Typical Entanglement in Quantum Gravity

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Typical Entanglement in Quantum Gravity

1. Entanglement Entropy in Constrained Systems

2. Typical Entanglement with Constraints

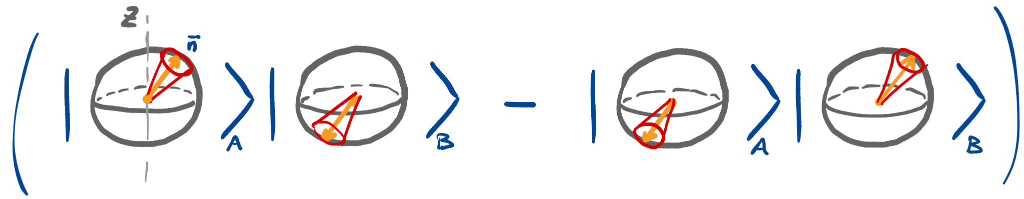
3. Applications:
   - CM
   - QFT
   - BH
   - QG

- Bianchi - Donà, PRD'19 [1904.08370]
- Bianchi - Hackl - Kieburg - Rigol - Vidmar, PRX '22 [2112.06959]

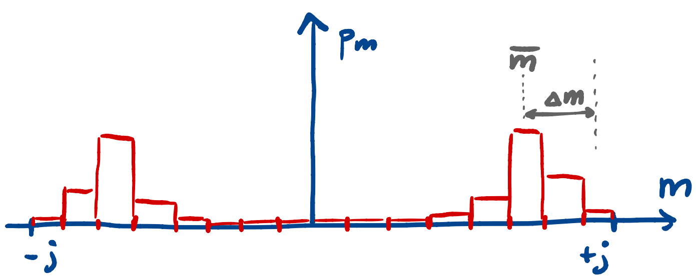
1 Entanglement Entropy & Observables: Example *with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- Pure State of Two Spins

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|j,+\vec{n}\rangle_A\,|j,-\vec{n}\rangle_B - |j,-\vec{n}\rangle_A\,|j,+\vec{n}\rangle_B\right) = \frac{1}{\sqrt{2}}\Big(\ \ \Big)$$



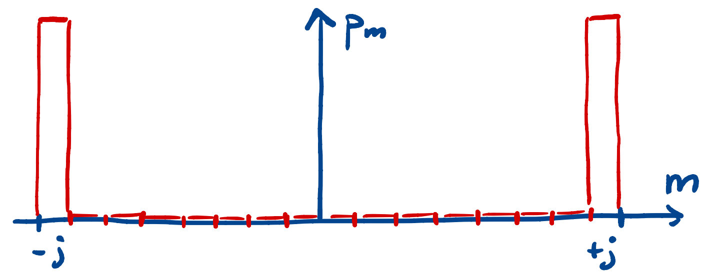
- Measure Observable J_A^z



- Entropy of outcomes

$$S(J_A^z \to \{m\}, |\psi\rangle) \underset{j \gg 1}{\approx} \log 2 + \log\left(\sqrt{2\pi e}\,\Delta m\right)$$

- Measure Observable $\vec{J}_A \cdot \vec{n}$



- Entropy of outcomes

$$S(\vec{J}_A\cdot\vec{n} \to \{m\}, |\psi\rangle) = -\sum_m p_m \log p_m = \log 2$$

- Entanglement Entropy of $|\psi\rangle$ restricted to A

$$S_A(|\psi\rangle) \equiv -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) = \log 2$$

* Note: Lower bound

$$\boxed{S_A(|\psi\rangle) \leq S(\vec{J}_A\cdot\vec{n}' \to \{m\}, |\psi\rangle) \quad \forall \vec{n}'}$$

1 Entanglement Entropy & Observables

* assuming $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Thm (a): Lower bound on the entropy of observables in a subsystem

$$\boxed{S_A(|\psi\rangle) = \min_{\mathcal{O}_A^{(i)}\ \text{c.c.s.}} S(\mathcal{O}_A^{(i)} \to \{k_i\}, |\psi\rangle)}$$

where:

- $\mathcal{O}_A^{(i)}$ = complete compatible set of observables in $\mathcal{H}_A$ $(i = 1, \dots, I)$
- $|k_1, \dots, k_I\rangle_A |\beta\rangle_B$ = o. n. basis of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ associated to $\mathcal{O}_A^{(i)}$
- $p(k_1, \dots, k_I) = \sum_\beta |\langle k_1, \dots, k_I; \beta | \psi \rangle|^2$ = probability of measurement outcome
- $S(\mathcal{O}_A^{(i)} \to \{k_i\}, |\psi\rangle) \equiv -\sum_{k_1, \dots, k_I} p(k_1, \dots, k_I) \log p(k_1, \dots, k_I)$ entropy
- $S_A(|\psi\rangle) \equiv -\mathrm{Tr}_A(\rho_A \log \rho_A)$ entanglement entropy

  with $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$

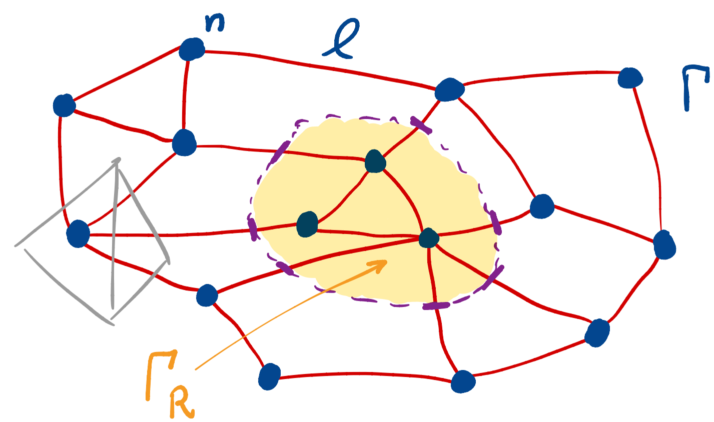
[1] Entanglement Entropy: How to generalize it to $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$?

$*$ Strategy: $\begin{cases} \text{- define subsystem in terms of observables} \\ \text{- require lower bound "} S_A = \min \text{" as in Thm (a)} \end{cases}$

▪ Example: Spin-Network basis state in $\mathcal{H}_\Gamma$

- $|\Gamma, j_\ell, i_n\rangle$ = simultaneous eigenstate of a c.c.s. of $\vec{E}_{n\ell} \cdot \vec{E}_{n\ell'}$

  (j_ℓ: area; i_n: dihedral angles) quantum polyhedra

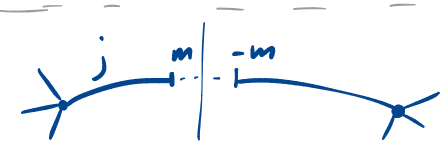


- subsystem ("region" Γ_R)
  $\longrightarrow$ subset of polyhedra

- probability of outcomes for areas & angles $\begin{cases} 1 \\ 0 \end{cases}$

$$S(\text{areas \& angles in } R, |\psi\rangle) = 0 \;\Rightarrow\; S_R(|\psi\rangle) \overset{!}{=} \min_{O_R^{(i)}} S(O_R^{(i)}, |\psi\rangle) = 0$$

$*$ Compare
- requirement $S_R(|\psi\rangle) = 0$
- to statement "$S_R(|\psi\rangle) = \sum_{\ell \in \partial R} \log(2j_\ell + 1)$" from non-gauge inv. observ.



1 Entanglement Entropy: Operational Definition of Subsystem

- $\mathcal{H}$ = Hilbert Space, $\mathcal{A} = \mathcal{B}(\mathcal{H})$ = Algebra of Observables
- $\mathcal{A}_R \subset \mathcal{A}$ subalgebra that defines the subsystem R
- $\mathcal{A}_{\bar{R}} \equiv \{ O_{\bar{R}} \in \mathcal{A} \mid [O_{\bar{R}}, O_R] = 0 \quad \forall O_R \in \mathcal{A}_R \}$ "rest of the system"

* note: in general $\mathcal{A}_R \cup \mathcal{A}_{\bar{R}} \neq \mathcal{A}$

* note: in general $\mathcal{Z} \equiv \mathcal{A}_R \cap \mathcal{A}_{\bar{R}} \neq \{\mathbb{1}\}$ non-trivial center

- Decomposition of the Hilbert space adapted to $\mathcal{A}_R$

$$\boxed{\mathcal{H} = \bigoplus_{\lambda \in \mathcal{Z}} \left(\mathcal{H}_R^{(\lambda)} \otimes \mathcal{H}_{\bar{R}}^{(\lambda)} \right)}$$

[Ohya & Petz (2004)]

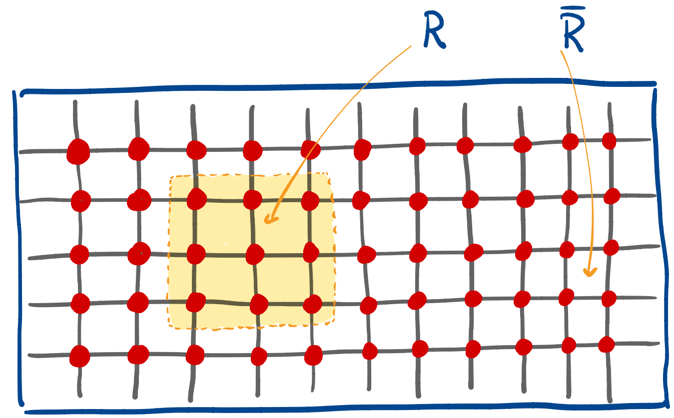
- State $|\psi\rangle \in \mathcal{H} \Rightarrow |\psi\rangle = \sum_{\lambda \in \mathcal{Z}} \sqrt{p_\lambda}\, |\phi_\lambda\rangle$ with $|\phi_\lambda\rangle \in \mathcal{H}_R^{(\lambda)} \otimes \mathcal{H}_{\bar{R}}^{(\lambda)}$
- Entropy:

$$\boxed{S_R(|\psi\rangle) = \sum_\lambda p_\lambda\, S_{\lambda R}(|\phi_\lambda\rangle) - \sum_\lambda p_\lambda \log p_\lambda}$$

Entanglement Entropy of $|\psi\rangle$ restricted to $\mathcal{A}_R$

▨ Example of system with constraint:

Bosonic lattice with N_0 excitations



- Kinematical Hilbert Space $\qquad \mathcal{H}_{kin} = \bigotimes_{i=1}^{V} \mathcal{H}_i$
- Constraint: fixed number of excitations $\qquad \boxed{\hat{C} = \hat{N} - N_0}$ $\quad$ where $\hat{N} = \sum_{i=1}^{V} a_i^\dagger a_i$
- Physical States: $\qquad \hat{C}\,|\psi\rangle = 0$
- Observables: $\qquad [\hat{O}, \hat{C}] = 0 \qquad$ (number preserving)
- Physical Hilbert Space $\qquad \mathcal{H}_{Phys}^{(N_0)} \subset \mathcal{H}_{kin}$
- Subsystem decomposition $\qquad \boxed{\mathcal{H}_{Phys}^{(N_0)} = \bigoplus_{N_R} \left(\mathcal{H}_{Phys\,R}^{(N_R)} \otimes \mathcal{H}_{Phys\,\bar{R}}^{(N_0-N_R)} \right)}$

number of excitations in R

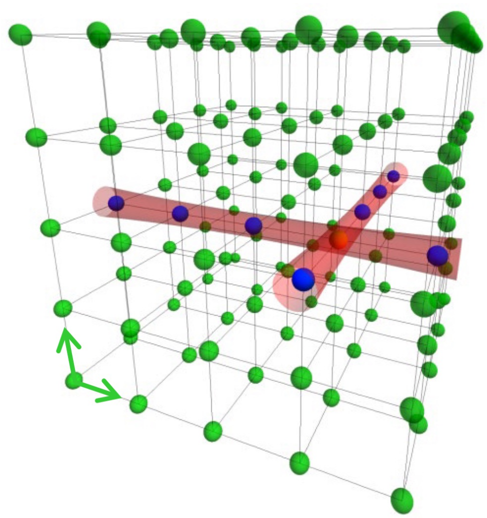
→ direct sum over of tensor products of Hilbert spaces at fixed N_R

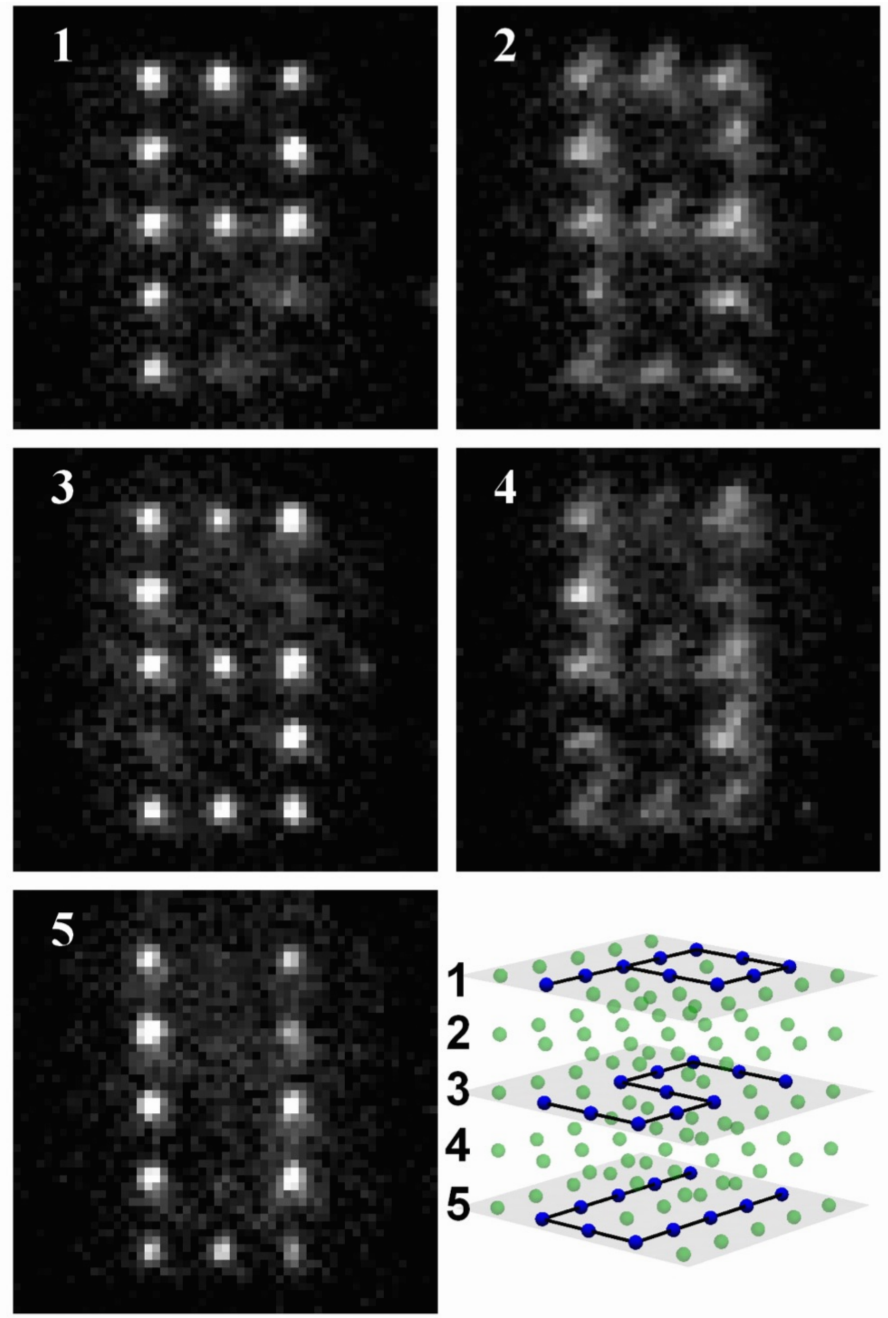
▨ Example of system with constraint:

Bosonic lattice with N_0 excitations

∗ Note: it can be realized in the lab as

ultracold atoms in an optical lattice

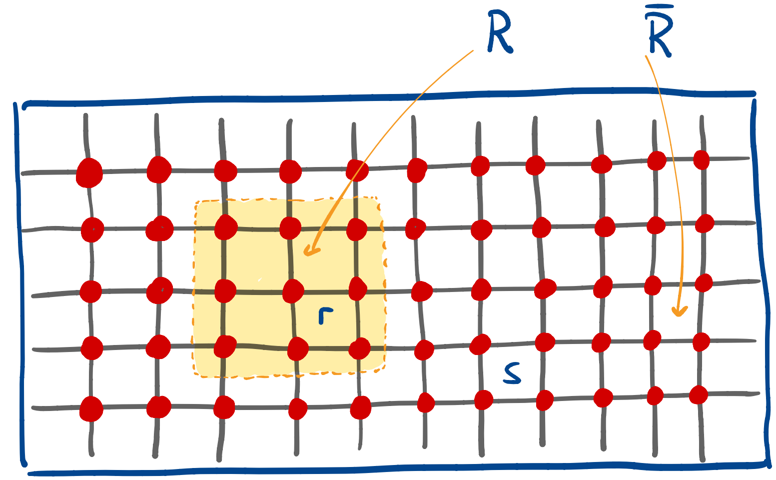




- 3D lattice: 5×5×5 sites (from ~850 nm lasers)
- lattice spacing ~5μm, wells ~200μK
- 32 atoms ^{133}Cs
- coherence time ~7s

[D. Weiss Lab at Penn State (2016)]

Bosonic lattice with N_0 excitations



- Observables, $[O, C] = 0$ $\quad C = \sum_{i=1}^{V} a_i^\dagger a_i - N_0$

  *note: $[a_i + a_i^\dagger, C] \neq 0$ not observable

- Observable Algebra $\mathcal{A}$ generated by hopping operator $\boxed{E_{ij} = \frac{1}{2}\left(a_i^\dagger a_j + a_j^\dagger a_i\right)}$

- Geometric Subsystem R: $\quad \mathcal{A}_R$ = generated by $E_{rr'}$ with $r \in R$
- Rest of the system: $\quad \mathcal{A}_{\bar{R}}$ = everything that commutes with $\mathcal{A}_R$

  $= \{$generated by $E_{ss'}$ with $s \in \bar{R}$ and $\hat{N}_R = \sum_{r \in R} a_r^\dagger a_r\}$

* Note: $\quad E_{rs} \notin \mathcal{A}_R \cup \mathcal{A}_{\bar{R}} \neq \mathcal{A}$

- Center $\mathcal{Z} = \mathcal{A}_R \cap \mathcal{A}_{\bar{R}} = \{$generated by $\hat{N}_R\}$
- Subsystem decomposition

$$\boxed{\mathcal{H}^{(N_0)}_{\text{Phys}} = \bigoplus_{N_R} \left(\mathcal{H}^{(N_R)}_{\text{Phys}\,R} \otimes \mathcal{H}^{(N_0 - N_R)}_{\text{Phys}\,\bar{R}} \right)}$$

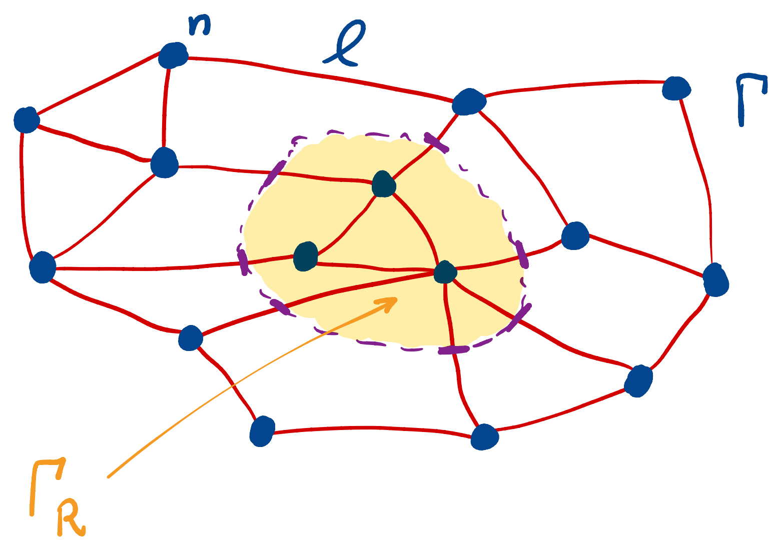
- o.n. basis adapted to the decomposition $\quad |\psi\rangle = \sum_{N_R} \sqrt{p(N_R)} \sum_{\alpha\beta} c_{\alpha\beta}(N_R)\, |\alpha, N_R\rangle \otimes |\beta, N_0 - N_R\rangle$

- Reduced density matrix $\quad \rho_R = \bigoplus_{N_R} \left(p(N_R) \;\; \rho_R(N_R) \right)$

- Entanglement Entropy:

$$\boxed{S_R(|\psi\rangle) = -\mathrm{Tr}(\rho_R \log \rho_R) = \sum_{N_R} p(N_R)\underbrace{\left(-\mathrm{Tr}(\rho_R(N_R) \log \rho_R(N_R)\right)}_{\text{Average over sectors}} \underbrace{- \sum_{N_R} p(N_R) \log p(N_R)}_{\text{Shannon Entropy of } p(N_R)}}$$

1 Geometric Subsystems & Spin-Networks



- Hilbert space of Spin Networks

$$
\boxed{
\begin{aligned}
\mathcal{H}_\Gamma &= \bigoplus_{\{j_e\}} \bigotimes_n \mathcal{H}_n^{(j_{n,e})} \\
&= \bigoplus_{\{\tilde{j}_e \in \partial R\}} \left(\mathcal{H}_R^{(\tilde{j}_e \in \partial R)} \otimes \mathcal{H}_{\bar{R}}^{(\tilde{j}_e \in \partial R)} \right)
\end{aligned}
}
$$

- Gauge-invariant state

$$
|\psi\rangle = \sum_{\{\tilde{j}_e\} \in \partial R} \sqrt{p(\tilde{j}_e)} \; \underbrace{\sum_{\{j'_e, i'_n\} \in R} \; \sum_{\{j''_e, i''_n\} \in \bar{R}} c(j'_e, i'_n, j''_e, i''_n) \, |j'_e, i'_n\rangle_R \, |j''_e, i''_n\rangle_{\bar{R}}}_{\equiv |\psi(\tilde{j}_e)\rangle}
$$

- Entanglement Entropy

$$
\boxed{
S_R(|\psi\rangle) = \underbrace{\sum_{\{\tilde{j}_e\}} p(\tilde{j}_e) \, S_R(|\psi(\tilde{j}_e)\rangle)} \; - \; \underbrace{\sum_{\{\tilde{j}_e\}} p(\tilde{j}_e) \log p(\tilde{j}_e)}
}
$$

Average over sectors at fixed $\{\tilde{j}_e\} \in \partial R$

Shannon Entropy of probability $p(\tilde{j}_e)$ of being in a sector

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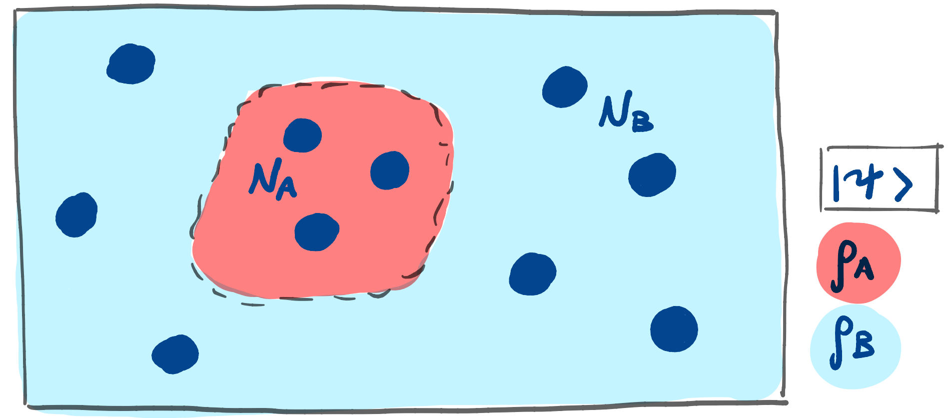
2 Typical Entanglement and the Page Curve

- N qubits in a pure state

$$|\psi\rangle = \sum_{i_1, \ldots, i_N = \pm 1} c_{i_1 \ldots i_n} |i_1\rangle \cdots |i_n\rangle$$



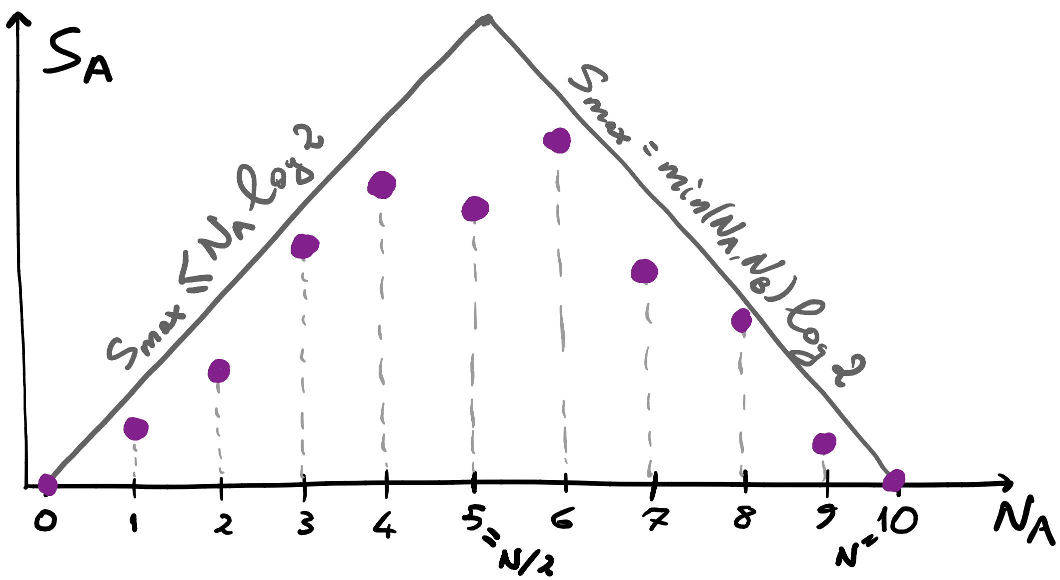
- Subsystem A of N_A spins

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$

- Entanglement Entropy in A

$$S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

- note: $S_A(|\psi\rangle) = S_B(|\psi\rangle)$



* Given a random pure state and a subsystem, what is the probability of finding entropy S_A?

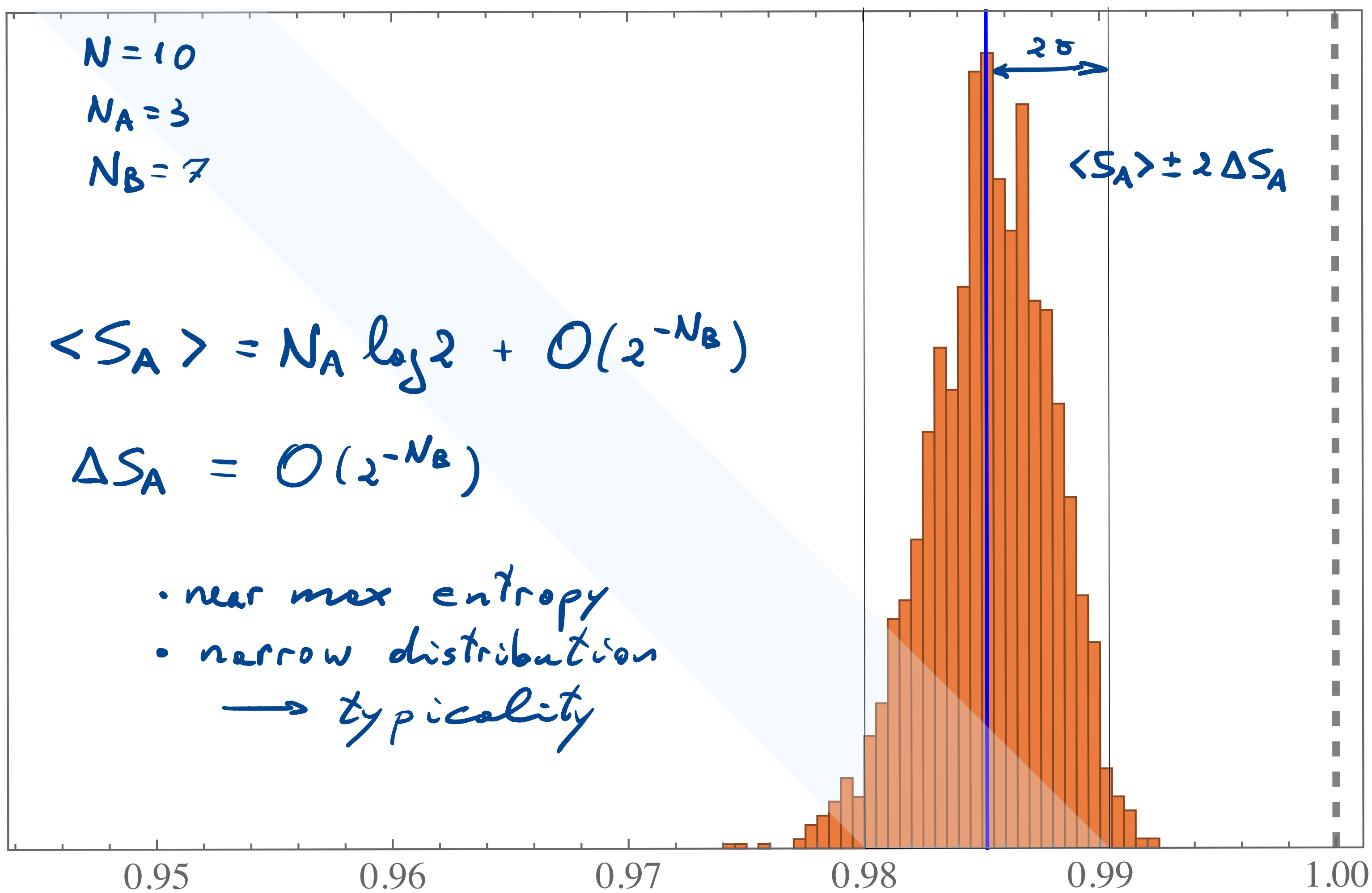
Example: N qubits

N: N_A | N_B

$$d_A = 2^{N_A}, \quad d_B = 2^{N_B}$$

$\langle S_A \rangle$

$P(S_A)$



$N = 10$
$N_A = 3$
$N_B = 7$

2σ

$\langle S_A \rangle \pm 2\Delta S_A$

$$\langle S_A \rangle = N_A \log 2 + O(2^{-N_B})$$

$$\Delta S_A = O(2^{-N_B})$$

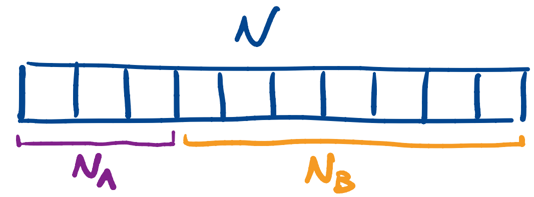
- near max entropy
- narrow distribution

$\longrightarrow$ typicality

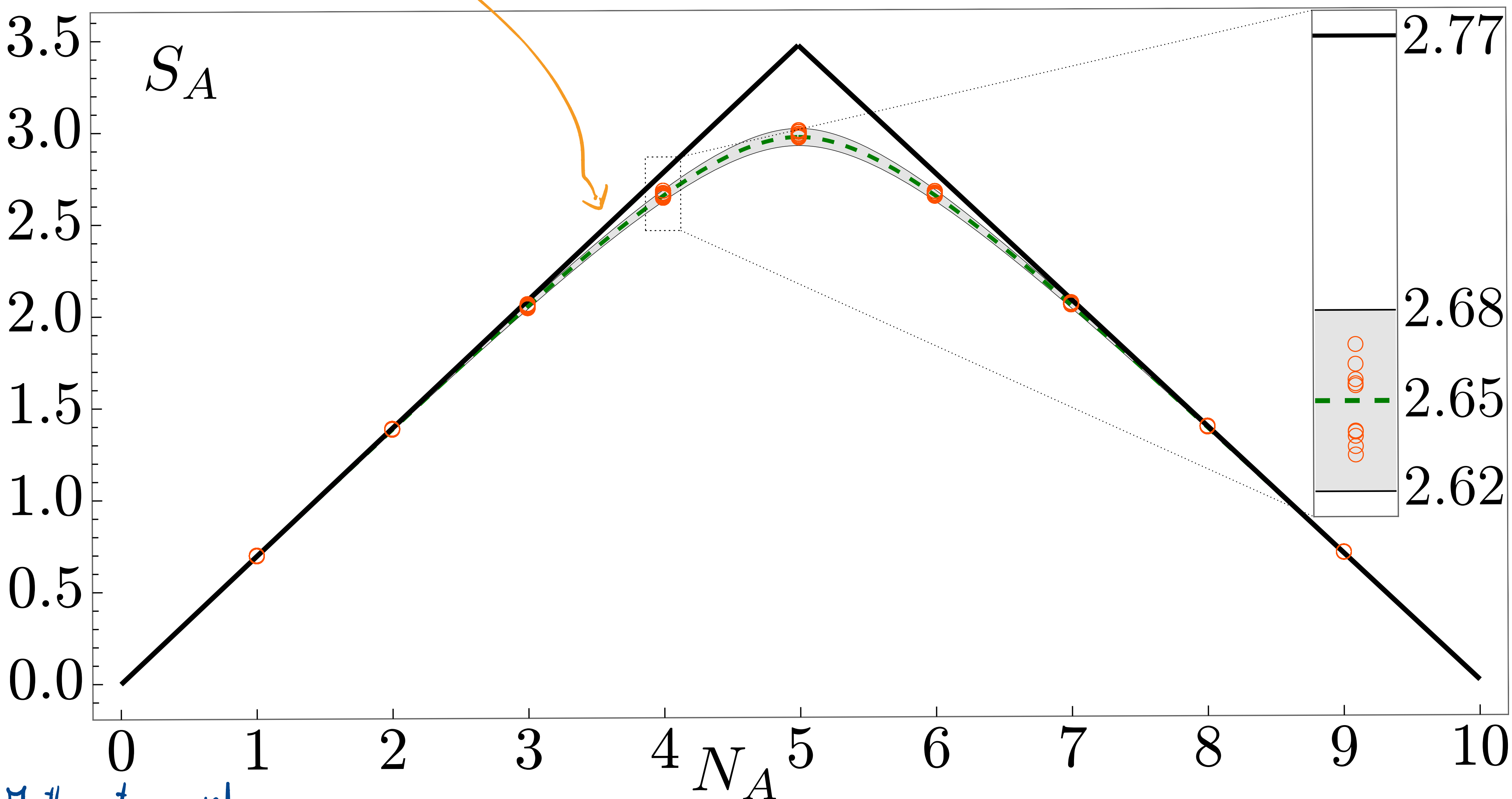
S_A / S_{max}

Mathematica .nb
bit.ly/LQG-Summer-School-2021

Example: N qubits



Page Curve



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2 Typical Entanglement and the Page Curve

- Hilbert Space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$d_A = \dim \mathcal{H}_A$
$d_B = \dim \mathcal{H}_B$

- Random Pure State $|\psi\rangle = \underline{U} \, |\psi_0\rangle$ ← Reference state

Random Unitary
from uniform probability distribution $d\mu(U)$ (Haar Measure)

- Entanglement Entropy $S_A(|\psi\rangle)$

* What is the probability of finding S_A?

→ compute $P(S_A)\, dS_A$

- Page, PRL '93: Average Entropy of a Subsystem $\langle S_A \rangle = \int S_A(U|\psi_0\rangle)\, d\mu(U)$

- Bianchi-Donà, PRD '19: Typical entropy $\langle S_A \rangle \pm \Delta S_A$

from moments $\mu_n = \int (S_A)^n \, P(S_A)\, dS_A$

- Bianchi-Hackl-Kieburg-Rigol-Vidmar, PRX '22: Ensembles & Constraints

2 Results:

- **Average Entropy** (Page, PRL '93)

$$\boxed{\langle S_A \rangle = \Psi(d_A d_B + 1) - \Psi(d_B + 1) - \frac{d_A - 1}{2 d_B}}$$

$1 < d_A \leq d_B$

$\Psi(x) = \Gamma'(x)/\Gamma(x)$ diGamma func.

- Asymptotics: $\langle S_A \rangle \underset{1 < d_A \ll d_B}{\approx} \underline{\log d_A} - \frac{1}{d_A d_B} \frac{d_A^2 - 1}{2} + O(1/d_B^2)$

- **Variance** (Bianchi-Donà, PRD '19) $(\Delta S_A)^2 = \langle S_A^2 \rangle - \langle S_A \rangle^2$

$$\boxed{(\Delta S_A)^2 = \frac{d_A + d_B}{d_A d_B + 1} \Psi'(d_B + 1) - \Psi'(d_A d_B + 1) - \frac{(d_A - 1)(d_A + 2 d_B - 1)}{4 d_B^2 (d_A d_B + 1)}}$$

- Asymptotics: $(\Delta S_A)^2 \underset{1 < d_A \ll d_B}{\approx} \frac{d_A^2 - 1}{2 d_A^2 d_B^2} + O(1/d_B^3)$

$\boxed{2}$ Technique:

- State $|\psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} w_{ab} \, |a\rangle |b\rangle$

  $\longrightarrow$ $(d_A \times d_B)$ matrix $W = [w_{ab}]$ with $\mathrm{Tr}(W W^\dagger) = 1$

- Density matrix $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) \longrightarrow \rho_A = W W^\dagger$

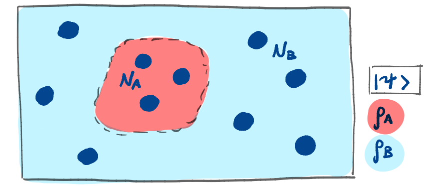
- Entropy $S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A) = -\partial_\varepsilon \mathrm{Tr}_A(\rho_A{}^\varepsilon)\Big|_{\varepsilon=1} = -\partial_\varepsilon \mathrm{Tr}(W W^\dagger)^\varepsilon \Big|_{\varepsilon=1}$

- Average Entropy

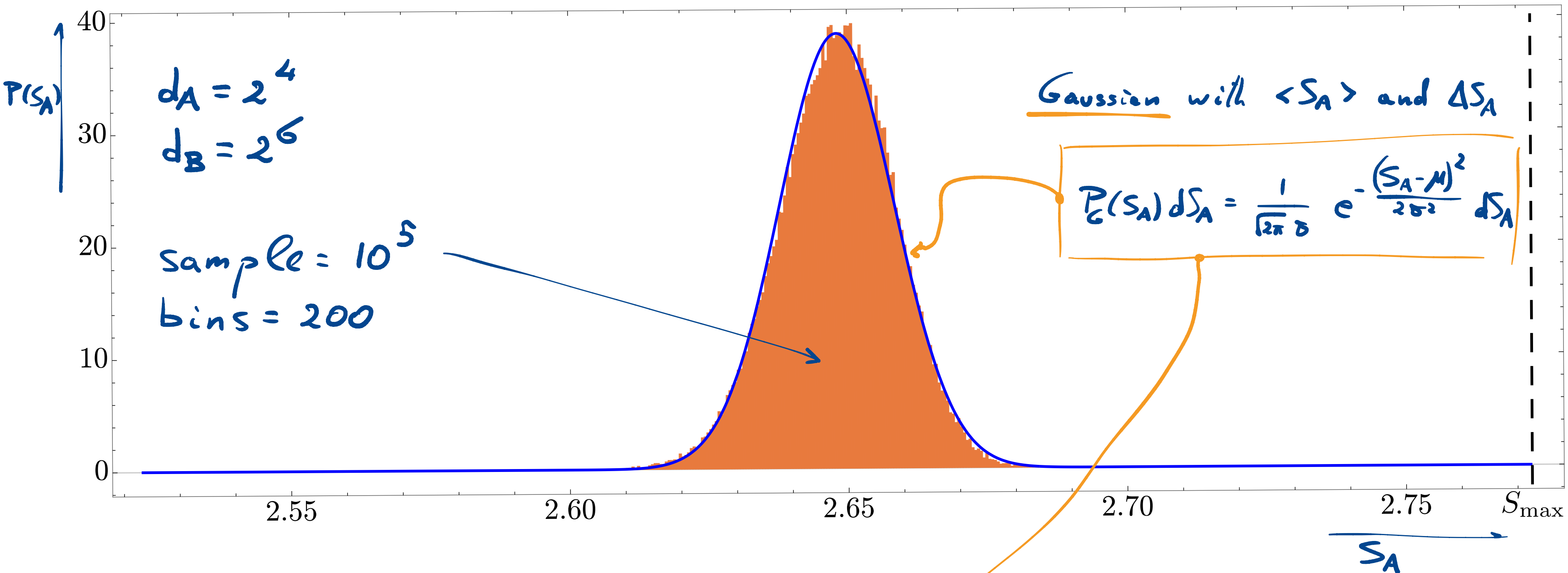
$$\langle S_A \rangle = \frac{1}{Z} \int \left(-\partial_\varepsilon \mathrm{Tr}(W W^\dagger)^\varepsilon \Big|_{\varepsilon=1} \right) \underbrace{\delta(1 - \mathrm{Tr}(W W^\dagger))}_{= \int_{-\infty}^{+\infty} dt \; e^{(1-it)(1-\mathrm{Tr} W W^\dagger)}} [dW \, dW^\dagger]$$

$\longrightarrow$ Fourier Transform of $e^{-\mathrm{Tr}(W W^\dagger)} [dW dW^\dagger]$

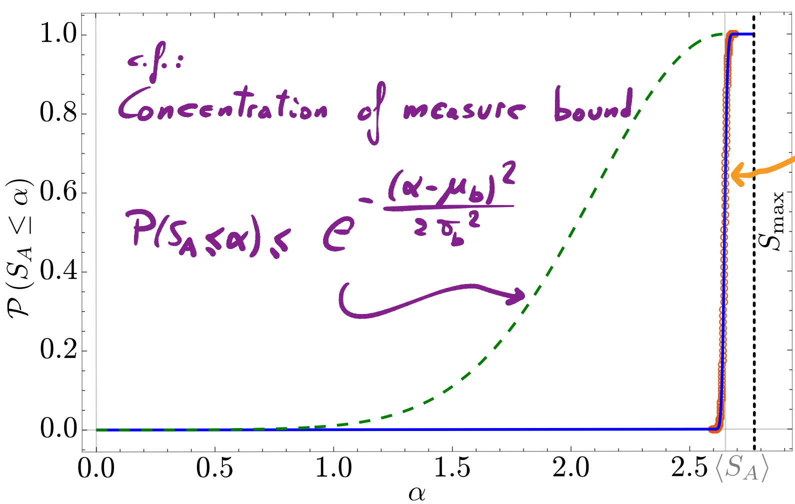
the Complex Wishart-Laguerre Ensemble in Random Matrix Th.



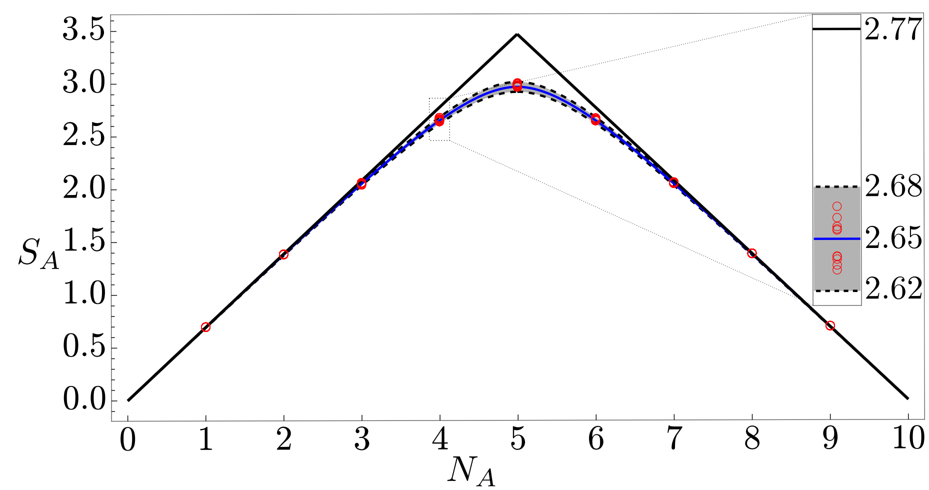
$\boxed{2}$ Probability Distribution $P(S_A)\,dS_A$



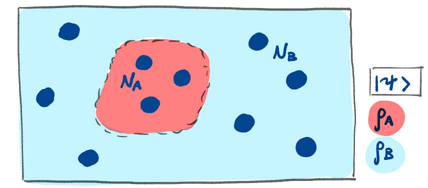
• Cumulative Distribution

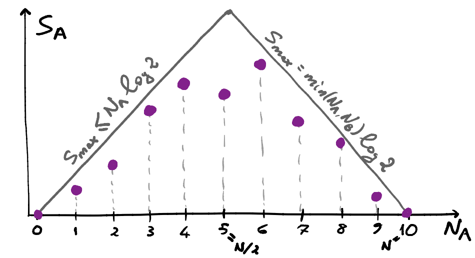


• Page Curve



2 Typical Entanglement with Constraints





- Page '93: qubit model of unitary black hole evaporation and typical entanglement entropy of a subsystem (note: no Hamiltonian)

* Assumptions:

- Finite Dimension $\dim \mathcal{H}$
$$\begin{cases} \text{qubits} & \checkmark \\ \text{QFT} & \times \\ \text{QG} & ? \end{cases}$$

- Factorization $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
$$\begin{cases} \text{qubits} & \checkmark \\ \text{QFT} & \times \quad \text{(Reeh-Schlieder Thm)} \\ \text{QG} & \times \end{cases}$$

- Constraints: $\hat{N}|\psi\rangle = N|\psi\rangle$, $\mathcal{H} = \bigoplus_N \mathcal{H}^{(N)}$

$\mathcal{H}^{(N)} \subset \mathcal{H}_A \otimes \mathcal{H}_B$

finite dim, non-factorized

e.g.:
$$\begin{cases} \text{qubits} \rightarrow \text{fixed } N \text{ excitations} \\ \text{QFT} \longrightarrow \text{fixed energy} \\ \text{QG} \longrightarrow \text{Diff \& H constraints} \\ \qquad \text{BH mass} \rightarrow \text{fixed energy} \end{cases}$$

2 Entanglement with Constraints : two perspectives

- Kinematical Hilbert Space with Constraints $\Rightarrow \mathcal{H}^{(phys)}$
  - Kinematical Hilbert Space with Tensor Prod. $\mathcal{H}^{(kin)} = \mathcal{H}_A \otimes \mathcal{H}_B$
  - Constraint $C = C_A + C_B$
  - Physical State $|\psi\rangle \in \mathcal{H}^{(kin)}$ s.t. $C|\psi\rangle = 0$

$$\Rightarrow \quad C_A|\psi\rangle = -C_B|\psi\rangle$$

  - Decomposition of the Hilbert space (with λ = eigenval C_A)

$$\boxed{\mathcal{H}^{(phys)} = \bigoplus_{\lambda \in \mathbb{Z}} \left(\mathcal{H}_A^{(\lambda)} \otimes \mathcal{H}_B^{(\lambda)}\right)}$$

  - State $|\psi\rangle \in \mathcal{H}^{(phys)} \Rightarrow |\psi\rangle = \sum_{\lambda \in \mathbb{Z}} \sqrt{p_\lambda}\, |\phi_\lambda\rangle$ with $|\phi_\lambda\rangle \in \mathcal{H}_A^{(\lambda)} \otimes \mathcal{H}_B^{(\lambda)}$
  - Entropy of $|\psi\rangle$ restricted to $\mathcal{A}_A$

$$\boxed{S_A(|\psi\rangle) = \sum_\lambda p_\lambda\, S_{\lambda A}(|\phi_\lambda\rangle) - \sum_\lambda p_\lambda \log p_\lambda}$$

2 Entanglement with Constraints : two perspectives

- **Operational def. of a subsystem and its entropy**

- $\mathcal{A}$ algebra of observables of the system
- $\mathcal{A}_A \subset \mathcal{A}$ observables that define the subsystem A
- $\mathcal{A}_B = \{ b \in \mathcal{A} \mid [a,b] = 0 \ \forall a \in \mathcal{A}_A \}$ rest of the system

* Note: $\mathcal{Z} = \mathcal{A}_A \cap \mathcal{A}_B$ center, in general non-trivial

- Decomposition of the Hilbert space

$$\boxed{\mathcal{H}^{(phys)} = \bigoplus_{\lambda \in \mathcal{Z}} \left(\mathcal{H}_A^{(\lambda)} \otimes \mathcal{H}_B^{(\lambda)} \right)}$$

Proof: in Ohya & Petz (2004)

- State $|\psi\rangle \in \mathcal{H}^{(phys)} \Rightarrow |\psi\rangle = \sum_{\lambda \in \mathcal{Z}} \sqrt{p_\lambda}\, |\phi_\lambda\rangle$ with $|\phi_\lambda\rangle \in \mathcal{H}_A^{(\lambda)} \otimes \mathcal{H}_B^{(\lambda)}$
- Entropy of $|\psi\rangle$ restricted to $\mathcal{A}_A$

$$\boxed{S_A(|\psi\rangle) = \sum_\lambda p_\lambda\, S_{\lambda A}(|\phi_\lambda\rangle) - \sum_\lambda p_\lambda \log p_\lambda}$$

2 Typical Entanglement with Constraints

- Direct-sum Hilbert space $\mathcal{H}^{(phys)} = \bigoplus_\lambda \left(\mathcal{H}_A^{(\lambda)} \otimes \mathcal{H}_B^{(\lambda)} \right)$

- Dimensions: $d_{A\lambda} = \dim \mathcal{H}_A^{(\lambda)}$, $d_{B\lambda} = \dim \mathcal{H}_B^{(\lambda)}$

$$d_{phys} = \dim \mathcal{H}^{(phys)} = \sum_\lambda d_{A\lambda}\, d_{B\lambda}$$

- Random Pure state $|\varphi\rangle \in \mathcal{H}^{(phys)}$

- Typical Entanglement Entropy for Subalgebra A

$$\boxed{\begin{aligned}\langle S_A \rangle_{phys} = & \sum_{\lambda=\lambda_{min}}^{\lambda_*} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{B\lambda}+1) - \frac{d_{A\lambda}-1}{2 d_{B\lambda}} \right) \\ & + \sum_{\lambda=\lambda_*+1}^{\lambda_{max}} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{A\lambda}+1) - \frac{d_{B\lambda}-1}{2 d_{A\lambda}} \right)\end{aligned}}$$

where: λ_* s.t. $\boxed{d_{A\lambda_*} = d_{B\lambda_*}}$ and $\Psi(x) = \Gamma'(x)/\Gamma(x)$

[Bianchi-Donà, PRD'19]

* C.f.: Page '93 for trivial $\sum_\lambda$

Typical Entanglement in Quantum Gravity

1 Entanglement Entropy in Constrained Systems

$$\boxed{\mathcal{H} = \bigoplus_{\lambda \in \mathbb{Z}} \left(\mathcal{H}_R^{(\lambda)} \otimes \mathcal{H}_{\bar{R}}^{(\lambda)} \right)}$$

2 Typical Entanglement with Constraints

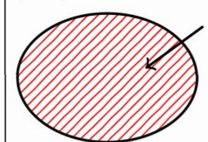
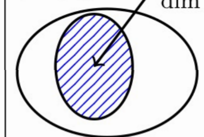
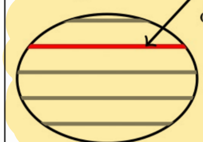
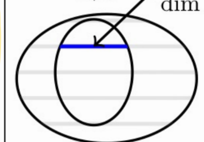
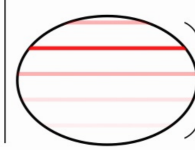
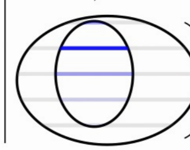
$$\boxed{\langle S_A \rangle_{phys} = \sum_{\lambda=\lambda_{min}}^{\lambda_*} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{B\lambda}+1) - \frac{d_{A\lambda}-1}{2 d_{B\lambda}} \right) + \sum_{\lambda=\lambda_*+1}^{\lambda_{max}} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{A\lambda}+1) - \frac{d_{B\lambda}-1}{2 d_{A\lambda}} \right)}$$

3 Applications:

- CM
- QFT
- BH
- QG

$\boxed{3}$ [CM]: Typical Entanglement in Energy-Eigenstates of Many-Body Hamiltonians

$\rightsquigarrow$ Exploration of new phenomena in constrained systems $|E, N\rangle$

	Physical Hamiltonians: (a) Interacting	Physical Hamiltonians: (b) Quadratic*	Characteristic ensembles: (a) General pure states	Characteristic ensembles: (b) Pure Gaussian states
(1) arbitrary N	$\hat{H} = \sum(t_{ij}\hat{a}_i^\dagger \hat{a}_j + d_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger + \text{H.c.}) + \sum(v_{ijkl}\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_l^\dagger + \cdots + \text{H.c.}) + \ldots$	$\hat{H}_{\text{G}} = \sum(t_{ij}\hat{a}_i^\dagger \hat{a}_j + d_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger + \text{H.c.})$	$\langle S_A \rangle$ full Hilbert space $\mathcal{H}$, $\dim \mathcal{H} = 2^V$ 	$\langle S_A \rangle_{\text{G}}$ Gaussian submanifold $\mathcal{M}$, $\dim \mathcal{M} = V(V-1)$ 
(2) fixed N	$\hat{H}_N = \sum t_{ij}\hat{a}_i^\dagger \hat{a}_j + \sum(v_{ijkl}\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l + \cdots + \text{H.c.}) + \ldots$	$\hat{H}_{\text{G},N} = \sum t_{ij}\hat{a}_i^\dagger \hat{a}_j$	$\langle S_A \rangle_N$ subspace $\mathcal{H}^{(N)}$, $\dim \mathcal{H}^{(N)} = \binom{V}{N}$ 	$\langle S_A \rangle_{\text{G},N}$ Gaussian submanifold $\mathcal{M}_N$, $\dim \mathcal{M}_N = 2N(V-N)$ 
	eigenstates of $\hat{H}_N$ or $\hat{H}_{\text{G},N}$ with fixed N			
(3) fixed w	weighted average over all eigenstates of $\hat{H}_N$ or $\hat{H}_{\text{G},N}$ (equal weight corresponds to $w = 0$)		$\langle S_A \rangle_w = \sum_N \binom{V}{N} \frac{e^{-wN}}{Z} \langle S_A \rangle_N$  weighted average over fixed N averages	$\langle S_A \rangle_{\text{G},w} = \sum_N \binom{V}{N} \frac{e^{-wN}}{Z} \langle S_A \rangle_{\text{G},N}$  weighted average over fixed N Gaussian averages

	(a) General pure states		**(b) Pure fermionic Gaussian states**	
(1) no particle number	$\langle S_A \rangle = aV - b + O(2^{-V})$ & **exact**	$\to$ (25), Fig. 3, [148]	$\langle S_A \rangle_{\text{G}} = aV + b + O(\frac{1}{V})$ & **exact**	$\to$ (90), Fig. 8, [155]
	$(\Delta S_A)^2 = \alpha e^{-\beta V} + o(e^{-\beta V})$	$\to$ (29), [161]	$(\Delta S_A)^2_{\text{G}} = a + o(1)$	$\to$ (94), [155]
(2) fixed particle number	$\langle S_A \rangle_N = aV - b\sqrt{V} - c + o(1)$	$\to$ (54), Fig. 6	$\langle S_A \rangle_{\text{G},N} = aV - \frac{b}{V} + O(\frac{1}{V^2})$ & **exact**	$\to$ (91)
	$(\Delta S_A)^2_N = \alpha\, V^{\frac{3}{2}}\, e^{-\beta V}$	$\to$ (60)	$(\Delta S_A)^2_{\text{G},N} = a + o(1)$	$\to$ (117), Fig. 9
(3) fixed weight	$\langle S_A \rangle_w = aV + b + c\sqrt{V} + o(1)$	$\to$ (67)	$\langle S_A \rangle_{\text{G},w} = aV + b + \frac{c}{\sqrt{V}} + \frac{d}{V} + o(\frac{1}{V})$	$\to$ (119), Fig. 10
	$(\Delta S_A)^2_w = aV + o(V)$	$\to$ (68)	$(\Delta S_A)^2_{\text{G},w} = aV + o(V)$	$\to$ (121)

[Bianchi-Hackl-Kieburg-Rigol-Vidmar, PRX '22]

3 [CM]: Eigenstates of Random Many-Body Hamiltonian with Number Conservation

- Fermionic system $\{a_i, a_j^\dagger\} = \delta_{ij}$, $i = 1, \dots, V$, $\dim \mathcal{H} = 2^V$

- Random many-body Hamiltonian with number conservation: $[H, \hat{N}] = 0$, $\hat{N} = \sum_i a_i^\dagger a_i$

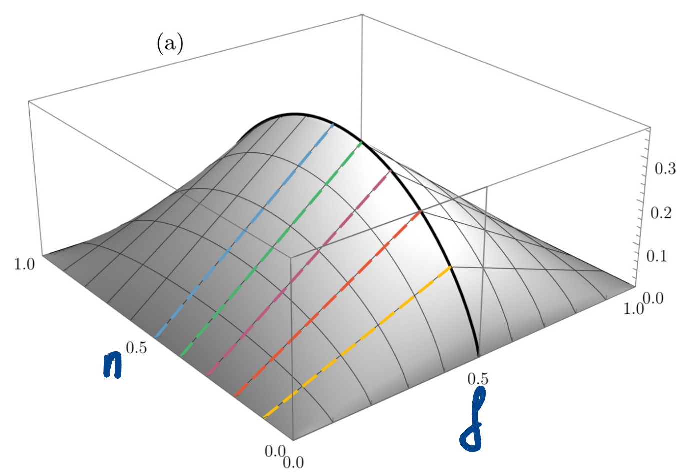
$$H = \sum_{\alpha,\beta}^{2^V} M_{\alpha\beta} \, |\alpha\rangle\langle\beta| = \sum_{q=2}^{2V} H^{(q)}_{SYK} \qquad \text{with} \qquad H^{(q)}_{SYK} = \sum_{i=1}^{2V} J_{i_1 \dots i_q} \, a^\dagger_{i_1} \dots a^\dagger_{i_{q/2}} \, a_{i_{q/2+1}} \dots a_{i_q}$$

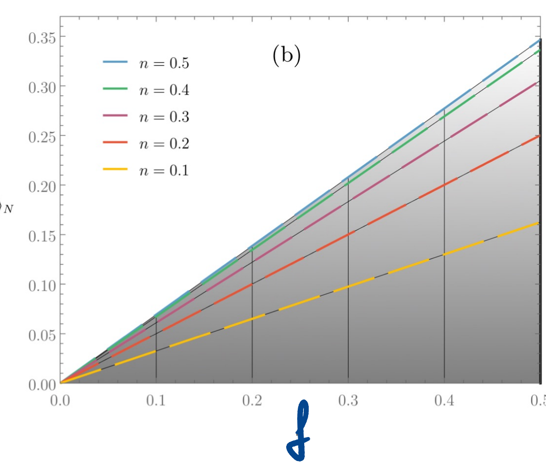
- Ensemble of energy eigenstates $|E, N\rangle$ at fixed N

  $\longrightarrow$ uniform distribution in $\mathcal{H}^{(N)} \subset \mathcal{H} = \bigoplus_N \mathcal{H}^{(N)}$

- Page curve $\langle S_A \rangle_N$ $*$note: $\mathcal{H}^{(N)} \neq \mathcal{H}_A \otimes \mathcal{H}_B$

- subsystem fraction $\boxed{f = \frac{V_A}{V}}$
- filling fraction $\boxed{n = \frac{N}{V}}$





[Bianchi-Hackl-Kieburg-Rigol-Vidmar, PRX '22]

3 [CM]: Eigenstates of Random Many-Body Hamiltonian with Number Conservation

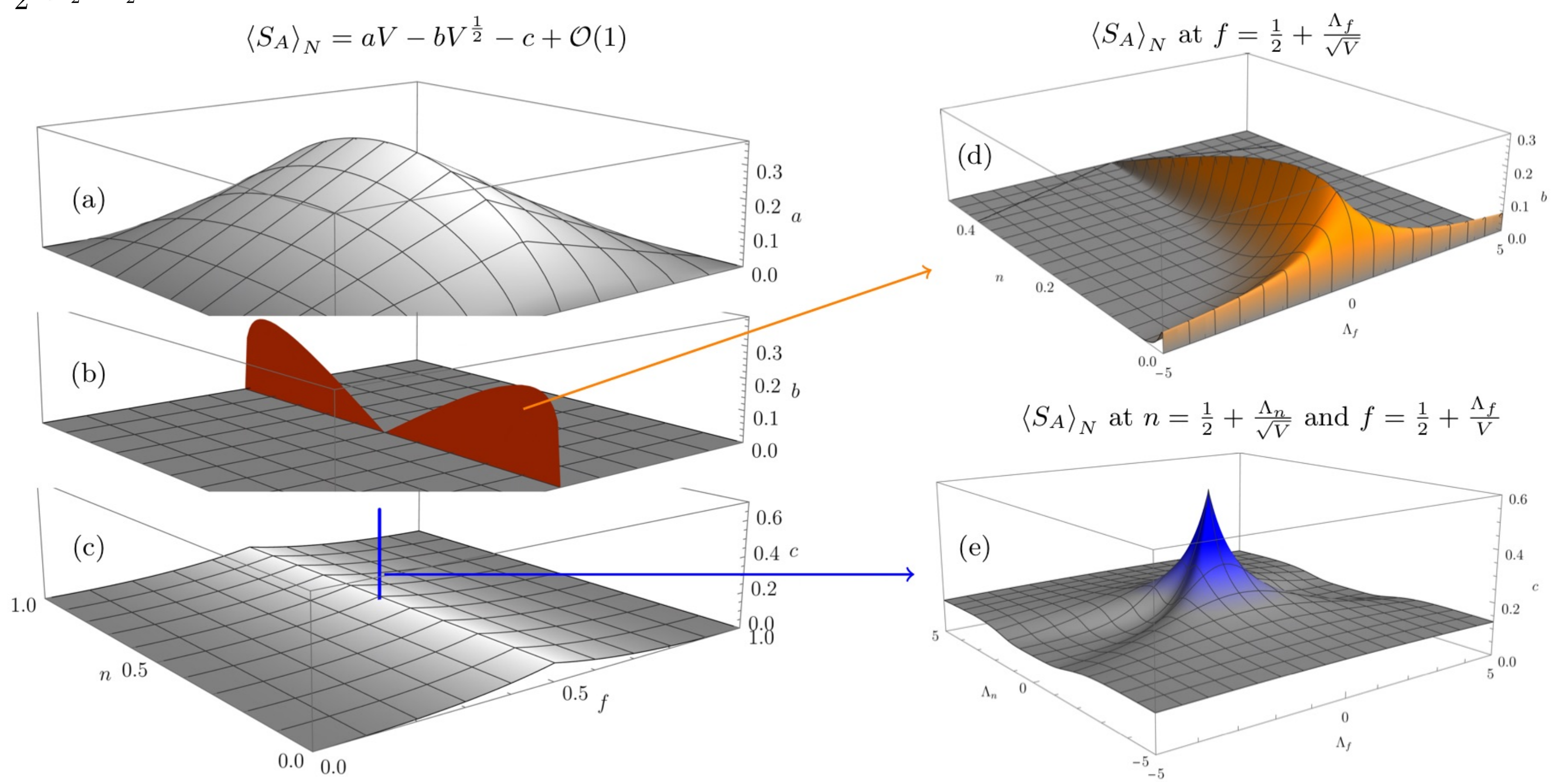
* Remarks : · volume law with n-dependent slope

→ Thermodynamics of a paramagnet from entanglement

· $\sqrt{V}$ subextensive term at half-system size

(first observed numerically by Vidmar & Rigol '17)

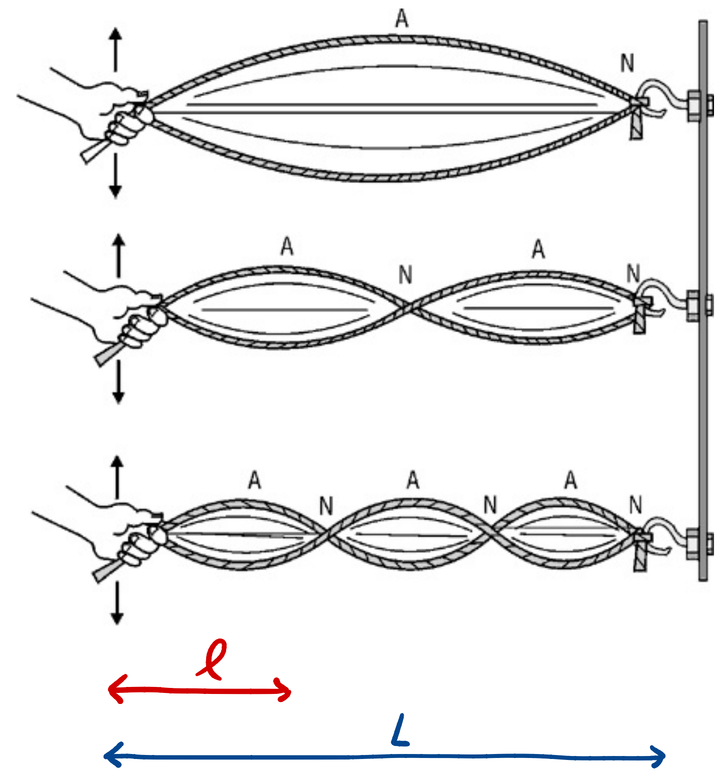
$$\langle S_A\rangle_N = [(n-1)\ln(1-n) - n\ln(n)]\, f\, V$$
$$- \sqrt{\frac{n(1-n)}{2\pi}}\left|\ln\left(\frac{1-n}{n}\right)\right|\delta_{f,\frac{1}{2}}\sqrt{V}$$
$$+ \frac{f+\ln(1-f)}{2} - \frac{1}{2}\delta_{f,\frac{1}{2}}\delta_{n,\frac{1}{2}} + o(1),$$



[Bianchi-Hackl-Kieburg-Rigol-Vidmar, PRX '22]

$\boxed{3}$ [QFT]: Typical Entanglement Entropy & Black Body Thermodynamics

* Can we derive black body thermodynamics from typical entanglement in an energy eigenstate?



- Electromagnetic field in a box, volume L^3, energy E, with $EL \gg \hbar c$
- Hilbert space: tensor prod. over wavelengths λ

$$\mathcal{H} = \bigotimes_\lambda \mathcal{H}_\lambda \qquad \text{with} \quad \lambda = \left(\frac{2L}{k_x}, \frac{2L}{k_y}, \frac{2L}{k_z}\right)$$

- Subsystem A: antenna of length ℓ

  $\rightsquigarrow$ picks wavelengths $\lambda_A = \left(\frac{2\ell}{k_x}, \frac{2\ell}{k_y}, \frac{2\ell}{k_z}\right)$

* What is the typical entropy measured by the antenna in a random eigenstate $|E,\alpha\rangle$?

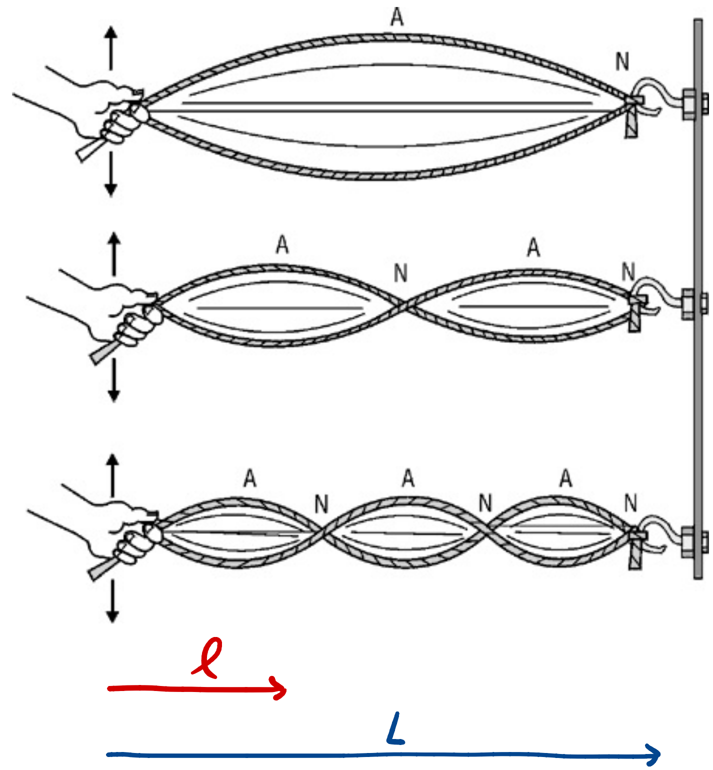
[Bianchi-Donà, PRD '19]

3 [QFT]: Typical Entanglement Entropy & Black Body Thermodynamics

- Energy Eigenspace and Partition of Energy

$$\boxed{\mathcal{H}(E) = \bigoplus_j \left(\mathcal{H}_A(\varepsilon_j) \otimes \mathcal{H}_B(E - \varepsilon_j) \right)}$$

antenna



- Typical Entanglement Entropy

(use exact, center formula + saddle approx)

$$\boxed{\langle S_A \rangle_E \approx \langle S_{A\bar{\varepsilon}} \rangle \approx \log d_{A\bar{\varepsilon}} \approx \frac{4\sqrt{\pi}}{3(15)^{1/4}} \left(\frac{\bar{\varepsilon}\,\ell}{\hbar c} \right)^{3/4}}$$

- Thermodynamics from Entanglement:

$$kT \equiv \left(\frac{\partial \langle S_A \rangle_E}{\partial \bar{E}_A} \right)^{-1}_{L,\ell} \Rightarrow \langle S_A \rangle_E \approx \frac{4\pi^2}{45} \left(\frac{kT}{\hbar c} \right)^3 \min(\ell^3, L^3 - \ell^3)$$

[Bianchi-Donà, PRD '19]

3 [BH]: Typical Entanglement and Black Hole Entropy

- Quantum Gravity in Asymptotically Flat Spacetime
  - Gravitational Hilbert Space at Fixed ADM Energy E

$$\dim \mathcal{H}_{grav}(E, E+\delta E) = \mathrm{Tr}\left(\delta(\hat{H}_{grav} - E)\right) \delta E = \nu(E)\, \delta E$$

  - Semiclassical Computation: Microcanonical from Canonical via Laplace

$$Z(\beta) = \int_0^\infty \nu(E)\, e^{-\beta E}\, dE = \int \mathcal{D}g_{\mu\nu}\; e^{-I_\beta[g_{\mu\nu}]/\hbar}$$

$$\downarrow$$

$$\nu(E) = \int_{\gamma - i\infty}^{\gamma + i\infty} Z(\beta)\, e^{+\beta E} \frac{d\beta}{2\pi i} \approx \mathcal{N}\, e^{+4\pi \frac{G E^2}{\hbar}}$$

Gibbons-Hawking '74
Brown-York '93

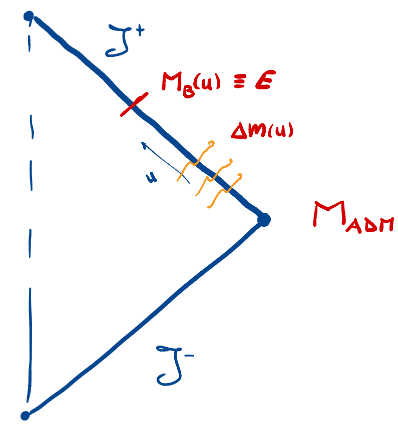
* Note: · spherically symmetric geom. $\Rightarrow$ $\nu_0(E) = e^{+4\pi \frac{GE^2}{\hbar}}$

· radiative perturbations $\nu_{pert}(E) = \mathcal{N} = \infty$

3 [BH]: Typical Entanglement and Black Hole Entropy

* Can we derive BH entropy from typical entanglement in a QG eigenstate $|E,\alpha\rangle$?

* What subalgebra of observables corresponds to BH thermodyn energy exchanges?



→ A tentative proposal

- Gravitational Subsystem: radiative perturbations

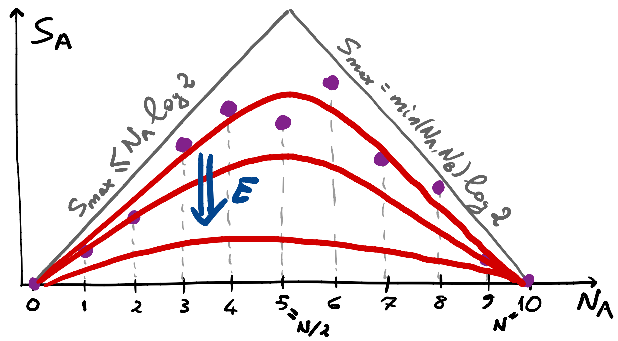
$$\boxed{\mathcal{H} = \bigoplus_{\Delta m} \left(\mathcal{H}_{\substack{\text{grav}\\ \text{coulombic}}}^{(M_B = M_{ADM} - \Delta m)} \otimes \mathcal{H}_{\substack{\text{grav}\\ \text{radiative}}}^{(\Delta m)} \right)}$$

- Typical Entanglement Entropy → Bekenstein-Hawking Formula

$$\boxed{\langle S_{\text{grav}}^{(\ell=0)} \rangle_E} \approx \log\Big(\min\big[\underbrace{\dim \mathcal{H}_{\substack{\text{grav}\\ \text{coulombic}}}^{(E,\bar{\Delta m})}}_{\text{finite}},\ \underbrace{\dim \mathcal{H}_{\substack{\text{grav}\\ \text{radiative}}}^{(\bar{\Delta m})}}_{=\infty}\big]\Big) \approx 4\pi \frac{G E^2}{\hbar} = \boxed{\frac{A(E)}{4G\hbar}}$$

2 [QG]: Entanglement & the Architecture of Spacetime

- In CM & QFT, as we lower the energy, we transition from volume-law to area-law
- In QG with asymptotically flat b.c.
  → expect similar behavior for the geometric entanglement entropy

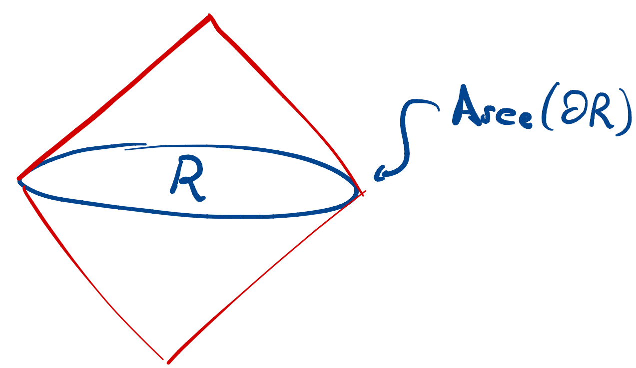


* In QG in finite regions (general boundary in H. Haggard's Lec.)
⤳ no* notion of energy (*see L. Freidel's Talk)

⟹ Reverse Perspective: Entanglement as a Probe

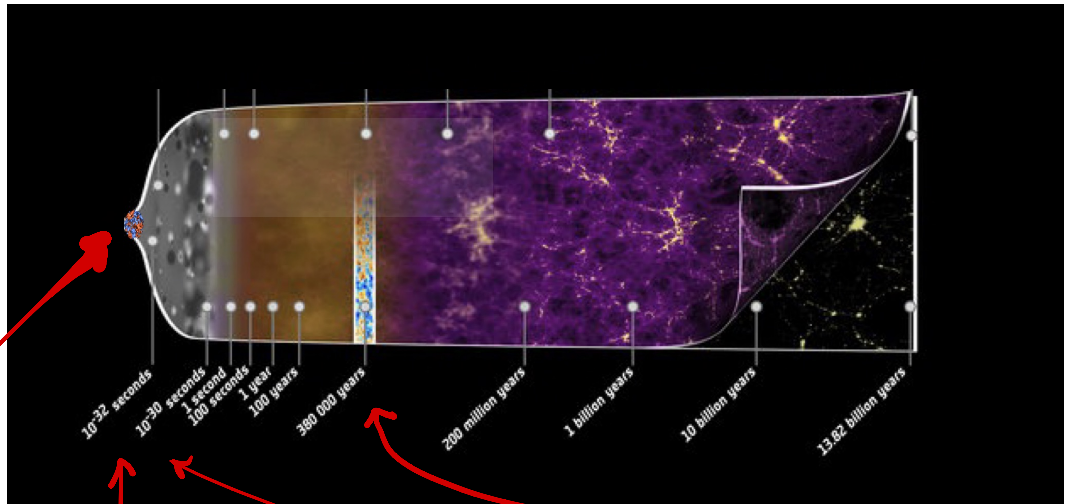
Architecture Conjecture

> Semiclassical $|\psi\rangle$ in QG belong to the area-law corner of $\mathcal{H}_{phys}$
>
> $$S_R(|\psi\rangle) = 2\pi \frac{\mathrm{Area}(\partial R)}{\ell_P^2} + \dots$$

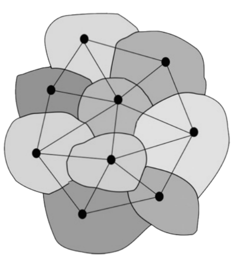
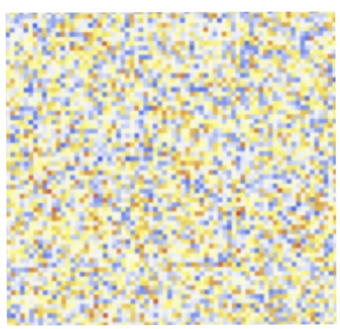
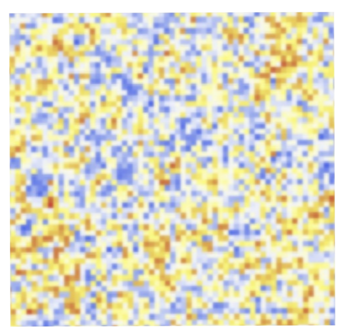
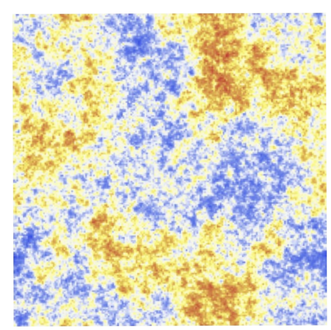
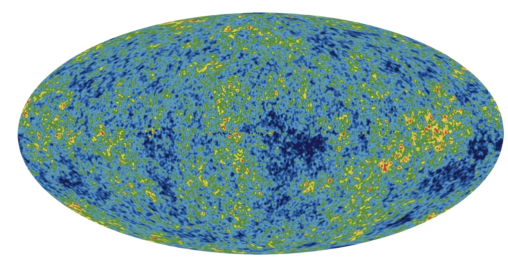


Bianchi - Myers [1212.5183]
Bianchi - Guglielmo - Hackl - Yokomizo [1605.05356]

$\boxed{2}$ [QG]: Scenario for the Production of Primordial Entanglement



Planck Scale → Pre-Inflationary Phase → Inflation → Hot Big Bang & CMB

Gozzini - Vidotto [1906.02211]

Bianchi - Hackl - Yokomizo [1512.08959]

Typical Entanglement in Quantum Gravity

1 Entanglement Entropy in Constrained Systems

$$\boxed{\mathcal{H} = \bigoplus_{\lambda \in \mathcal{Z}} \left(\mathcal{H}_R^{(\lambda)} \otimes \mathcal{H}_{\bar{R}}^{(\lambda)} \right)}$$

2 Typical Entanglement with Constraints

$$\boxed{\langle S_A \rangle_{phys} = \sum_{\lambda=\lambda_{min}}^{\lambda_*} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{B\lambda}+1) - \frac{d_{A\lambda}-1}{2 d_{B\lambda}} \right) + \sum_{\lambda=\lambda_*+1}^{\lambda_{max}} \frac{d_{A\lambda}\, d_{B\lambda}}{d_{phys}} \left(\Psi(d_{phys}+1) - \Psi(d_{A\lambda}+1) - \frac{d_{B\lambda}-1}{2 d_{A\lambda}} \right)}$$

3 Applications:

- **CM**: Paramagnet Thermodyn. from Typical Entanglement
- **QFT**: Black Body Thermodyn. from Typical Entanglement
- **BH**: Black Hole Thermodyn. from Typical Entanglement
- **QG**: Architecture of Spacetime Geometry