INSTITUTO DE FÍSICA—UFRGS
Programa de Pós-Graduação em Física

PROGRAMA DE PÓS-GRADUAÇÃO
PPG FÍSICA UFRGS

Holonomy Observables for the Turaev–Viro model

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Overview

Current goals (work in progress — ~2 months behind schedule):

1. Extend the definition of holonomy observables for the Ponzano–Regge model (Barrett and Hellmann 2012) to the Turaev–Viro model (~)
2. Perform explicit computations of these observables (✗)

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Why?

- Quantum groups in LQG are not (yet) fully understood
- Work on the canonical formulation points to the origins of quantum groups (Dupuis et al. 2020; Noui, Perez and Pranzetti 2010, 2011, 2012, 2014), but the root of unity case is still murky (Rennert 2018)
- Developing a "holonomy representation" of the TV model is in parallel to the problem above (covariant vs. canonical)
- Adapt tools that use holonomies to the TV model, e.g., coupling of particles

The Turaev–Viro model as a TQFT

The Turaev–Viro model is

- a TQFT built from (the category of representations of) the quantum group $u_q(\mathfrak{sl}(2, \mathbb{C}))$
- a spinfoam model for 3d Euclidean LQG with $\Lambda > 0$ and

$$q = \exp\left(i\frac{8\pi G\hbar\sqrt{\Lambda}}{c}\right)$$

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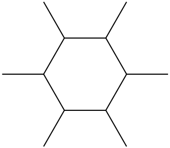
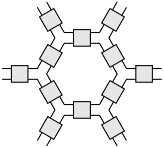
From Hopf algebras to TQFT (Oeckl 2005)

$\triangle^* =$  $\rightsquigarrow$  $= \mathrm{d}_\triangle$

$$\triangle^* \ni \mathrm{f} \mapsto V_\mathrm{f} \in \mathrm{ob}(\mathscr{C}), \qquad (\partial\triangle)^* \ni \ell \mapsto V_\ell \in \mathrm{ob}(\mathscr{C}),$$

$$Z^\triangle = \kappa^{-\chi_{\triangle^*}} \sum_{V_\mathrm{f}} \left(\prod_\mathrm{f} \mathrm{cdim}(V_\mathrm{f}) \right) \mathrm{d}_\triangle(V_\mathrm{f}, o_\mathrm{f}), \qquad Z\colon n\mathbf{Cob} \to \mathbf{Hilb}$$

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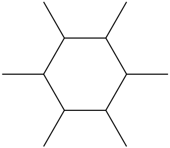
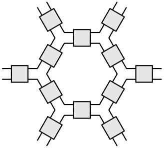
$\triangle^* =$  $\rightsquigarrow$  $= \mathrm{d}_\triangle$

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TV model is the theory with $\mathscr{C} = {}_{u_q(\mathfrak{sl}(2,\mathbb{C}))}\mathcal{M}$ at q a root of unity

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$U_q(\mathfrak{sl}(2,\mathbb{C}))$ is the q-deformation of $U(\mathfrak{sl}(2,\mathbb{C}))$ with generators $\{K, K^{-1}, E, F\}$

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$$u_q(\mathfrak{sl}(2,\mathbb{C})) = U_q(\mathfrak{sl}(2,\mathbb{C}))/I; \qquad K^{p'} = 1, \;\; E^{p'} = 0, \;\; F^{p'} = 0,$$

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Turaev–Viro model:

$$Z_{\mathrm{TV}} = \left(-\frac{2r}{(q-q^{-1})^2}\right)^{-P} \sum_{j_{\mathrm{f}}} \left(\prod_{\mathrm{f}} (-1)^{2j_{\mathrm{f}}} [2j_{\mathrm{f}}+1]_q\right) \prod_{\mathrm{v}} (-1)^{J_{\mathrm{v}}} \begin{Bmatrix} j_{\mathrm{v}1} & j_{\mathrm{v}2} & j_{\mathrm{v}3} \\ j_{\mathrm{v}4} & j_{\mathrm{v}5} & j_{\mathrm{v}6} \end{Bmatrix}_q,$$

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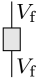

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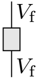

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The TV model has even $p = 2r \implies p' = r$

Holonomy observables

Holonomy observables $F(g_\mathrm{e})$ computed for the PR model in (Barrett and Hellmann 2012)

$$\langle F\rangle \propto \lim_{q\to 1}\langle\Gamma(j_\mathrm{e})\rangle_\mathrm{K}\langle\Gamma(j_\mathrm{e},O_\mathrm{e})\rangle_\mathrm{K}, \qquad F(g_\mathrm{e}) = \prod_{\mathrm{e}\in\Gamma} D^{(j_\mathrm{e})}_{m_\mathrm{e} m'_\mathrm{e}}(U_\mathrm{e}) O_\mathrm{e}^{m_\mathrm{e} m'_\mathrm{e}};$$

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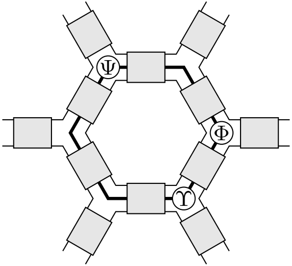
These are $q \to 1$ limits of spin observables for the TV model: Wilson networks in the TQFT formalism

$$\langle \Gamma_W \rangle = \frac{Z^\triangle[\Gamma_\mathrm{W}]}{Z^\triangle}, \quad (\Gamma_\mathrm{W}, V_\mathrm{e}, \Phi_\mathrm{v}), \;\; \mathrm{e} \mapsto V_\mathrm{e}, \;\; \mathrm{v} \mapsto \Phi_\mathrm{v}$$



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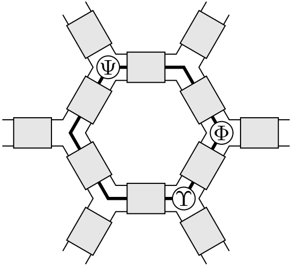

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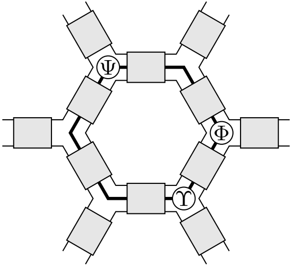

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How to extend to the TV model?

The argument

- If $\mathscr{C} \cong \mathscr{C}'$, the TQFTs are isomorphic
- If $H \cong H'$, ${}_H\mathcal{M} \cong {}_{H'}\mathcal{M}$
- For $\mathfrak{su}(2)$, $U(\mathfrak{su}(2)) \cong (U(\mathfrak{su}(2)))^* = \mathcal{O}(\mathrm{SU}(2))$ (Fourier transform between $\mathfrak{su}(2)$ and $\mathrm{SU}(2)$; between spins and holonomies)
- Finding a 1-to-1 duality for $u_q(\mathfrak{sl}(2,\mathbb{C}))$ allows for the definition of the holonomy observables
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For odd p ($q^p = 1$): (Aziz and Majid 2019; Glushchenkov and Lyakhovskaya 1997; Lyubashenko and Majid 1994):

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For even p (the TV case), things are (unfortunately) not so simple...

What is next?

- If possible, gain more information on the case of even p
- If impossible, work on simpler cases (e.g., $\Lambda < 0 \implies q \in \mathbb{R}$)
- Compute the observables explicitly
- Study the contributions to the canonical approach and compare results

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Summary and Plans for the Future

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Thank you!

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