Scalar curvature operator for loop quantum gravity on a cubical graph

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Scalar curvature in loop quantum gravity

The object of interest: Ricci scalar integrated over the spatial manifold

$$\int_\Sigma d^3x\,\sqrt{q}\,{}^{(3)}R$$

Relevant to loop quantum gravity

- As a geometrical observable
- As an alternative to the Lorentzian part of the Hamiltonian constraint

$$C = \frac{1}{\beta^2}\frac{\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab}}{\sqrt{|\det E|}} + (1+\beta^2)\sqrt{|\det E|}\,{}^{(3)}R$$

Previously: The "Regge" curvature operator (Alesci, Assanioussi, Lewandowski 2014)

$$\int d^3x\,\sqrt{q}\,{}^{(3)}R \simeq \sum_{\text{hinges}} (\text{hinge length}) \times (\text{deficit angle})$$

Refers classically to an auxiliary manifold of singular geometry (curvature concentrated on one-dimensional hinges) instead of the actual physical manifold.

Ricci scalar as a function of the Ashtekar variables

The starting point of our construction: Express the Ricci scalar as ${}^{(3)}R(A, E)$

$$q^{ab} = \frac{E^a_i E^b_i}{|\det E|} \qquad {}^{(3)}R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc}$$

$$\begin{aligned}
|\det E|\, {}^{(3)}R &= -2E^a_i \mathcal{D}_{(a}\mathcal{D}_{b)} E^b_i + 2Q^{ab} E^i_c \mathcal{D}_{(a}\mathcal{D}_{b)} E^c_i \\
&- (\mathcal{D}_a E^a_i)(\mathcal{D}_b E^b_i) - \frac{1}{2}(\mathcal{D}_a E^b_i)(\mathcal{D}_b E^a_i) \\
&+ \frac{5}{2} Q^{ab} (\mathcal{D}_a E^c_i)(\mathcal{D}_b E^i_c) - \frac{1}{2} Q^{ab} Q_{cd} (\mathcal{D}_a E^c_i)(\mathcal{D}_b E^d_i) \\
&+ 2\mathcal{A}^{ab}{}_a \mathcal{B}_{cb}{}^c + 2\mathcal{A}^{ab}{}_b \mathcal{B}_{ca}{}^c + \mathcal{A}^{ab}{}_c \mathcal{B}_{ba}{}^c \\
&+ \frac{1}{2} Q_{ab} \mathcal{A}^{ca}{}_d \mathcal{A}^{db}{}_c - Q^{ab} \mathcal{B}_{ca}{}^c \mathcal{B}_{db}{}^d \\
&+ 2\big(Q^{ab}\mathcal{B}_{ca}{}^c - \mathcal{A}^{ab}{}_a - \mathcal{A}^{ba}{}_a\big) \frac{\partial_b |\det E|}{|\det E|} \\
&+ \frac{3}{2} Q^{ab} \frac{\partial_a |\det E|}{|\det E|} \frac{\partial_b |\det E|}{|\det E|} - 2Q^{ab} \frac{\partial_a \partial_b |\det E|}{|\det E|}
\end{aligned}$$

$$\begin{aligned}
Q^{ab} &= E^a_i E^b_i \\
Q_{ab} &= E^i_a E^i_b \\
\mathcal{A}^{ab}{}_c &= E^a_i \mathcal{D}_c E^b_i \\
\mathcal{B}_{ab}{}^c &= E^i_a \mathcal{D}_b E^c_i
\end{aligned}$$

Gauge covariant derivatives:

$$\begin{aligned}
\mathcal{D}_a E^b &= \partial_a E^b + \left[A_a, E^b\right] \\
\mathcal{D}_a \mathcal{D}_b E^c &= \partial_a \big(\mathcal{D}_b E^c\big) + \left[A_a, \mathcal{D}_b E^c\right]
\end{aligned}$$

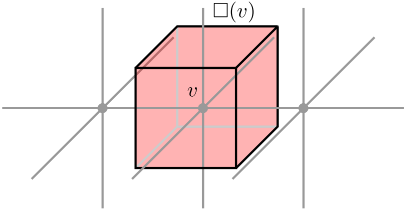
Regularization on a cubical graph

Main assumption/simplification:

Aim to construct the operator on the Hilbert space of a fixed cubical graph

May seem like a severe restriction. However, several approaches in LQG make extensive use of states defined on cubical graphs:

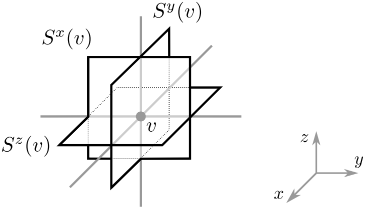
- Algebraic quantum gravity
- Quantum-reduced loop gravity
- Models of effective dynamics



The integrated Ricci scalar is regularized as a Riemann sum over the cubical partition:

$$\int d^3x\,\sqrt{q}\,{}^{(3)}R$$

$$\simeq \sum_{\square} \epsilon^3 \sqrt{|\det E|}(v)\,{}^{(3)}R(v)$$



Regularization of gauge covariant derivatives

We use parallel transported flux variables (also known as gauge covariant fluxes) to regularize the gauge covariant derivatives of the triad.

$$\widetilde{E}(S, x_0) = \int_S d^2\sigma \, n_a(\sigma) h_{x(\sigma)\to x_0} E^a\big(x(\sigma)\big) h^{-1}_{x(\sigma)\to x_0}$$

The holonomies $h_{x(\sigma)\to x_0}$ perform parallel transport from points on the surface to a fixed point x_0 along a chosen family of paths.

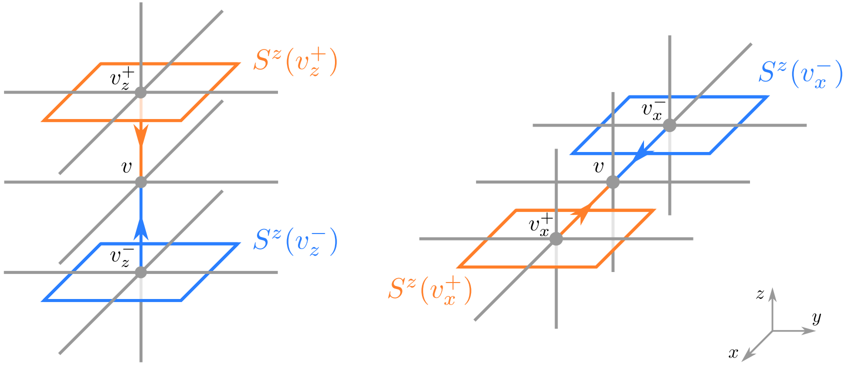
Covariant derivatives of the triad are approximated as finite differences of neighboring parallel transported fluxes (all transported to the same node v).

First derivatives: $$f'(x) \simeq \frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$$

Second derivatives: $$f''(x) \simeq \frac{f(x+\epsilon) - 2f(x) + f(x-\epsilon)}{\epsilon^2}$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \simeq \frac{f(x+\epsilon, y+\epsilon) - f(x+\epsilon, y-\epsilon) - f(x-\epsilon, y+\epsilon) + f(x-\epsilon, y-\epsilon)}{4\epsilon^2}$$

First derivatives

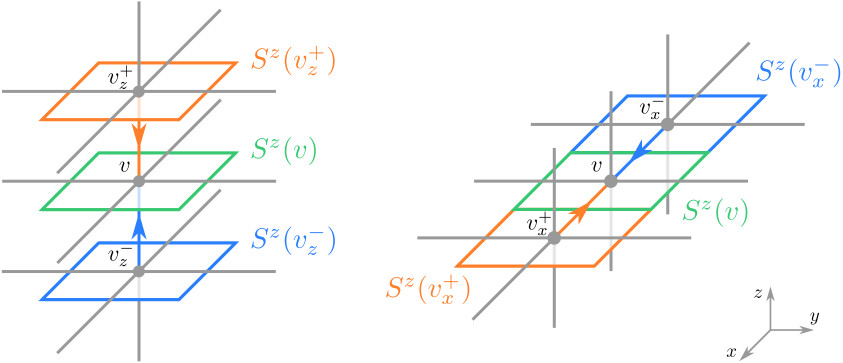


The covariant derivative $\mathcal{D}_a E^b(v)$ is approximated by the discretized variable

$$\boxed{\Delta_a E\big(S^b, v\big) \equiv \frac{\widetilde{E}\big(S^b(v_a^+), v\big) - \widetilde{E}\big(S^b(v_a^-), v\big)}{2}}$$

For small regularization parameter: $\Delta_a E\big(S^b, v\big) = \epsilon^3 \mathcal{D}_a E^b(v) + \mathcal{O}\big(\epsilon^4\big)$

Second derivatives



The diagonal components of the second covariant derivative are discretized as

$$\Delta_{aa} E\big(S^b, v\big) \equiv \widetilde{E}\big(S^b(v_a^+), v\big) - 2\widetilde{E}\big(S^b(v), v\big) + \widetilde{E}\big(S^b(v_a^-), v\big)$$

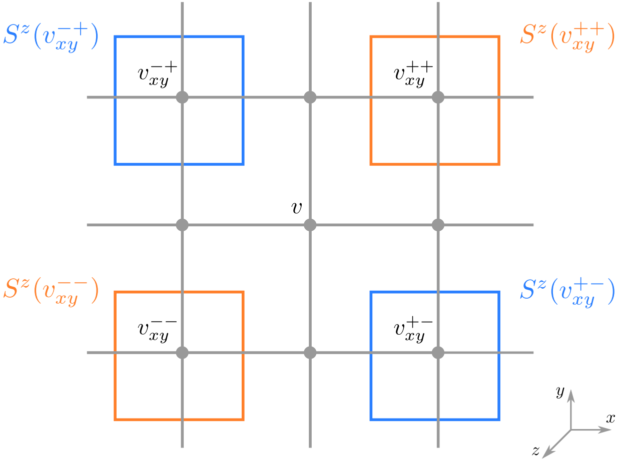
For small regularization parameter: $\Delta_{aa} E\big(S^b, v\big) = \epsilon^4 \mathcal{D}_a^2 E^b(v) + \mathcal{O}\big(\epsilon^5\big)$

Mixed second derivatives

The regularization of the mixed second derivative

$$\mathcal{D}_a \mathcal{D}_b E^c(v)$$

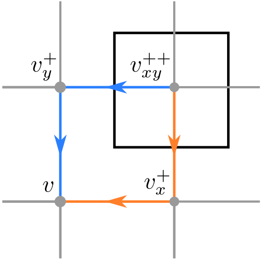
involves the four nodes diagonally adjacent to v in the (a, b) coordinate plane.



Symmetric regularization of mixed second derivatives

Two equally good paths are now available for parallel transport to the central node. Hence introduce

$$\widetilde{E}\big(S^z(v_{xy}^{++}), v\big)_{\text{sym.}} \equiv \frac{1}{2}\Big(\widetilde{E}\big(S^z(v_{xy}^{++}), v\big)_{v_{xy}^{++}\to v_x^{+}\to v} + \widetilde{E}\big(S^z(v_{xy}^{++}), v\big)_{v_{xy}^{++}\to v_y^{+}\to v}\Big)$$



Then we define the discretized variable

$$\Delta_{ab}E\big(S^c, v\big) \equiv \frac{1}{4}\Big(\widetilde{E}\big(S^c(v_{ab}^{++}, v\big)_{\text{sym.}} - \widetilde{E}\big(S^c(v_{ab}^{+-}, v\big)_{\text{sym.}} - \widetilde{E}\big(S^c(v_{ab}^{-+}, v\big)_{\text{sym.}} + \widetilde{E}\big(S^c(v_{ab}^{--}, v\big)_{\text{sym.}}\Big)$$

This approximates the symmetric part of the mixed second derivative at v:

$$\Delta_{ab}E\big(S^c, v\big) = \epsilon^4 \mathcal{D}_{(a}\mathcal{D}_{b)}E^c(v) + \mathcal{O}(\epsilon^5)$$

Quantization

After regularization on the cubical lattice, we have

$$\sqrt{q}\,{}^{(3)}R = \mathcal{R}\big(E^a, \mathcal{D}_a E^b, \mathcal{D}_{(a}\mathcal{D}_{b)} E^c, \sqrt{|\det E|}, \partial_a \sqrt{|\det E|}\big)$$

$$\int d^3x\,\sqrt{q}\,{}^{(3)}R \simeq \sum_{\Box} \mathcal{R}\Big(\widetilde{E}(S^a), \Delta_a E(S^b), \Delta_{ab} E(S^c), V(\Box), \Delta_a V(\Box)\Big)$$

Every factor appearing here can now be promoted into an operator in LQG.

Negative powers of the volume are quantized using the regularized inverse volume operator:

$$\frac{1}{V(\Box)} \quad\longrightarrow\quad \widehat{\mathcal{V}_v^{-1}} \equiv \lim_{\delta\to 0} \frac{\hat{V}_v}{\hat{V}_v^2 + \delta^2}$$

(Previous uses in LQG: Length operator, Regge curvature operator, Warsaw Hamiltonian)

The result is an operator of the form

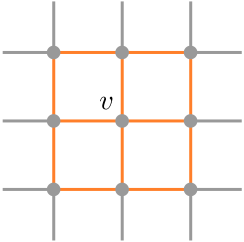
$$\int \widehat{d^3x\,\sqrt{q}\,{}^{(3)}R} = \sum_{v\in\Gamma_{\rm cubical}} \hat{\mathcal{R}}_v$$

on the Hilbert space of the chosen cubical graph.

Properties of the curvature operator

- Gauge invariant (under internal $SU(2)$ transformations)
- Adjoint operator is densely defined on the Hilbert space of the fixed graph. Hence it is possible to construct a symmetric factor ordering (e.g. for the physical Hamiltonian in deparametrized models).
- Action of the operator

$\hat{\mathcal{R}}_v$:



The degree of complexity is roughly similar to the Euclidean part of Thiemann's Hamiltonian in the usual graph-preserving regularization.

- The operator acts by coupling holonomies of spin 1. There is no regularization ambiguity related to the spin.

Summary and outlook

We have defined a new operator representing the three-dimensional scalar curvature in loop quantum gravity.

- The operator is restricted to the Hilbert space of a fixed cubical graph
- Classical starting point: Ricci scalar expressed directly as a function of the densitized triad and its gauge covariant derivatives
- The covariant derivatives are quantized by using parallel transported flux variables to discretize them on the cubical lattice provided by the graph

Open questions/topics for future work:

- Generalize the construction to other kinds of graphs (e.g. four-valent nodes)
- Extension to the entire kinematical Hilbert space of LQG? Different regularization of derivatives needed to obtain a symmetric operator
- Curvature operator for quantum-reduced loop gravity (work in progress)
- Semiclassical properties?
- Physical applications?

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Thank you for your attention!