Fermions in Loop Quantum Gravity

and

resolution of doubling problem

Cong Zhang

Based on works in collaboration with Muxin Han, Jerzy Lewandowski and Hongguang Liu

Introduction

Quantum Gravity (QG) needs matter couplings:

- **matter fields are dominant in the universe**
- **gravity-matter interaction makes matter serve as probes for QG effects, especially for studying the UV behaviors of QG**
- **matter coupling plays an important role in cosmology, black holes, asymptotic safety, QG phenomenology**

Loop quantum gravity (LQG) sets a stage for matter coupling with QG:

Alesci, Ashtekar, Assanioussi, Banerjee, Bianchi, Bodendorfer, Bojowald, Domagala, Eder, Han ,Giesel, Lewandowski, Liu, Lee, Ma, Makinen, Perez, Rovelli, Sahlmann, Thiemann, Xu, Yang, CZ, …

Our work focus on chiral fermions in LQG

Introduction

LQG has the fundamental discreteness that quantum states are defined on graphs.

This fundamental discreteness of LQG $\longrightarrow$ LQG resembles the lattice field theory (LFT)

- **an interesting issue of fermion in LFT is the fermion doubling problem**
- **suspicion of fermion doubling problem in LQG**
- **the doubling problem is a crucial issue revealing the contradiction between the fundamental discreteness of QG and fermions**

In contract to LFT, graphs in LQG have the geometric meaning of quantum states. We can consider the superposition of quantum states w.r.t. graphs

The superposition of graphs helps to resolve the doubling problem

We investigate the fermion propagator in LQG. In this model, we needs to define the vacuum state $|\Omega\rangle$. Our work defines $|\Omega\rangle$ s.t. it is a superstition of quantum states on many graphs.

Introduction

Earlier results of fermions in LQG：

- **The framework of fermion coupling in LQG:**

  Ashtekar, Bianchi, Eder, Bordendorfer, Bojowald, Han, Giesel, Lewandowski, Liu, Perez, Sahlmann, Thiemann, Rovelli, CZ, …

- **The argument for fermion doubling problem in LQG: No-go theorem on a fixed lattice**

  Barnett and Smolin 2015

- **The argument that LQG should be doubling-free: Quantum geometry is a superposition of graphs**

  Gambini and Pullin 2015,
  Bianchi, Han, Rovelli, Wieland, Magliaro, Perini, 2011

Outline

① **Fermion doubling in lattice field theory**

② **A resolution resulting from LQG: superposition of graphs**

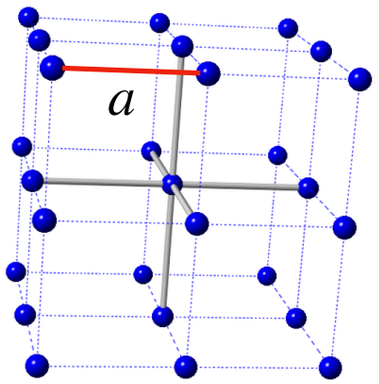
③ **Fermions in LQG**

④ **Constructing fermion propagator in LQG**

⑤ **Conclusion and outlook**

A review of doubling problem in LFT

A review of doubling problem in LFT



Consider the Minkowski spacetime with 3d cubic lattice γ in the spatial slice but continuous time. The lattice spacing is a.

Fermion propagator: $$G_\gamma\left(\omega, \overrightarrow{k}\right) = \frac{-\omega \mathbf{1}_{2\times 2} + \frac{1}{a}\sum_{I=1}^{3}\sigma^I \sin\left(ak_I\right)}{\omega^2 - \frac{1}{a^2}\sum_{I=1}^{3}\sin^2\left(ak_I\right) + i\epsilon}$$

Fix $\omega = \omega_o$, the poles of $G_\gamma(\omega_o, k)$ in fundamental Brillouin zone $k^I \in \left[-\pi/a, \pi/a\right)$.
The poles are interpreted as the fermion particles.

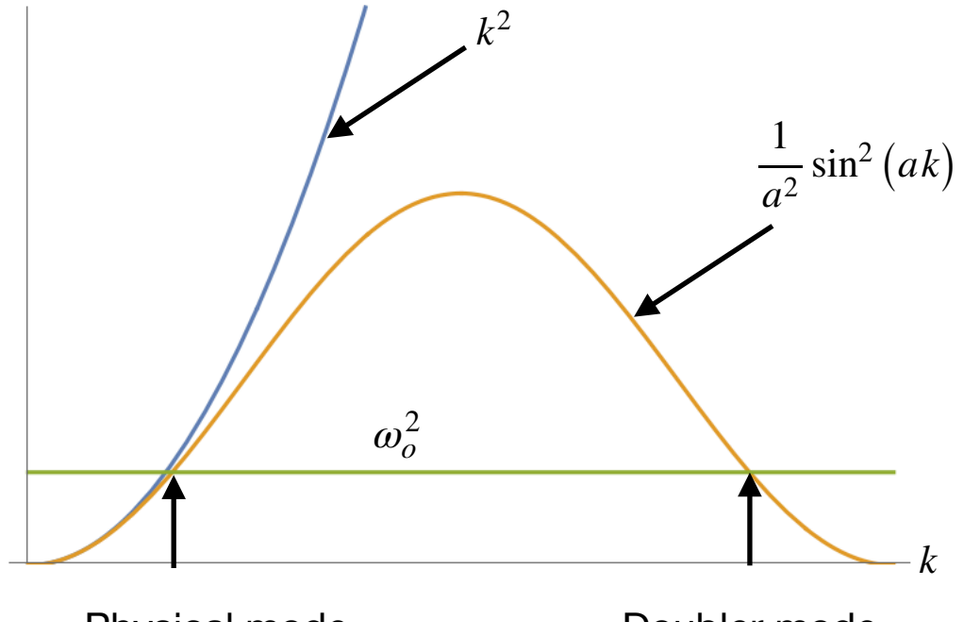
a physical pole: $\overrightarrow{k}_o = (k_o, 0, 0)$ with $k_o \simeq \omega_o$ for $a \ll 1$

a spurious doubler pole: $\overrightarrow{k} = \left(\frac{\pi}{a} - k_o, 0, 0\right)$ because of $\sin^2\left(\pi - ak_o\right) = \sin^2(ak_o)$

A review of doubling problem in LFT

$$G_\gamma\left(\omega, \overrightarrow{k}\right) = \frac{-\omega \mathbf{1}_{2\times 2} + \frac{1}{a}\sum_{I=1}^{3}\sigma^I \sin\left(ak_I\right)}{\omega^2 - \frac{1}{a^2}\sum_{I=1}^{3}\sin^2\left(ak_I\right) + i\epsilon}$$

$$G_c\left(\omega, \overrightarrow{k}\right) = \frac{-\omega \mathbf{1}_{2\times 2} + \sum_{I=1}^{3}\sigma^I k_I}{\omega^2 - \sum_{I=1}^{3}(k_I)^2 + i\epsilon}$$



Nielsen-Ninomiya no-go theorem:

> ***Any even dimensional local, hermitian, translationally invariant fermion theory always has the same number of left-handed and right-handed Weyl fermions on a lattice***

In our model, there are 7 doublers.

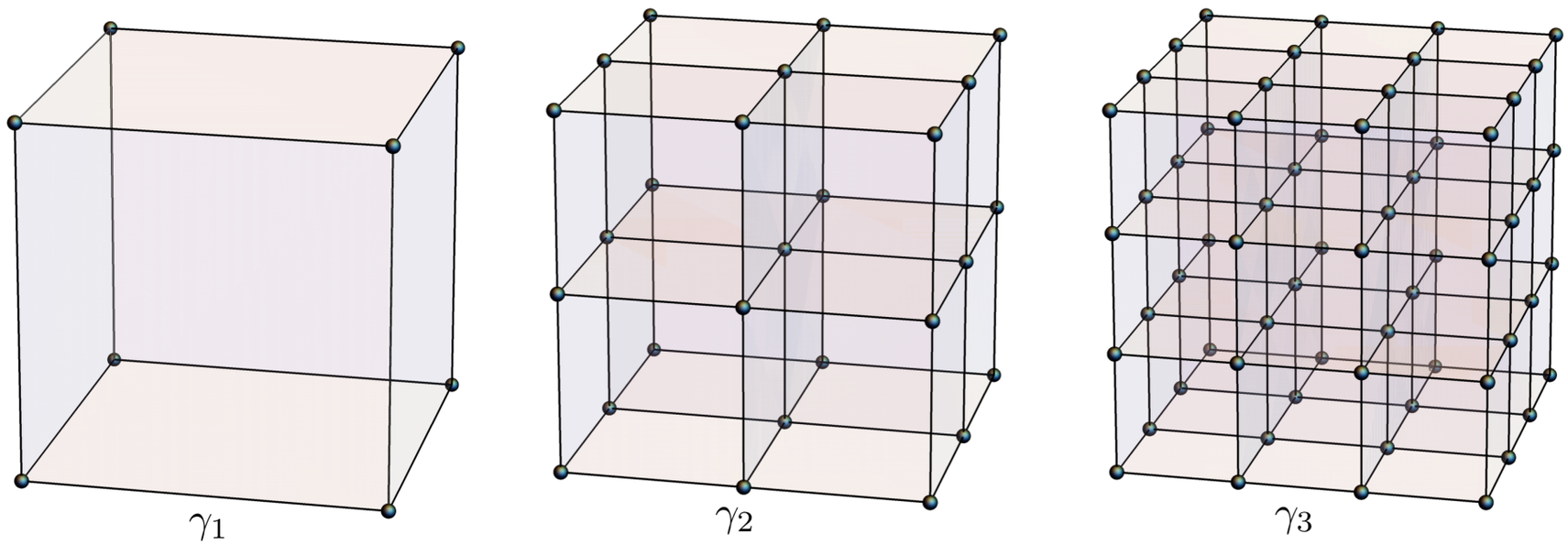
The doubling problem indicates that the naive chiral fermion in LFT fails to give the correct continuum limit

Resolutions in lattice field theory: Wilson fermion, Twisted-mass fermion, etc

A resolution from LQG : superposition of graphs

A resolution from LQG

A sequence of refined cubic lattices γ_n with flat geometry. $a_n = a_1/n$ denote the lattice spacing of γ_n:



$G_{\gamma_1}(\omega, k)$ $G_{\gamma_2}(\omega, k)$ $G_{\gamma_3}(\omega, k)$

$\vec{k} \in [-\pi/a_1, \pi/a_1)^3$ $\vec{k} \in [-\pi/a_2, \pi/a_2)^3$ $\vec{k} \in [-\pi/a_3, \pi/a_3)^3$

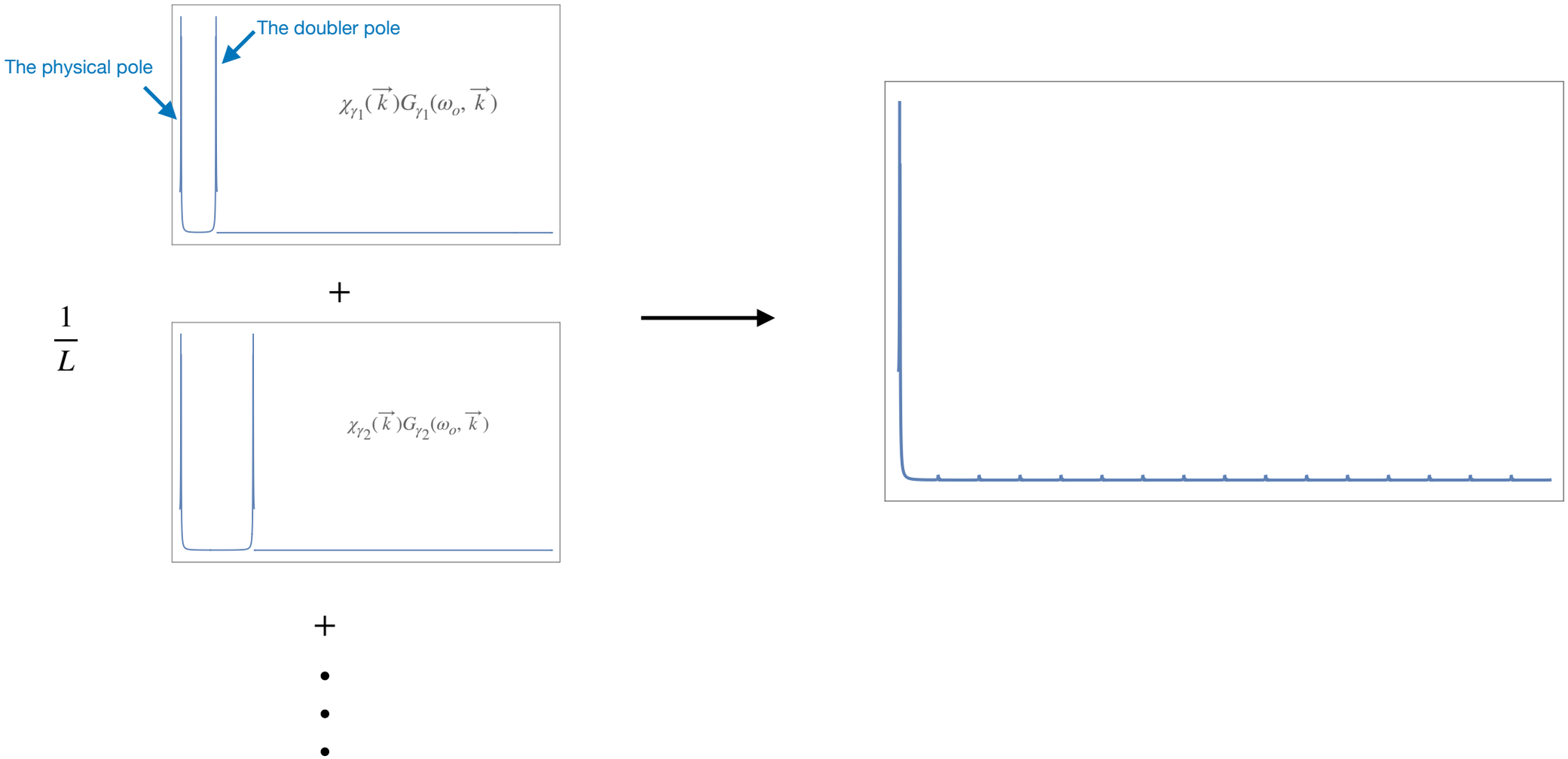
The superposition of quantum geometries w.r.t. the graphs —**to be derived**→ **the average of propagators over graphs**

A resolution from LQG

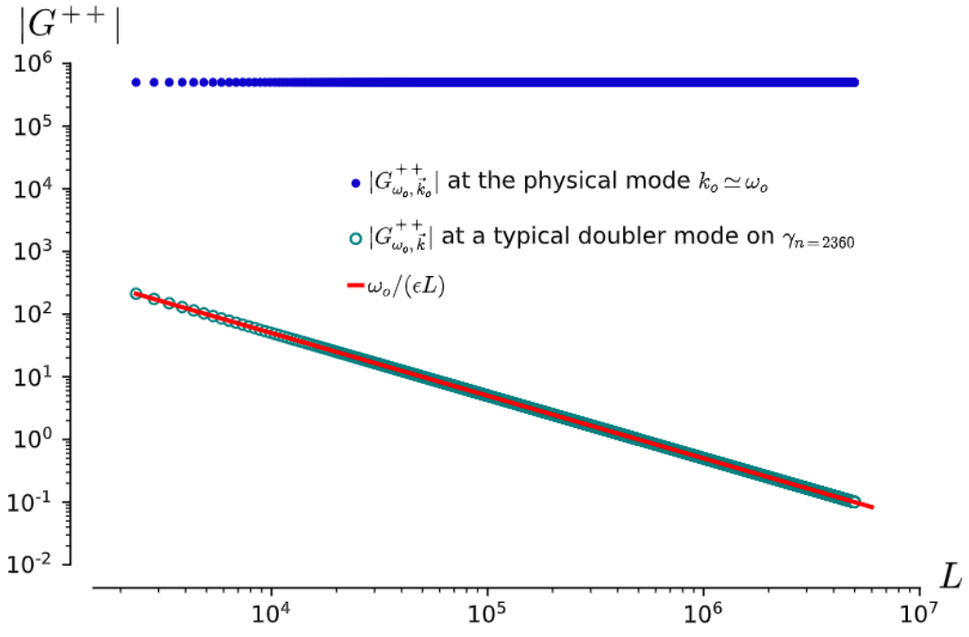
$$G(\omega, \vec{k}) = \frac{1}{L}\sum_{n=1}^{L} \chi_{\gamma_n}(\vec{k})\, G_{\gamma_n}\left(\omega, \vec{k}\right), \qquad L \gg 1, \qquad \chi_{\gamma_n}(\vec{k}) = \begin{cases} 1, & \text{if} \quad \vec{k} \in \mathrm{FBZ}(\gamma_n) \\ 0, & \text{otherwise} \end{cases}$$

The average suppresses doubler's contribution to $G(\omega, \vec{k})$ as L goes large:

a qualitative explanation:



A resolution from LQG



The contribution from the physical mode stays the same

$$\omega_o = 50$$
$$a_1 = 2^{-50}$$
$$\epsilon = 10^{-4}$$

The contribution from the doubler mode is suppressed

At fixed $L = 1.2 \times 10^5$, with $\omega_o = 50,\ a_1 = 2^{-50},\ \epsilon = 10^{-4}$, the numerics shows that $|G^{++}(\omega_o, \vec{k})| < 4.2$ at all doubler modes, while $|G^{++}(\omega_o, k_o)| \simeq 5 \times 10^5$ at the physical mode.

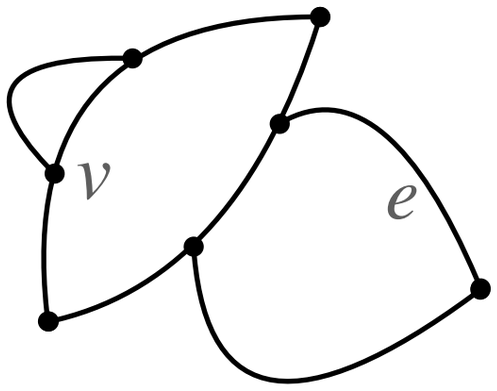
In absence of doubling mode, $G(\omega_o, \vec{k})$ is peaked at the physical mode $\vec{k} = \vec{k}_o$.

In any neighborhood of $\vec{k}_o$ and with sufficiently large L, $G(\omega_o, k)$ approximates the continuum fermion propagator arbitrarily well, because of $\lim_{L\to\infty} \frac{1}{L} \sum_{n=1}^{L} f_n = \lim_{n\to\infty} f_n$.

Chiral fermions in LQG

Chiral fermions in LQG: kinematics

- **The 4d spacetime topology is assumed to be $\mathbb{R}\times\mathbb{T}^3$.**
- **A graph γ in the spatial slice $\mathbb{T}^3$ consists of oriented edges e and their sources and targets as vertices.**



$$\mathscr{H}_\gamma = \mathscr{H}^G_\gamma \otimes \mathscr{H}^F_\gamma$$

  - **$\mathscr{H}^G_\gamma$ is spanned by spin-networks on γ with nonzero spins $j_e \neq 0$.** Thiemann 2000
  - **$\mathscr{H}^F_\gamma = \otimes_v \mathscr{H}^F_v$ with $\mathscr{H}^F_v \simeq \mathbb{C}^4$ is the Hilbert space of fermions.** Giesel, Thiemann 2007 Lewandowski, CZ 2021

- **The LQG Hilbert space $\mathscr{H}$ is given by the direct sum over all graphs: $\mathscr{H} = \oplus_\gamma \mathscr{H}_\gamma \big/ \mathscr{G}$**
  - **$\oplus_\gamma \mathscr{H}_\gamma$ carries the representation of the holonomy-flux algebra and the Weyl fermion oscillators**

$$\left[\hat{h}_e, \hat{h}_{e'}\right] = 0, \quad \left[\hat{h}_e, \hat{p}^a_{e'}\right] = \beta\ell_P^2\delta_{e,e'}(\sigma^a/2)h_e, \quad \left[\hat{p}^a_e, \hat{p}^b_{e'}\right] = -i\beta\ell_P^2\delta_{e,e'}\epsilon^{abc}p^c_e, \quad \left[\hat{\zeta}_v\,,\,\hat{\zeta}^\dagger_{v'}\right]_+ = \delta_{v,v'}\mathbf{1}_{2\times 2}$$

  - **$\mathscr{G}$ are SU(2) gauge transformations, and $\mathscr{H}$ only contains SU(2) gauge-invariant states.**

Chiral fermions in LQG: dynamics

- **The theory semiclassically relates to the reduced phase space dynamics of gravity coupled to Weyl fermion and Gaussian dust (clock fields).** Talked by Giesel K.

- **The physical Hamiltonian:**

$$\hat{\mathbf{H}} = \oplus_\gamma \hat{\mathbf{H}}_\gamma, \quad \hat{\mathbf{H}}_\gamma = \hat{\mathbf{P}}_\gamma(\hat{H}^G_\gamma + \hat{H}^F_\gamma)\hat{\mathbf{P}}_\gamma, \qquad \hat{\mathbf{P}}_\gamma \text{ kills the states with } j_e = 0 \text{ on } \gamma$$

$\hat{H}^G_\gamma$ is Giesel-Thiemann graph-preserving Hamiltonian:

Thiemann 2000
Giesel, Thiemann 2007
Lewandowski, CZ 2021

$$\hat{H}^G_\gamma = \hat{E}_\gamma - (1+\beta^2)\hat{L}_\gamma$$

$$\hat{E}_\gamma = \frac{-1}{4i\kappa\beta\ell_p^2} \sum_{v\in V(\gamma)} \sum_{e(I),e(J),e(K) \text{ at } v} \epsilon^{IJK} \mathrm{Tr}\left(h_{\alpha_{IJ}} h^{-1}_{e(K)} [h_{e(K)}, \hat{V}_v]\right)$$

$$\hat{L}_\gamma = -\frac{4\kappa}{i\beta^7\ell_p^{10}} \sum_{v\in V(\gamma)} \sum_{e(I),e(J),e(K) \text{ at } v} \epsilon^{IJK} \mathrm{Tr}\left([h_{e(I)}, [\hat{V}_\gamma, \hat{E}_\gamma]] h^{-1}_{e(I)} [h_{e(J)}, [\hat{V}_\gamma, \hat{E}_\gamma]] h^{-1}_{e_J} [h_{e_K}, \hat{V}_v] h^{-1}_{e_K}\right)$$

$\hat{H}^F_\gamma$ is the fermion Hamiltonian:

$$\hat{H}^F_\gamma / \hbar = \sum_{\langle v,v'\rangle} \hat{\zeta}^\dagger_v M_{v,v'}(\hat{h},\hat{p})\, \hat{\zeta}_{v'}, \qquad \text{quadratic to fermion } \hat{\zeta}$$

$$M_{v,v'} = \delta_{v,v'} \hat{V}_v^{-\frac{1}{2}} \left(-\frac{(1+3\beta^2)}{4\beta} \sum_{e \text{ at } v} \hat{p}^a_e \boldsymbol{\sigma}_a - \frac{\kappa}{i\ell_p^2\beta} \left[\hat{E}_\gamma, \hat{V}_v\right] \mathbf{1}_{2\times 2} \right) \hat{V}_v^{-\frac{1}{2}} - \frac{\beta - i}{4} \sum_{e \text{ at } v} \delta_{v',t_e} \hat{V}_v^{-\frac{1}{2}} \boldsymbol{\sigma}_a \hat{h}_e \hat{p}^a_e \hat{V}_v^{-\frac{1}{2}} - \frac{\beta + i}{4} \sum_{e' \text{ at } v'} \delta_{v,t_{e'}} \hat{V}_v^{-\frac{1}{2}} \hat{p}^a_{e'} h^{-1}_{e'} \boldsymbol{\sigma}_a \hat{V}_v^{-\frac{1}{2}}$$

$$\hat{V}_v^{-\frac{1}{2}} = -\frac{16}{3} \left(\frac{1}{i\ell_p^2\beta}\right)^3 \sum_{e(I),e(J),e(K) \text{ at } v} \epsilon^{IJK} \mathrm{Tr}\left(h^{-1}_{e(I)} [h_{e(I)}, \hat{V}_v^{1/2}] h^{-1}_{e(J)} [h_{e(J)}, V_v^{1/2}] h^{-1}_{e(K)} [h_{e(K)}, V_v^{1/2}]\right).$$

Construct the fermion propagator in LQG

Fermion vacuum state in LQG

The fermion propagator needs 2 ingredients: The vacuum state $|\Omega\rangle$ and the local field operator $\hat{\Psi}(\vec{x},\tau)$

The vacuum state $|\Omega\rangle \in \mathscr{H}$ is a LQG analog of the fermion ground on the semiclasscal flat spacetime

To resemble the semiclassical geometry:

- **coherent state $|\psi_{g_0(\gamma)}\rangle$ over cubic lattices** Dapor, Liegner, 2018, CZ, Song and Han 2022
- **$|\psi_{g_0(\gamma)}\rangle$ is Thiemann's coherent state peaked at the internally and externally flat geometries**

$$\frac{\langle\psi_{g_0(\gamma)}|\hat{h}_e|\psi_{g_0(\gamma)}\rangle}{\|\psi_{g_0(\gamma)}\|^2} = \mathbf{1}_{2\times 2} + O(\ell_P^2), \qquad \frac{\langle\psi_{g_0(\gamma)}|\hat{p}^a_{e(I)}|\psi_{g_0(\gamma)}\rangle}{\|\psi_{g_0(\gamma)}\|^2} = a^2\delta^a_I + O(\ell_P^2)$$

a is the lattice spacing of the semiclassical geometry given by $|\psi_{g_0(\gamma)}\rangle$

To define the fermion ground associated to $|\psi_{g_0(\gamma)}\rangle$:

- **effective fermion Hamiltonian:**

$$\hat{H}^F_\gamma\left(\psi_{g_0(\gamma)}\right)/\hbar = \sum_{\langle v,v'\rangle}\hat{\zeta}^\dagger_v \frac{\langle\psi_{g_0(\gamma)}|M_{v,v'}(\hat{h},\hat{p})|\psi_{g_0(\gamma)}\rangle}{\|\psi_{g_0(\gamma)}\|^2}\hat{\zeta}_{v'} = \text{fermion Hamiltonian on the flat lattice} + O(\ell_P^2)$$

- **$|\omega_\gamma\rangle$ is the ground state of the "effective fermion Hamiltonian" on the semiclassical geometry**

The similar idea exists in the literature e.g. Sahlmann and Thiemann 2002, Stottmeister and Thiemann 2015, etc

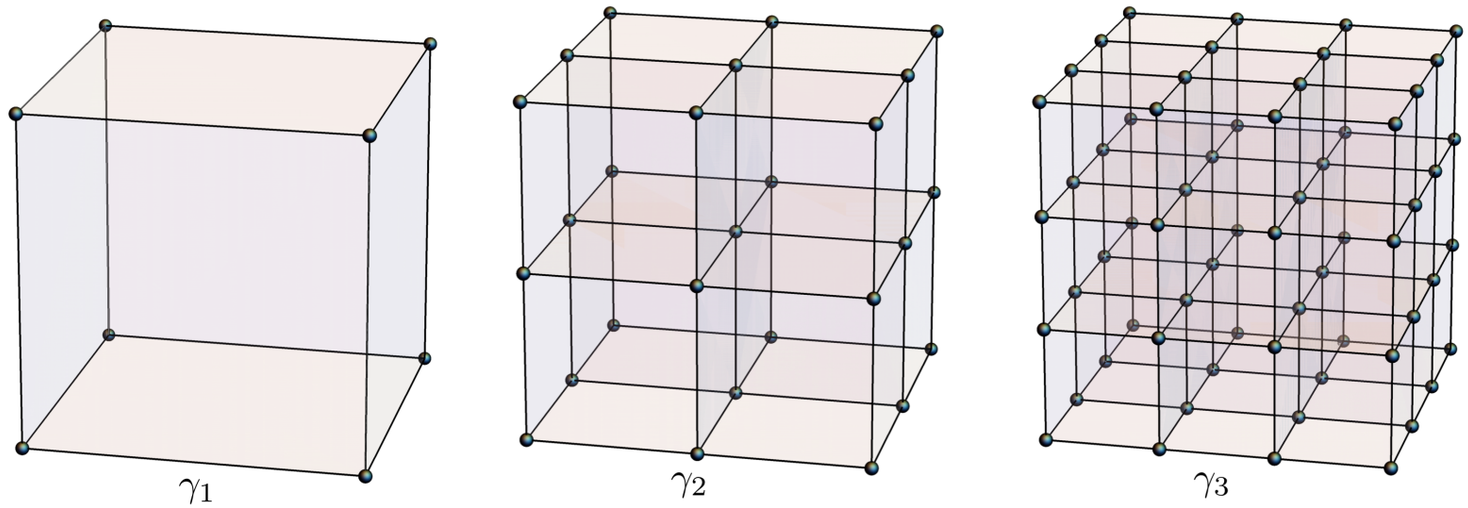
$|\Omega_\gamma\rangle = \hat{\mathbf{P}}_{\mathscr{G}}(\hat{\mathbf{P}}_\gamma|\psi_{g_0(\gamma)}\rangle \otimes |\omega_\gamma\rangle) \in \mathscr{H}^G_\gamma \otimes \mathscr{H}^F_\gamma$ is the vacuum state associated to γ

Fermion propagator in LQG: the vacuum state of fermion

The smooth flat geometry admits many different discretizations on different lattices

The semiclassical flat geometry should be a superposition of the semiclassical discrete geometries over different lattices.

A sequence of refined cubic lattice γ_n : $N_n = nN_1$ is the number of vertices along each direction



$$|\Omega_{\gamma_1}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_1}|\psi_{g_0(\gamma_1)}\rangle \otimes |\omega_{\gamma_1}\rangle) \quad |\Omega_{\gamma_2}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_2}|\psi_{g_0(\gamma_2)}\rangle \otimes |\omega_{\gamma_2}\rangle) \quad |\Omega_{\gamma_3}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_3}|\psi_{g_0(\gamma_3)}\rangle \otimes |\omega_{\gamma_3}\rangle)$$

$|\psi_{g_0(\gamma_n)}\rangle$ gives the semiclassical lattice spacing $a_n = a_1/n$ on γ_n (consistent discretizations of the flat geometry)

$|\omega_{\gamma_n}\rangle$ is the fermion ground state on the corresponding semiclassical lattice geometry.

Fermion propagator in LQG: the vacuum state of fermion

The smooth flat geometry admits many different discretizations on different lattices

The semiclassical flat geometry should be a superposition of the semiclassical discrete geometries over different lattices.

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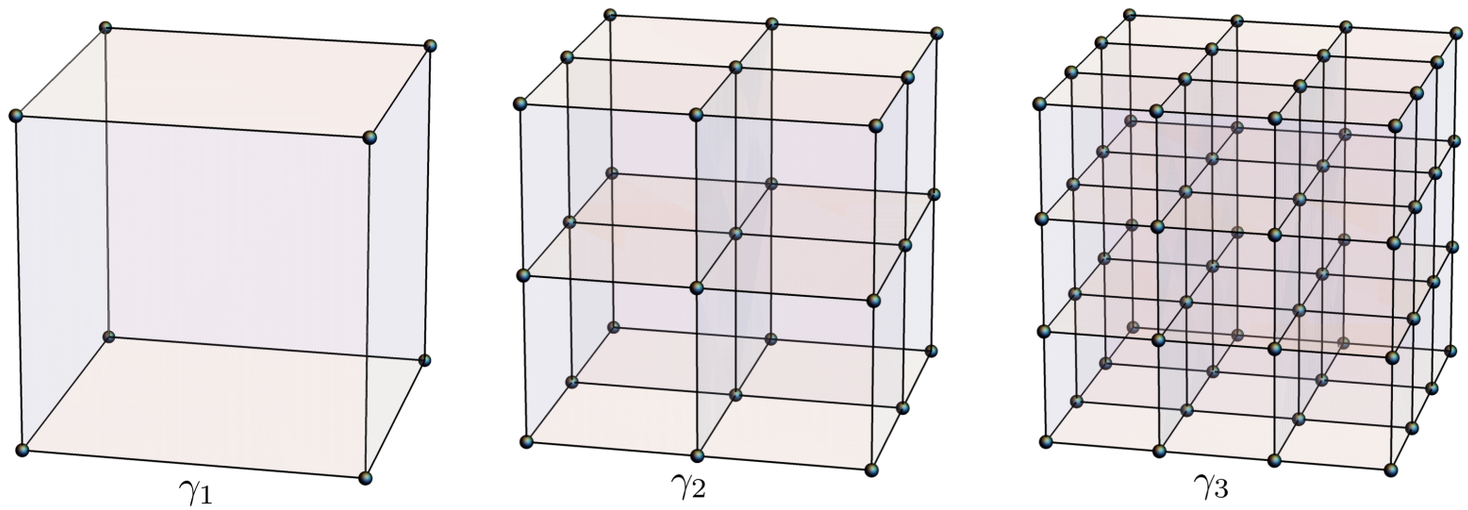

$$|\Omega_{\gamma_1}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_1}|\psi_{g_0(\gamma_1)}\rangle \otimes |\omega_{\gamma_1}\rangle) \quad |\Omega_{\gamma_2}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_2}|\psi_{g_0(\gamma_2)}\rangle \otimes |\omega_{\gamma_2}\rangle) \quad |\Omega_{\gamma_3}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}(\hat{\mathbf{P}}_{\gamma_3}|\psi_{g_0(\gamma_3)}\rangle \otimes |\omega_{\gamma_3}\rangle)$$

$|\Omega\rangle$ is a superposition of $|\Omega_{\gamma_n}\rangle \in \mathscr{H}^G_\gamma \otimes \mathscr{H}^F_\gamma$ over the set of cubic graphs

$$|\Omega\rangle = \mathscr{N}\sum_{n=1}^{L}\omega_n|\Omega_{\gamma_n}\rangle, \quad |\Omega_{\gamma_n}\rangle = \hat{\mathbf{P}}_{\mathcal{G}}\left(\hat{\mathbf{P}}_{\gamma_n}|\psi_{g_0(\gamma_n)}\rangle \otimes |\omega_{\gamma_n}\rangle\right).$$

$\omega_n = 1, \quad \mathscr{N} = \dfrac{1}{\sqrt{L}}$ for instance in the talk, but the mechanism is valid for a general family of weight ω_n

Fermion propagator in LQG: field operator

The fermion propagator needs 2 ingredients: The vacuum state $|\Omega\rangle$ and the local field operator $\hat{\Psi}(\overrightarrow{x},\tau)$

The fermion operators $\hat{\zeta}_v$, $\hat{\zeta}_v^\dagger$ come from half-densities Thiemann 2000, Giesel, Thiemann 2007, Lewandowski, CZ 2021

$$\zeta(x) = \int \sqrt{\delta(x,y)}\sqrt[4]{\det\left(q(y)\right)}\ \Psi(y) \qquad \left[\hat{\zeta}_v\,,\,\hat{\zeta}_{v'}^\dagger\right]_+ = \delta_{v,v'}\mathbf{1}_{2\times 2}$$

We define the local field operator $\hat{\Psi}(x)$:

$$\hat{\Psi}(v) = \hat{\zeta}_v V_v^{\frac{1}{2}}$$

The Heisenberg operator:

$$\hat{\Psi}(\tau, v) = e^{\frac{i}{\hbar}\hat{\mathbf{H}}(\tau - T_i)}\,\hat{\Psi}(v)\,e^{-\frac{i}{\hbar}\hat{\mathbf{H}}(\tau - T_i)}$$

Fermion propagator in LQG

The LQG fermion propagator:

$$\mathcal{G}^{AB}(\tau_1,\tau_2;v_1,v_2)=\langle\Omega'|\,\mathcal{T}[\hat{\Psi}^{A}(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\,\Omega\rangle$$

v_1, v_2 are vertices in γ_1 , and thus belong to all γ_n .

$|\,\Omega'\rangle$ is gauge non-invariant vacuum state

The fermion propagator $\mathcal{G}^{AB}(\tau_1,\tau_2;v_1,v_2)=\langle\Omega'|\,\mathcal{T}[\hat{\Psi}^{A}(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\,\Omega\rangle$ is not SU(2) gauge invariant, similar to the usual gauge theories.

But the observables constructed from the propagator are gauge invariant. For gauge invariant observables, $|\,\Omega'\rangle\to|\,\Omega\rangle$

An example of gauge invariant observable

$$\mathcal{G}_h(\tau_1,\tau_2;v_1,v_2)=\langle\Omega\,|\,(\hat{h}_c)_{BA}\mathcal{T}[\hat{\Psi}^{A}(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\,\Omega\rangle$$

$\hat{h}_c$ is the holonomy operator on γ_1 connecting v_1 , v_2 .

We will show $\mathcal{G}_h=\mathcal{G}^{AB}\delta_{AB}\left[1+O(\ell_P^2)\right]$ **(contracting the propagator with other ingredients)**

Fermion propagator in LQG: computing

$$\mathcal{G}^{AB}(\tau_1,\tau_2;v_1,v_2) = \langle\Omega'|\,\mathcal{T}[\hat{\Psi}^A(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\Omega\rangle$$
$$= \frac{1}{L}\sum_{n,n'}\langle\Omega'_{\gamma_{n'}}|\,\mathcal{T}[\hat{\Psi}^A(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\Omega_{\gamma_n}\rangle$$

Recall that the Hamiltonian is a direct sum: $\hat{\mathbf{H}} = \oplus_\gamma \hat{\mathbf{H}}_\gamma, \quad \hat{\mathbf{H}}_\gamma = \hat{\mathbf{P}}_\gamma(\hat{H}^G_\gamma + \hat{H}^F_\gamma)\hat{\mathbf{P}}_\gamma,$

$$\langle\Omega'_{\gamma_{n'}}|\,\mathcal{T}[\hat{\Psi}^A(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\Omega_{\gamma_n}\rangle \propto \delta_{n,n'}$$

The fermion propagator becomes an average of the propagators on graphs

$$\mathcal{G}^{AB}(\tau_1,\tau_2;v_1,v_2) = \frac{1}{L}\sum_{n=1}^{L}\mathcal{G}^{AB}_n(\tau_1,\tau_2;v_1,v_2)$$
$$\mathcal{G}^{AB}_n(\tau_1,\tau_2;v_1,v_2) = \langle\Omega'_{\gamma_n}|\,\mathcal{T}[\hat{\Psi}^A(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\Omega_{\gamma_n}\rangle$$

$\mathcal{G}^{AB}_n(\tau_1,\tau_2;v_1,v_2)$ **on the fixed graph** γ_n **can be computed with coherent state path integral method**

Fermion propagator in LQG: computing

$\mathscr{G}_n^{AB}(\tau_1, \tau_2; v_1, v_2)$ **on the fixed graph** γ_n **can be computed with coherent state path integral method**

$$\hat{\Psi}(\tau, v) = e^{\frac{i}{\hbar}\hat{\mathbf{H}}_{\gamma_n}(\tau - T_i)}\, \hat{\Psi}(v)\, e^{-\frac{i}{\hbar}\hat{\mathbf{H}}_{\gamma_n}(\tau - T_i)} \qquad e^{-\frac{i}{\hbar}\hat{\mathbf{H}}_{\gamma}(\tau - T_i)} \simeq \left[1 - \frac{i}{\hbar}\hat{\mathbf{H}}_{\gamma}\delta\tau\right]^{\mathcal{N}}$$

Insert the overcomplete relation

$$\int dZ(\gamma_n)\, |Z(\gamma_n)\rangle\langle Z(\gamma_n)| = 1, \qquad |Z(\gamma_n)\rangle = |\psi_{g(\gamma_n)}\rangle \otimes |\phi_{\zeta(\gamma_n)}\rangle$$

Thiemann's coherent state — **coherent state of fermion oscillators**

A path integral formula of the propagator on γ_n **:**

$$\mathscr{G}_n^{AB}(\tau_1, \tau_2; v_1, v_2) = \lim_{n\to\infty} \int \prod_{j=0}^{N} dZ_j(\gamma_n)\, e^{S^G_{\gamma_n}(h,p) + S^F_{\gamma_n}(h,p,\zeta)}\, (V_{v_1}^{-\frac{1}{2}}\zeta^A_{v_1})_{j_1}\, (V_{v_2}^{-\frac{1}{2}}\zeta^{B*}_{v_2})_{j_2}\, \langle \Omega_{\gamma_n} | Z_N(\gamma_n)\rangle\langle Z_0(\gamma_n) | \Omega_{\gamma_n}\rangle$$

Fermion effective action

$$S^F_{\gamma_n} = \sum_{j,j',v,v'} \zeta^\dagger_{v,j} \mathscr{K}_{v,j;v',j'}(h,p)\zeta_{v',j'}, \qquad \mathscr{K}_{v,j;v',j'} = \delta_{v',v}\delta_{j,j'+1} - \delta_{j',j}\delta_{v',v} - i\delta_{j,j'+1}\delta\tau_j \left\langle \psi_{g_j(\gamma_n)} \middle| M_{v,v'}(\hat{h},\hat{p}) \middle| \psi_{g_{j'}(\gamma_n)} \right\rangle$$

Fermion propagator in LQG: computing

$$\mathscr{G}_n^{AB}(\tau_1,\tau_2;v_1,v_2)=\int\prod_{j=0}^{N}dZ_j(\gamma_n)\,e^{S^G_{\gamma_n}(h,p)+S^F_{\gamma_n}(h,p,\zeta)}\,(V_{v_1}^{-\frac{1}{2}}\zeta^A_{v_1})_{j_1}\,(V_{v_2}^{-\frac{1}{2}}\zeta^{B*}_{v_2})_{j_2}\,\langle\Omega_{\gamma_n}\,|\,Z_N(\gamma_n)\rangle\langle Z_0(\gamma_n)\,|\,\Omega_{\gamma_n}\rangle$$

$$S^F_{\gamma_n}=\sum_{j,j',v,v'}\zeta^\dagger_{v,j}\mathscr{K}_{v,j;v',j'}(h,p)\zeta_{v',j'},\qquad \mathscr{K}_{v,j;v',j'}=\delta_{v',v}\delta_{j,j'+1}-\delta_{j',j}\delta_{v',v}-i\delta_{j,j'+1}\delta\tau_j\left\langle\psi_{g_j(\gamma_n)}\right|M_{v,v'}(\hat{h},\hat{p})\left|\psi_{g_{j'}(\gamma_n)}\right\rangle$$

- **Perform fermion Gaussian integral**

$$\mathscr{G}_n^{AB}(\tau_1,\tau_2;v_1,v_2)=\int\prod_{j=0}^{N}dg_j(\gamma_n)\,e^{S^G_{\gamma_n}(h,p)}\,V^{-\frac{1}{2}}_{v_1,j_1}V^{-\frac{1}{2}}_{v_2,j_2}\,I_n^{AB}(h,p)$$

The fermion integral, independent of $\hbar$

- $S^G_{\gamma_n}\sim\ell_P^{-2}$, **stationary phase approximation:**

$$\delta S^G_{\gamma_n}(h,p)=0\quad\Longrightarrow\quad\frac{dh_e}{d\tau}=\{h_e\,,\,H^G\},\quad\frac{dp^a_e}{d\tau}=\{p^a_e\,,\,H^G\},$$ Han and Liu 2020

The only solution compatible to the initial data is the flat spacetime $\bar{h}_e=\mathbf{1}_{2\times2},\ \ \bar{p}^a_{e(I)}=a_n^2\delta^a_I$

$$\mathscr{G}_n^{AB}(\tau_1,\tau_2;v_1,v_2)=\,a_n^{-3}\,I_n^{AB}(\bar{p},\bar{h})\big[1+O(\ell_P^2)\big]\int\prod_{j=0}^{N}dg_j(\gamma_n)\,e^{S^G_{\gamma_n}(h,p)}$$

It reduces to the naive fermion propagator on the flat lattice

$$=\frac{1}{N_n^3a_n^3}\sum_{k^I\in\mathrm{FBZ}(\gamma_n)}\int_{\mathbb{R}}\frac{d\omega}{2\pi i}\,e^{-i\omega(\tau_1-\tau_2)+ia_nk_I\cdot(\vec{v}_1-\vec{v}_2)}\,G^{AB}_{\gamma_n}(\omega,\vec{k})\big[1+O(\ell_P^2)\big]$$

$$\int\prod_{j=0}^{N}dg_j(\gamma_n)\,e^{S^G_{\gamma_n}(h,p)}=\langle\psi_{g_0(\gamma_n)}\,|\,U(-\tau_2)U(\tau_2-\tau_1)U(\tau_1)\,|\,\psi_{g_0(\gamma_n)}\rangle=1$$

Fermion propagator in LQG: computing

$$\mathcal{G}^{AB}(\tau_1,\tau_2;v_1,v_2) = \frac{1}{L}\sum_{n=1}^{L}\mathcal{G}_n^{AB}(\tau_1,\tau_2;v_1,v_2)$$

$$= \frac{1}{V_{tot}}\sum_{k^I\in \mathrm{FBZ}(\gamma_L)}\int_{\mathbb{R}}\frac{d\omega}{2\pi i}\, e^{-i\omega(\tau_1-\tau_2)+ik_I\cdot(\vec{x}_1-\vec{x}_2)}\,\frac{1}{L}\sum_{n=1}^{L}\chi_{n,k}G_{\gamma_n}(\omega,\vec{k})\left[1+O(\ell_P^2)\right]$$

This reduces to the propagator free of doublers: $G_{\omega,k} = \frac{1}{L}\sum_{n=1}^{L}\chi_{n,k}G_{\omega,k}\left(\gamma_n, a_n\right).$

The gauge invariant observable:

$$\mathcal{G}_h(\tau_1,\tau_2;v_1,v_2) = \langle\Omega\,|\,(\hat{h}_c)_{BA}\mathcal{T}[\hat{\Psi}^A(\tau_1,v_1)\hat{\Psi}^{B\dagger}(\tau_2,v_2)]\,|\,\Omega\rangle$$

$$= \frac{1}{L}\sum_{n=1}^{L}\mathcal{G}_{n,h}(\tau_1,\tau_2;v_1,v_2)$$

$$\mathcal{G}_{n,h}(\tau_1,\tau_2;v_1,v_2) = \int\prod_{j=0}^{N}dg_j(\gamma_n)\, e^{S^G_{\gamma_n}(h,p)}\, V^{-\frac{1}{2}}_{v_1,j_1}V^{-\frac{1}{2}}_{v_2,j_2}\,(h_c)_{BA}\,I_n^{AB}(h,p)$$

At the solution of equations of motion: $(h_c)_{BA} = \delta_{BA} \quad\Longrightarrow\quad \mathcal{G}_h = \mathcal{G}^{AB}\delta_{AB}\left[1+O(\ell_P^2)\right].$

Some interesting remarks

- **The fact that the superposition of lattices brings the fermion propagator close to the continuum limit suggests that the continuum limit of the full LQG should also relate to the superposition of lattices. This is similar to the approach of group field theory**

- **The Gaussian dust energy E_{dust} is nonzero and positive:**

$$E_{dust} + \mathbf{H} = 0, \quad \text{where} \quad \mathbf{H} = \langle \psi_{g_0(\gamma)} \otimes \omega_\gamma | \hat{\mathbf{H}} | \psi_{g_0(\gamma)} \otimes \omega_\gamma \rangle = \text{fermion zero-point energy} < 0$$

Non-normal-ordering fermion Hamiltonian in phase space reduced model

- **The LQG propagator gives a soft UV cut-off to the fermion propagator:**

$$G(\omega, k) = \frac{1}{L} \sum_{n=1}^{L} \chi_{\gamma_n}(\vec{k}) G_{\gamma_n}(\omega, \vec{k}), \qquad L \gg 1.$$

As the physical mode $(\omega_o, \vec{k}_o)$ goes large, $|G^{AB}(\omega_o, k_o)|$ is suppressed because less and less terms survive in the sum, because of $\chi_{\gamma_n}(\vec{k})$.

This result is consistent with the expectation that QG should regularize the UV behavior of matters.

Conclusion and outlook

- **The superposition of quantum geometries resolves the tension between the fermion and the fundamental discreteness**

- **The fermion propagator from LQG is an average of the discrete fermion propagators over lattices, and is free of the fermion doubling problem**

- **The fermion propagator has an interesting UV behavior, which may relates to the expected UV completion of QG**

Todo list:

- **Chiral anomaly on quantum geometry and continuum limit**
- **Higher order corrections to the fermion propagator, 1-loop correction, and UV behavior**
- **n-point functions**

Thanks for your attention !