functions s.t.: $\{H_i, H_j\} = f_{ij}^k H_k$ with $f_{ij}^k(x) \in C^\infty(M)$.

Try to define Lie algebroid anchor using Hamiltonian v.f. $X_i = \{\cdot, H_i\}$:

$$\rho\colon \mathrm{A} \doteq M \times \mathbb{R}^n \to TM \qquad s = s^i u_i \mapsto \rho(s) \doteq s^i X_i = s_i\{H_i, \cdot\}, \quad s_i \in C^\infty(M)$$

with Lie algebra structure on $\Gamma(\mathrm{A})$:

$$[\![s_1, s_2]\!] \doteq \left(f_{ij}^k s_1^i s_2^j + s_1^i X_i(s_2^k) - s_2^i X_i(s_1^k) \right) u_k .$$

Involutivity for the Hamiltonian vector fields is such that $\forall g \in C^\infty(M)$:

$$[X_i, X_j](g) = f_{ij}^k X_k(g) + \{f_{ij}^k, g\} H_k$$

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1^{st} **class constraints generally define Lie algebroids only on $\mathcal{C}$!**
($\rho|_{\mathcal{C}} : \mathrm{A}|_{\mathcal{C}} \to T\mathcal{C}$ is a nontrivial Lie algebroid: $X_i|_{\mathcal{C}} \neq 0$ even if $H_i \approx 0$).

Note: $\rho(s) \doteq \{s_i H_i, \cdot\}$ [BBBD] is not a bundle morphism, and hence **does not define a Lie algebroid**.

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Cohomological Methods - BFV framework

One step back: What can we associate to a constraint set?

The "physical"/reduced phase space is the coisotropic reduction $\underline{\mathcal{C}}$ of the constrained locus $\mathcal{C} = \mathrm{Zero}(H_i) \subset (F^\partial, \omega^\partial)$.

BFV theory: $\mathfrak{F} = (\mathcal{F}^\partial, S^\partial, Q^\partial, \varpi^\partial)$ a graded 0-symplectic manifold $(\mathcal{F}^\partial, \varpi^\partial)$, endowed with a cohomological vector field $[Q^\partial, Q^\partial] = 0$ and Hamiltonian (degree 1) function S^∂ - the BFV charge:

$$\iota_{Q^\partial}\varpi^\partial = \delta\mathcal{S}^\partial, \qquad (\mathcal{S}^\partial, \mathcal{S}^\partial) = 0.$$

Describe reduced phase space with the cohomology of BFV complex $(\mathfrak{BFV}^\bullet = C^\infty(\mathcal{F}^\partial), Q^\partial)$:

$$H^0(\mathfrak{BFV}^\bullet) \simeq C^\infty(F^\partial)/\mathcal{I}_{\mathcal{C}} \simeq C^\infty(\underline{\mathcal{C}}).$$

Question: what structure is associated to this type of problem?
Answer: The output of the BFV construction is an L_∞ algebroid.
Think of as: **off-shell extension of an on-shell Lie algebroid.**

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L_∞-algebroids

[Vai; Bru; KV; Vor; BCSX]

Proposition/Definition

An L_∞-algebroid is a vector bundle of $\mathbb{Z}$-graded manifolds $\mathcal{A} \to \mathcal{B}$ with a cohomological vector field $Q \in \mathcal{X}(\mathcal{A}[1])$ that is tangent to the zero section $0\colon \mathcal{B} \hookrightarrow \mathcal{A}[1]$.

In practice, we have higher brackets and higher anchors on $\Gamma(\mathcal{A})$ that satisfy higher Jacobi identities, encoded by $[Q, Q] = 0$.
(We will see an example shortly.)

All Lie algebroids are L_∞ algebroids. Certain operations, such as deformation, land you outside of Lie algebroids and into L_∞!

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BFV data for Einstein–Hilbert gravity

[FV; CS; BSW]

BFV data: $\mathcal{F}^\partial := T^*\left(\mathcal{R}(\Sigma)\times\mathfrak{X}[1](\Sigma)\times C^\infty[1](\Sigma)\right)$, with cohomological Q^∂:

$$Q^\partial(\xi^n) = \mathcal{L}_{\xi^\partial}\xi^n,$$

$$Q^\partial(\xi^\partial) = \xi^n\,\mathrm{grad}_h\xi^n + \frac{1}{2}[\xi^\partial,\xi^\partial]$$

$$Q^\partial(h) = \frac{\delta\mathbf{H}_{\mathrm{ham}}(\xi^n)}{\delta\Pi} - \mathcal{L}_{\xi^\partial}h,$$

$$Q^\partial(\Pi) = -\frac{\delta\mathbf{H}_{\mathrm{ham}}(\xi^n)}{\delta h} - \mathcal{L}_{\xi^\partial}\Pi - (\chi_\partial\otimes_s d\xi^n)^{\sharp\sharp}\xi^n$$

$$Q^\partial(\chi_\partial) = \frac{\delta\mathbf{H}_{\mathrm{mo}}(\xi^\partial)}{\delta\xi^\partial} + \mathcal{L}_{\xi^\partial}\chi_\partial - \chi_n d\xi^n,$$

$$Q^\partial(\chi_n) = \frac{\delta\mathbf{H}_{\mathrm{ham}}(\xi^n)}{\delta\xi^n} + \mathcal{L}_{\xi^\partial}\chi_n - 2\mathcal{L}_{\chi^\sharp}(\xi^n\mathrm{vol}_h^{-\frac{1}{2}})\mathrm{vol}_h^{\frac{1}{2}},$$

where

$\xi^n\in C^\infty[1](\Sigma)$,	$\xi^\partial\in\mathfrak{X}[1](\Sigma)$	ghosts
$\chi_n\in C^\infty[-1](\Sigma)\otimes\mathrm{Dens}(\Sigma)$,	$\chi_\partial\in\Omega^1[-1](\Sigma)\otimes\mathrm{Dens}(\Sigma)$	antifields

The BFV charge/ boundary action is:

$$\mathcal{S}^\partial_{EH} = \mathbf{H}_{\mathrm{ham}}(\xi^n) + \mathbf{H}_{\mathrm{mo}}(\xi^\partial) + \int_\Sigma \chi_n\mathcal{L}_{\xi^\partial}\xi^n + \langle\chi_\partial,\xi^n\,\mathrm{grad}_h\xi^n\rangle + \frac{1}{2}\langle\chi_\partial,[\xi^\partial,\xi^\partial]\rangle.$$

L_∞ algebroid for Einstein–Hilbert gravity

Observations

$(\mathcal{F}^\partial, Q^\partial)$ defines an L_∞-algebroid $\mathcal{F}^\partial \to \mathcal{B}$, where

$$\mathcal{B} \doteq \underbrace{T^*\mathcal{R}(\Sigma)}_{(h,\Pi)} \times \underbrace{\Omega^1[-1](\Sigma) \otimes \mathrm{Dens}(\Sigma)}_{\chi_\partial} \times \underbrace{C^\infty[-1](\Sigma) \otimes \mathrm{Dens}(\Sigma)}_{\chi_n},$$

with fibres $C^\infty(\Sigma) \times \mathfrak{X}(\Sigma) \ni (f, X)$. It has a nontrivial Multianchor:

$$\rho^{(2)}((f_1, X_1), (f_2, X_2))(\Pi) = \chi_\partial^\sharp \otimes_s (f_1 \mathrm{grad}_h f_2 - f_2 \mathrm{grad}_h f_1),$$

and Ternary brackets on sections: trivial only on *constant* sections but not on "field-dependent gauge transformations" [BSW]

$$[s_1, s_2, f s_3]_{(3)} = \rho^{(2)}(s_1, s_2)(f) s_3 + f[s_1, s_2, s_3] = \rho^{(2)}(s_1, s_2)(f) s_3.$$

Note: This is akin to the action algebroid for an abelian Lie group, only "3-bracket Abelian".

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Algebroid or not?

In 2011 [BFW] constructed a Lie algebroid:

$$\mathsf{A}_{BFW} \simeq C^\infty(\mathbb{R}, \mathcal{R}(\Sigma)) \times C^\infty(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\mathbb{R}, \mathcal{R}(\Sigma))$$

The anchor takes (f, X) as initial value for the Gaussian vector field $\tilde{X} \in \mathfrak{X}(\Sigma \times \mathbb{R})$, which acts on Gaussian 4-metrics $g = -dt^2 + h(t)$, with $h(t) \in C^\infty(\mathbb{R}, \mathcal{R}(\Sigma))$.

The bracket of (constant!) sections of A_{BFW} reproduces the HDA.

Note: This algebroid is supported on $C^\infty(\mathbb{R}, \mathcal{R}(\Sigma))$, **not on** $T^*\mathcal{R}(\Sigma)$**!**

> *There is a natural projection π to $T\mathcal{R}(\Sigma)$ from $C^\infty(\mathbb{R}, \mathcal{R}(\Sigma))$, assigning to every Σ-universe the 1-jet with respect to t at $t = 0$ of its gaussian representation as a path in $\mathcal{R}(\Sigma)$. (Note that this is well defined even if the Σ-universe is not cylindrical.) But there is no way to push our Lie algebroid forward under this projection, essentially because the value of the anchor at a given 1-jet would have to depend on the 2-jet.* [BFW]

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Conclusions

L_∞-morphisms and transfer

The natural projection relating the bulk (covariant), to the boundary (canonical) description **of the same data** does not preserve the Lie algebroid structure **presenting the HDA**.

Natural in the L_∞ setting (homotopy transfer interpretation).

Take-home message:

1. First class constraints do not generally define Lie algebroids, but only up to homotopy [Sta; BSW].
2. Restriction to Lie algebroid morphisms for deformations does not seem natural. [MS, Loops '22...]
3. L_∞-algebroids control deformations of coisotropic submanifolds [OP] and the BFV cohomology is the fundamental object that should be preserved by deformations. [Sch]

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Thanks!

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Formulas

The *Momentum and Hamiltonian* constraints are

$$H_{\mathsf{mo}} := \mathrm{div}_\gamma(\Pi) = 0 \tag{1a}$$

$$H_{\mathsf{ham}} := \mathrm{Tr}_\gamma[\Pi^2] - \frac{1}{d-1}\mathrm{Tr}_\gamma[\Pi]^2 + R[\gamma] = 0 \tag{1b}$$

and, for $\phi \in C^\infty(\Sigma)$ and $X \in \mathfrak{X}(\Sigma)$ the constraint functions:

$$\mathbf{H}_{\mathsf{mo}}(X) = \int_\Sigma \gamma(X, H_{\mathsf{mo}})\,\mathrm{dVol}_\gamma; \qquad \mathbf{H}_{\mathsf{ham}}(\phi) = \int_\Sigma \phi H_{\mathsf{ham}}\,\mathrm{dVol}_\gamma$$

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