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The loop representation and r-Fock measures

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Once upon a time ...

M. Varadarajan, [PRD '00, '01, '02] – $U(1)^N$ gauge theory

- Why:
  - understanding the relation between qu. states of linearized gravity and states in LQG.
  - Bridging the gap between the Fock quantization and the loop quantization.

- What is it:

  Roughly: r–Fock reps. are reps. for matter fields propagating on Minkowski spacetime, which connect the standard Fock reps. to the background ind. loop reps.

- Role:
  - measurements at a given "scale": $\exists$ r–Fock rep. $\equiv$ Fock rep.
  - Provide new measures for the loop states
  - Provide mappings of Fock states to the Loop rep. Space

Abelian gauge theory

Smearing function (Schwartz)

$$\lim_{r\to 0} f_r(x-y) = \delta^{(3)}(x,y)$$

$$X^a_{\gamma,r}(x) := \int_\gamma ds f_r(x-\gamma(s))\dot{\gamma}^a(s)$$

A

$$\{A_a(x), E^b(y)\} = \frac{1}{q}\delta^b_a \delta^{(3)}(x,y)$$

Fock rep. = rep.

HA: (h, E_r)

$$h_\gamma(A) := \exp\left[i\int_\gamma ds\ \dot{\gamma}^a(s)A_a(\gamma(s))\right]$$

$$E^a_r(x) := \int_{R^3} d^3y f_r(x-y)E^a(\vec{y})$$

Isomorphic

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"smeared holonomy"

The Fock rep. of HA_r induces the r-Fock rep. of HA

~~Abelian~~ gauge theory

Non-Abelian

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The Fock rep. of HA_r ~~induces the r-Fock rep. of HA~~

induces an r-Fock measure on HA

The r-Fock representation of *HA*[U(1)]

- Fock vacuum expectation values:

r-Fock vacuum

$$\langle 0|\hat{h}^r_\gamma|0\rangle = \exp\left[-\int \frac{d^3k}{4q^2|k|}|X^a_{\gamma,r}(k)|^2\right] =: \langle 0_r|\hat{h}_\gamma|0_r\rangle$$

r-Fock rep. = cyclic rep. generated from the r-Fock vacuum

- r-Fock measure & AL-measure:

$$G_{IJ} := \int \frac{d^3k}{|k|}\ \tilde{X}^a_{e_I,r}(k)\ \overline{\tilde{X}^a_{e_J,r}(k)}$$

$$d\mu^r_{U(1)} = \left(\sum_{\Gamma,\vec{n}} \exp\left[-\frac{1}{4q^2}\sum_{I,J} n_I n_J G_{IJ}\right]\ \overline{\mathcal{N}_{\Gamma,\vec{n}}}\right) d\mu^o_{U(1)}$$

- Fock states mapped to Cyl*, in particular the vacuum st. & canonical coherent states:

$$\mathcal{Z}^r_F = \sum_{\Gamma,\vec{n}} \exp\left[-\frac{1}{q^2}\sum_I n_I \int d^3k\ \overline{\tilde{X}^a_{e_I,r}(k)} Z_a(k)\right] \exp\left[-\frac{1}{q^2}\sum_{I,J} n_I n_J G_{IJ}\right] \langle \mathcal{N}_{\Gamma,\vec{n}}|$$

- Shadow states = projections of Fock states on separable sub-Hilbert spaces:

    exp. : fixed graph, dynamical super-selected sector, ...

Beyond U(1)^N ? ...

- R–Fock representation for scalar field:

[A. Ashtekar, J. Lewandowski, H. Sahlmann, CQG '03.]

- Shadow states – operator generalization:

[A. Ashtekar, J. Lewandowski, CQG '01.]

$$|\mathcal{Z}^r_\Gamma\rangle = \sum_{\vec{n}} e^{-\frac{1}{q^2}\sum_I n_I \mathcal{Z}^r_I} e^{-\frac{1}{q^2}\sum_{I,K} n_I n_K G_{IK}} |\mathcal{N}_{\Gamma,\vec{n}}\rangle = \sum_{\vec{n}} e^{-\frac{1}{q^2}\sum_I \hat{E}_I \mathcal{Z}^r_I} e^{-\frac{1}{q^2}\sum_{I,K} \hat{E}_I \hat{E}_K G_{IK}} |\mathcal{N}_{\Gamma,\vec{n}}\rangle$$

$$\longrightarrow \qquad SU(2): \quad |\mathcal{Z}^r_\Gamma\rangle = \sum_{\vec{n}} e^{-\frac{1}{q^2}\sum_I \hat{J}^i_I \mathcal{Z}^r_{i,I}} e^{-\frac{1}{q^2}\sum_{I,K} \hat{J}^i_I \hat{J}^i_K G_{IK}} |\mathcal{N}_{\Gamma,\vec{n}}\rangle$$

- Shadow states v.s. Complexifier coherent states:

[T. Thiemann CQG '06]

"In the Abelian case: shadow states are complexifier coherent states!"

- SU(N) r–Fock measure:

[M.A., J. Lewandowski, PRD '22]

Details in the following...

r-Fock measures for SU(N) gauge theories

- Defining the measure in the case of U(1):

$$\int d\mu^r_{U(1)}\ h_\gamma(A) := \langle 0|\hat{h}^r_\gamma|0\rangle \qquad \Rightarrow \qquad \int_{\bar{\mathcal{A}}/\bar{\mathcal{G}}} d\mu^r_{U(1)}\ \Psi(A) := \langle 0|\psi(\hat{h}^r_{\gamma_1},\dots,\hat{h}^r_{\gamma_K})|0\rangle$$

consequence of Mandelstam identities for U(1): every U(1) cylindrical function can be expressed as a linear combination of Wilson loops.

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- Defining the measure in the case of SU(N):
  - We "cannot" use smeared holonomies: gauge transformations are peculiar.
  - ***Mandelstam identities*** for SU(N) imply that the natural generalization is in terms of ***Wilson loops***:

$$\int d\mu^r_{SU(N)}\, W^J_{\gamma_1}(A)\ldots W^J_{\gamma_{N-1}}(A) := \langle 0|\hat{W}^{r,J}_{\gamma_1}\ldots\hat{W}^{r,J}_{\gamma_{N-1}}|0\rangle$$

To calculate
Not easy!

$\hat{W}^{r,J}_\gamma := \mathrm{Tr}\left[\hat{h}^{r,J}_\gamma\right]$ "r-Wilson loop operator"

r–Fock measures for SU(N) gauge theories

- Linear functional:

$$\Phi_F^r : \mathcal{H}\mathcal{A} \longrightarrow \mathbb{C}$$

$$\Phi_F^r \left[\sum_{i=1}^{M} a_i W_{\gamma_1^i}^{j_o} \dots W_{\gamma_{N-1}^i}^{j_o} \right] := \sum_{i=1}^{M} a_i \langle 0 | \hat{W}_{\gamma_1^i}^{r,j_o} \dots \hat{W}_{\gamma_{N-1}^i}^{r,j_o} | 0 \rangle$$

  - Definiteness : *boundedness of the coefficients* implies the convergence of the expansion of the expectation value
  - Positivity : *Mandelstam identities* for the smeared Wilson loop operators;

- Existence of a measure on $\overline{A/G}$: continuity w.r.t. the C*–norm on $\overline{A/G}$

$$\left| \Phi_F \left[\sum_{i=1}^{M} a_i W_{\gamma_i}^{r,1/2} \right] \right| \leq \sup_{A \in \bar{\mathcal{A}}/\bar{\mathcal{G}}} \left| \sum_{i=1}^{M} a_i W_{\gamma_i}^{1/2}(A) \right|$$

the smearing : $A \in \mathcal{S}^* \longrightarrow A^r \in \mathcal{S}^* \cap \bar{\mathcal{A}}$

r-Fock measures for SU(N) gauge theories

- Mapping between measures:

$$d\mu^r_{SU(N)} = \left(\sum_{\Gamma} \sum_{\{j,\iota\}_\Gamma} \Phi^r_F \left[\sum_{(\{\gamma^i_k\},a_i)\in\mathcal{I}_\Gamma\left(\Psi_{\Gamma,\{j,\iota\}}\right)} a_i W^{j_o}_{\gamma^i_1} \dots W^{j_o}_{\gamma^i_{N-1}} \right] \overline{\Psi_{\Gamma,\{j,\iota\}}} \right) d\mu^o_{SU(N)}$$

lift to a gauge invariant measure on $\overline{A}$ via group averaging.

SU(2) case:

$$\tilde{X}^a_{\gamma,r}(s,k) := \dot{\gamma}^a(s) e^{-i\vec{k}.\vec{\gamma}(s)} \ \tilde{f}_r(k)$$

$$d\mu^r_{SU(2)} = \left(\sum_{\Gamma} \sum_{\{j,\iota\}_\Gamma} \Phi^r_F \left[\sum_{(\{\gamma^i\},a_i)\in\mathcal{I}_\Gamma\left(\Psi_{\Gamma,\{j,\iota\}}\right)} a_i W^{1/2}_{\gamma^i} \right] \overline{\Psi_{\Gamma,\{j,\iota\}}} \right) d\mu^o_{SU(2)}$$

$$\Phi^r_F\left[W^{1/2}_{\gamma^i}\right] = \left\langle \hat{W}^{r,1/2}_{\gamma^i} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \mathcal{P}\int_\gamma ds_1 \dots ds_{2n} \sum_{\sigma} \Upsilon^{(1/2)}_{\sigma(2n)} \left(\prod_{m=1}^{n} \int \frac{d^3k}{2q^2|k|} \tilde{X}^a_{\gamma,r}(s_{\sigma(2m-1)},k) \tilde{X}^a_{\gamma,r}(s_{\sigma(2m)},-k) \right)$$

Path ordered integral

Permutations

Comments & outlook

Results:

✔ Construction of SU(N) r-Fock measures:

- Use of Wilson loops and their properties provides a systematic procedure;

Implications:

✔ Fock states as shadow states for SU(N) gauge fields (details to appear soon) :

- Matter states encoding Minkowski geometry;
- Non-local coefficients;
- Graphs superposition could be restricted by the dynamics;

To explore:

- 🔍 Extension of the construction to fermions; (in progress)
- 🔍 Non-locality (entanglement) & semi-classical prop. of shadow states; (in progress)
- 🔍 Effective dynamics for the shadow states as approximate physical states;
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Thank you!