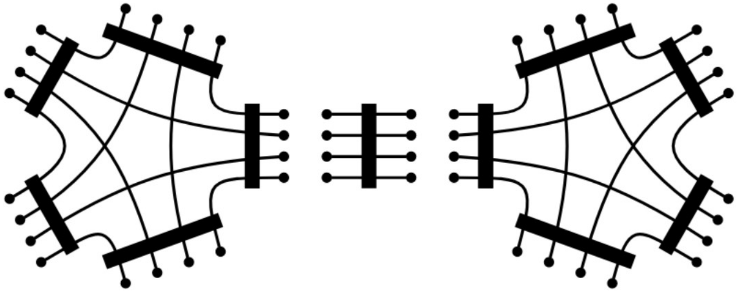
# Towards a hybrid algorithm for spin-foam amplitudes

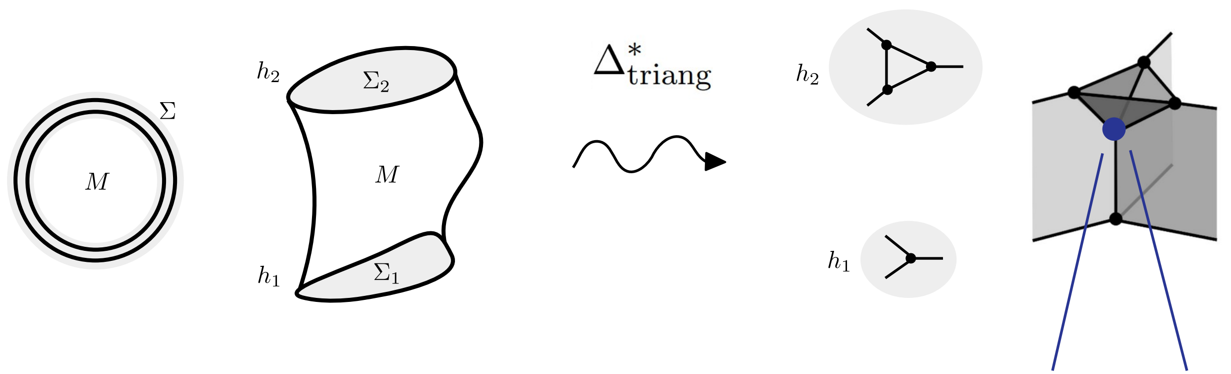
*Gluing constraints and asymptotics*



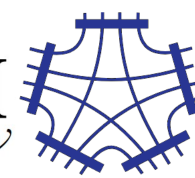
ArXiv 2206.13540, *based on joint work with S. Asante and S. Steinhaus.*

# Nutshell-ed spin-foams

- Quantization of gravity via "sum over histories", reminiscent of transition amplitudes between 3d boundaries. [Perez '03]




$$Z_{\text{grav.}}(\Delta^*_{\text{triang.}}) = \sum_{\{j\}\to\{f\}} \int |_{\text{simpl.}} \left[\prod_{f\in\mathcal{F}} \sim \dim(j_f)\right] \left[\prod_{v\in\mathcal{V}} \quad \right]$$



- The vertex amplitude – which involves many unbounded integrals of fast-oscillating functions - becomes increasingly hard to compute as spins increase.

→ Difficulty of operational viability

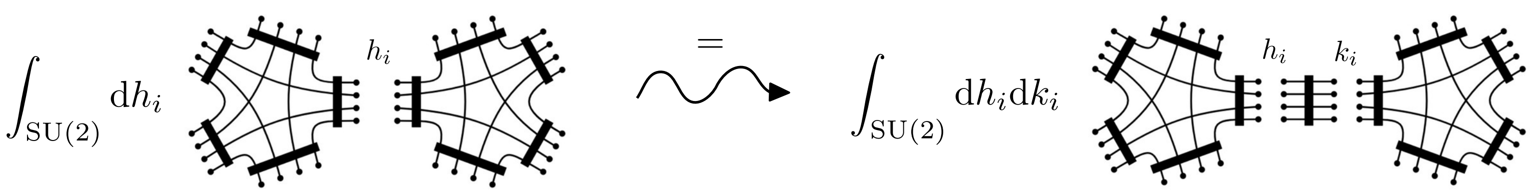
# Sketching a hybrid algorithm

To compute the spin-foam for a general complex $\Delta^*$:

- use the full amplitude at small spins [Donà, Gozzini, Sarno, ... ]
- use an asymptotic approximation at appropriately large spins

- Asymptotics of full spin-foam with many vertices is difficult to obtain on general grounds
- But very good control of 1-vertex asymptotic approximation [Barret et al., Kaminski et al., Liu, Han, JDS, Steinhaus, Dona, Speziale...]

⟶ **proposal**: decouple vertices, apply asymptotics locally, and glue results



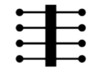

Introduce *gluing constraints*

charactering how critical points of different vertices overlap

*Named after the constraints in effective spin-foams of Asante, Dittrich, Haggard ...*

# Gluing constraints for different models

- General form of the constaints:

$$G(h_i,k_i) = \int_{\mathcal{G}} \mathrm{d}\mu(g) \bigotimes_{i=1}^{4} \langle J \triangleright \chi_i, h_i | D^{\chi_i}(g) | \chi_i, k_i \rangle = \; h_i$$  $k_i$

integration over Haar measure of $\mathcal{G}$, depending on the model

boundary data are group elements $h_i, k_i \in \mathrm{SU}(2)$, $\mathrm{SU}(2)/\mathrm{U}(1) \simeq S^2$ [Livine, Speziale '10]

states constructed from unitary irred. reps. labbeled by $\chi$ as $|\chi, h\rangle = D^{\chi}(h)\,|\mathrm{ref}_{\chi}\rangle$ *(coherent states)*

- Constraints for SU(2) *BF* theory: [Baez '97]

$$G_{\mathrm{SU}(2)}(h_i,k_i) = \int_{\mathrm{SU}(2)} \mathrm{d}\mu(g) \prod_{i=1}^{4} \langle J h_i | g | k_i \rangle^{2j_i}$$

- Constraints for SL(2,C) EPRL theory: [Engle, Livine, Pereira, Rovelli '07, (see also Freidel-Krasnov)]

$$G_{\mathrm{SL}(2,\mathbb{C})}(\overline{h}_i,\overline{k}_i) = \int \mathrm{d}\mu(g) \prod_{i=1}^{4} \int_{\mathbb{C}P} \omega(z_i) \frac{\langle g z_i, g z_i \rangle^{j_i(i\gamma-1)-1}}{\langle z_i, z_i \rangle^{j_i(i\gamma+1)+1}} \langle g z_i, k_i \rangle^{2j_i} \langle J h_i, z_i \rangle^{2j_i}$$

*$\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{C}^2$*

$|h\rangle = h \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

*$g$, $h_i$, $k_i$ in the defining rep.*



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# Asymptotic constraints for all data

- There is a theorem on the asymptotic evaluation of integrals subject to free parameters (i.e. *boundary data*, here the variable $y$):

**Theorem IV.1** (Hörmander). *Let $S(x,y)$ be smooth and complex-valued in a neighbourhood $K$ of $(0,0) \in \mathbb{R}^{n+m}$, such that $\Im S \geq 0$, $\Im S(0,0) = 0$, $S'_x(0,0) = 0$ and $\det S''_{xx}(0,0) \neq 0$. Consider furthermore $u \in \mathcal{C}_0^\infty(K)$. Then*

$$\int dx\, u(x,y) e^{i\lambda S(x,y)} = \left(\frac{2\pi i}{\lambda}\right)^{n/2} \frac{u^0(y) e^{i\lambda S^0(y)}}{\sqrt{\left(\det S''_{xx}\right)^0(y)}} + \mathcal{O}\left(\lambda^{-n/2-1}\right), \tag{4.1}$$

*where the superscript $f^0(y)$ denotes an $x$-independent residue in the residue class of $f(x,y) \bmod \mathcal{I}$, for $\mathcal{I}$ the ideal generated by the partial derivatives $\partial_{x^i} S$.*

- Hörmander further shows that there locally exist smooth $f^\alpha(y)$, $X(y)$ near the origin s.t.

$$f(x,y) = \sum_{|\alpha|<N} f^\alpha(y)(x - X(y))^\alpha \bmod \mathcal{I}^N$$

the expansion being unique up to the ideal $\mathcal{I}^N = \left\{\sum_{|\alpha|=N} s^\alpha \left(x - X(y)\right)^\alpha \mid s^\alpha \in \mathcal{C}^\infty(K)\right\}$.

Need to identify $f^0(y)$ generically

# Asymptotics of SL(2,C) EPRL constraints

- A sketch of the procedure for EPRL-type SL(2,C) constraints:

$$S(g,z,y) = \sum_{i=1}^{4} 2j_i \left( \ln \langle g z_i, k_i \rangle \langle J h_i, z_i \rangle + \ln \frac{\langle g z_i, g z_i \rangle^{\frac{i\gamma-1}{2}}}{\langle z_i, z_i \rangle^{\frac{i\gamma+1}{2}}} \right)$$

1) Second order Taylor series to identify $x$ monomials in Hörmander's theorem

$$S^0(y) = S(x_c, y) - \frac{1}{2} \partial^I S(x_c, y)\, \partial^J S(x_c, y)\, {H_c^{-1}}_{IJ}$$

2) Pick useful coordinates and evaluate measure

$$\partial_I g = \frac{i}{2} \sigma_I g \longrightarrow \mathrm{d}\mu(g) = N \mathrm{Tr}\left[ (g g^{-1})^{\wedge 3} \wedge (g g^{-1})^{\dagger \wedge 3} \right]$$

3) Solve critical point equations and exploit gauge symmetry

$$g_c = \pm\, e^{-i \hat{\phi}_i \vec{\sigma} \cdot \vec{h}_i} h_i (-i\sigma_2) k_i^\dagger \longrightarrow g_c = \pm \mathbb{1}$$

- Asymptotic formula given by

$$G_{j_i}(\overline{h}_i, \overline{k}_i) \simeq \overline{\mathcal{N}}^4_{j_i} \frac{(1 + (-1)^{2\sum_i j_i}) \prod_i (2j_i + 1) \langle J h_i \,|\, k_i \rangle^{2j_i}}{32\pi \sqrt{-\det H}} \exp\left\{ V_\alpha (H^{-1})^{\alpha\beta} V_\beta \right\}$$

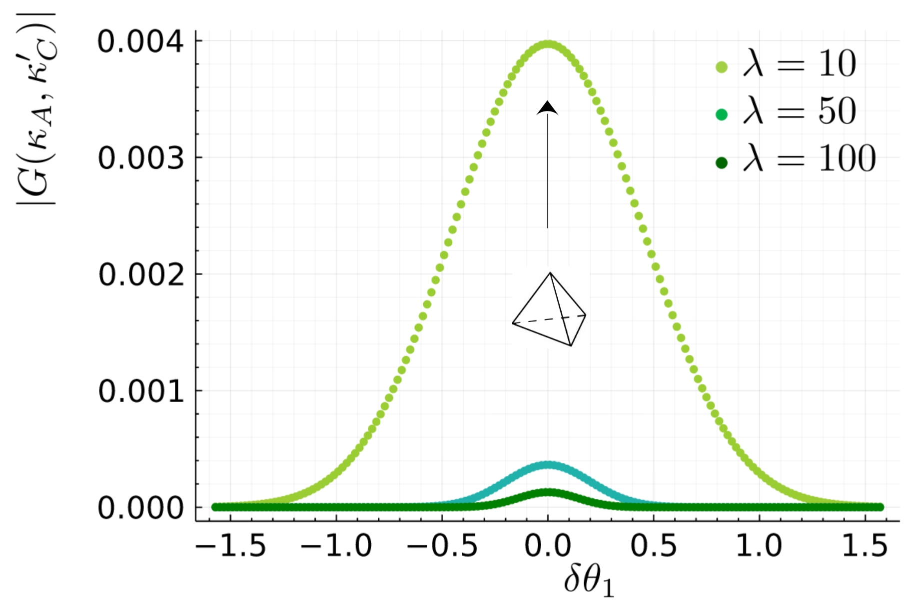
*→ Matches constraints imposition à la Bianca, Seth, Hal and José!*

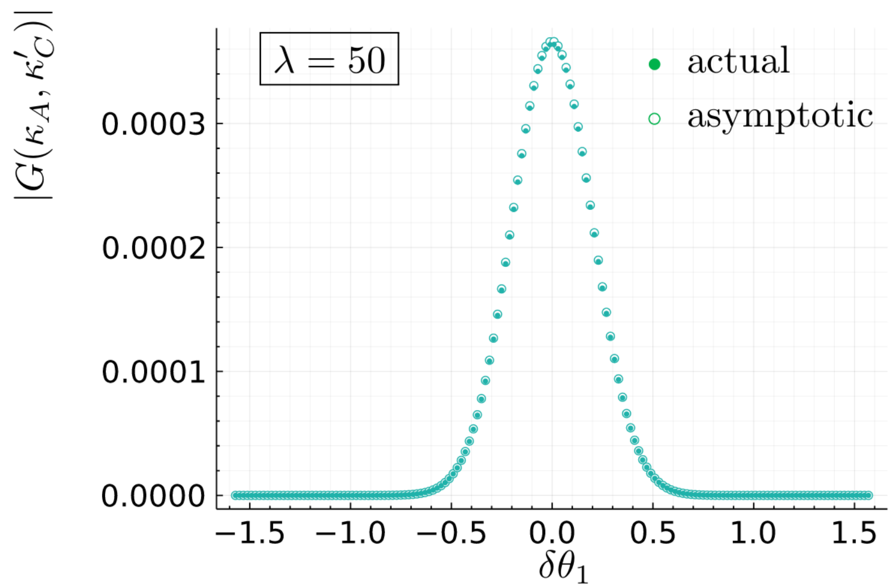
measures how much data fails to close

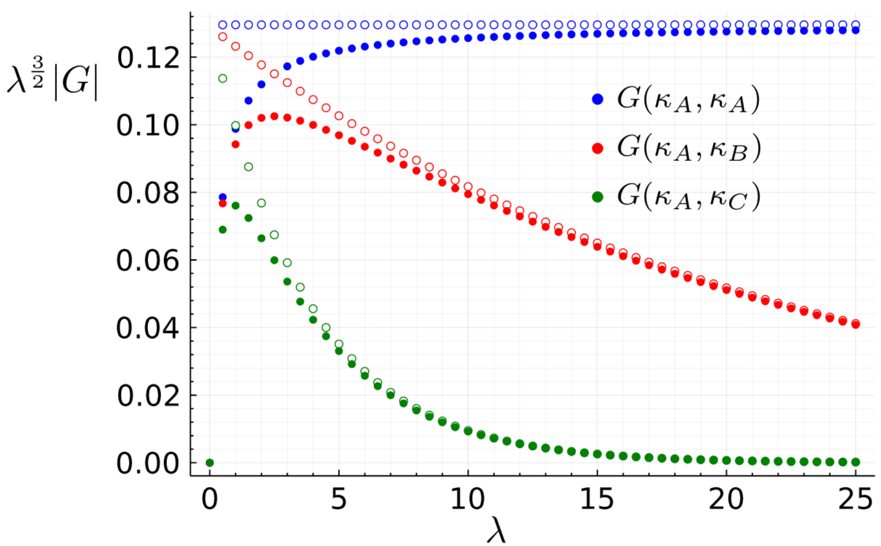
$$V = \left( -i\vec{\Gamma}, -i\gamma\vec{\Gamma}, -2\vec{\Theta}, \vec{0}_4 \right)$$

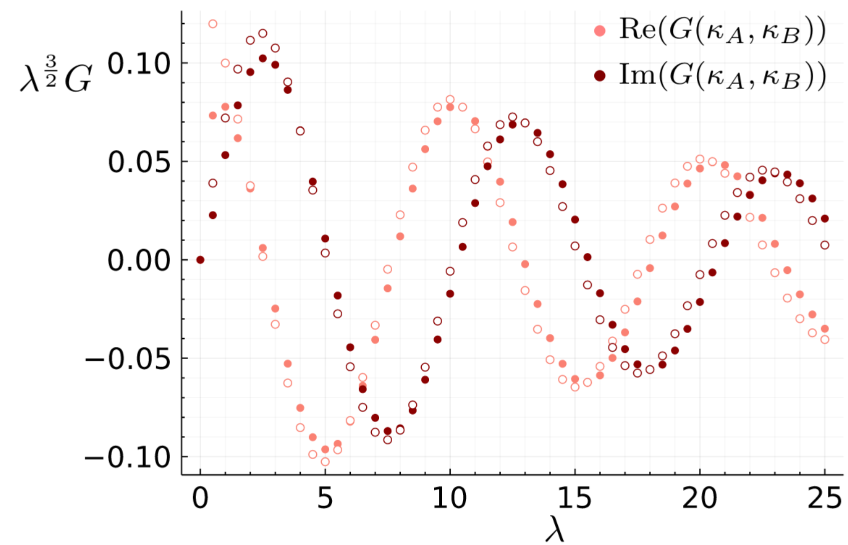
measures how much data fails to glue

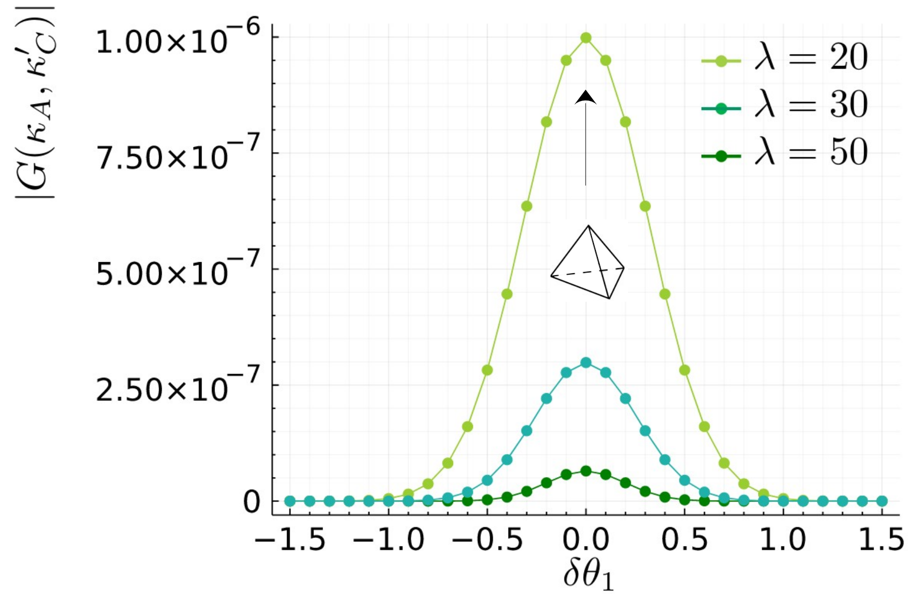
# Some numerical results - SU(2)

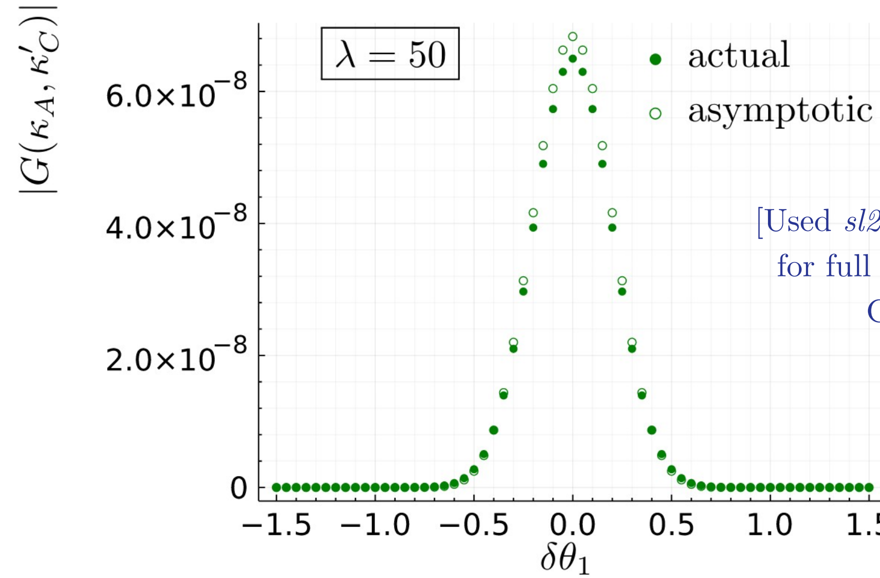


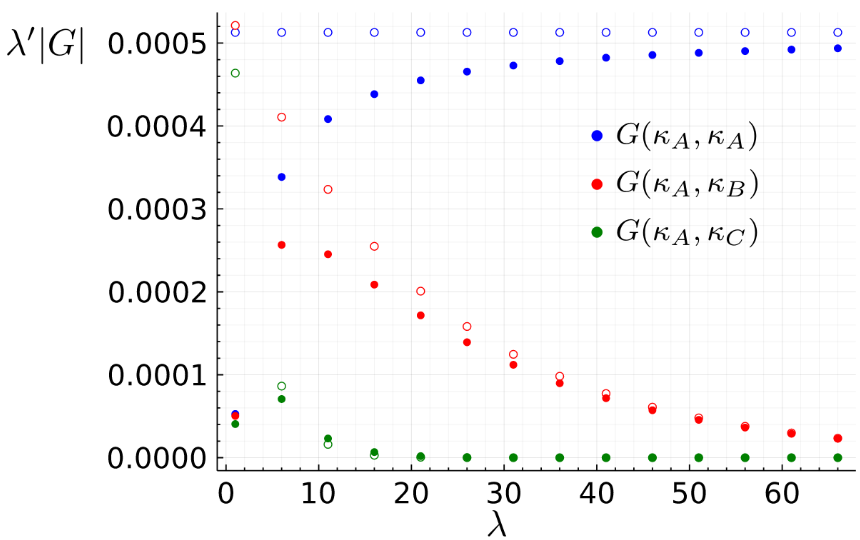






[Used *sl2cfoam-next* for full amplitude, Gozzini '21]




No good control over phases in *sl2cfoam-next*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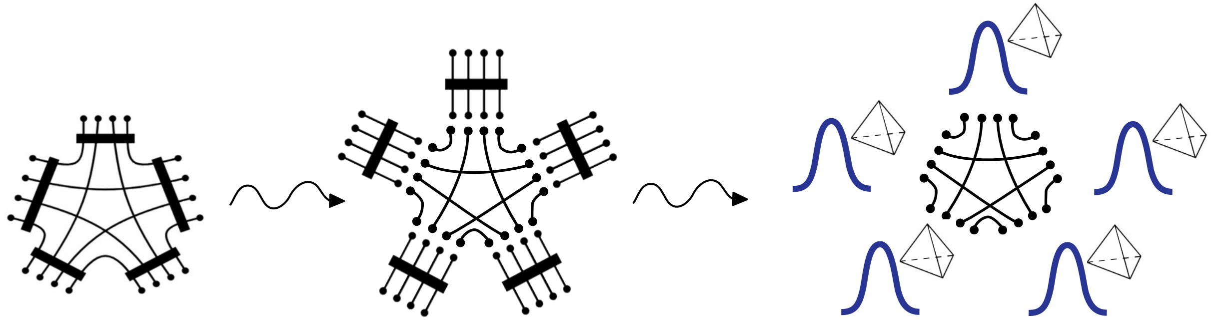
# Summary and Outlook

- We propose to design an hybrid algorythm that leverages both the full spin-foam amplitude as well insight from its asymptotics;
- Gluing constrints measure the overlap between two vertices, and behave like normal distributions peaked on gluing tetrahedra;
- Asymptotic formula for general boundary data – including away from critical points – recovers the full constraints remarkably well;



→ hope to extend asymptotic analysis away from critical points to vertex amplitude;

→ next steps for hybrid algorythm:

- Identify regime of transition
- Construct all critical configurations compatible with given spins