All-Loop Scattering

as a

Counting Problem

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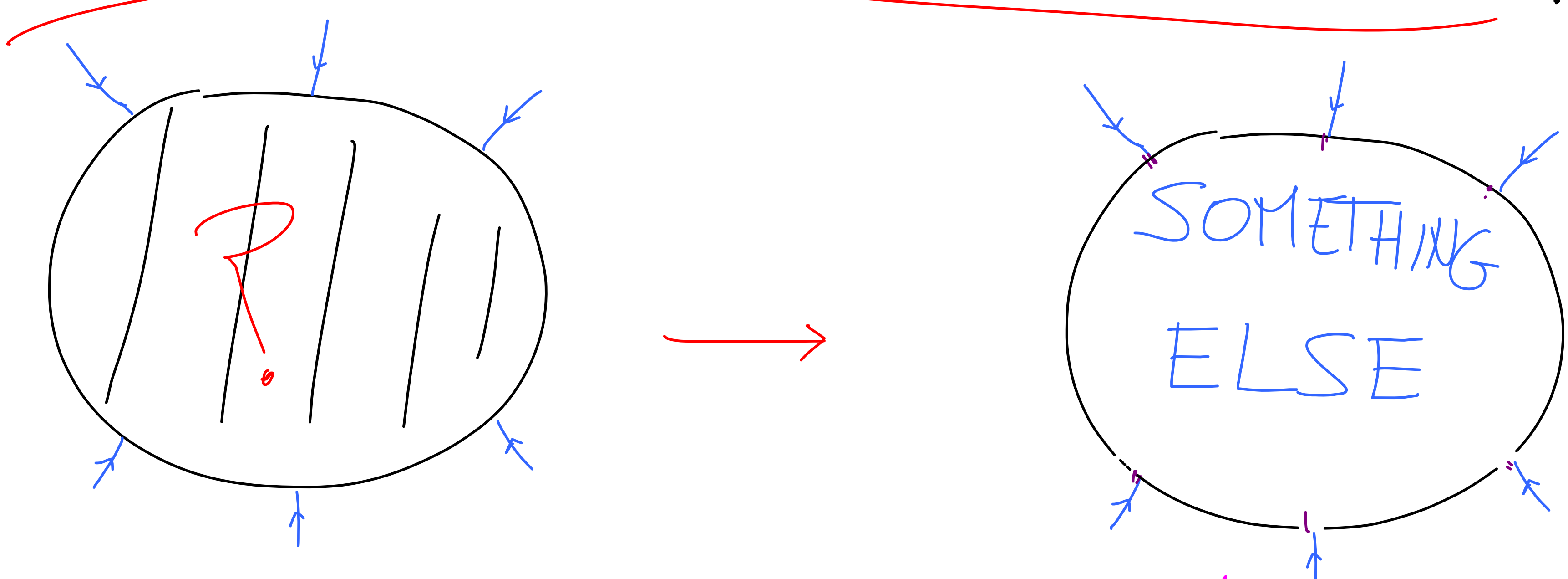
H. Thomas

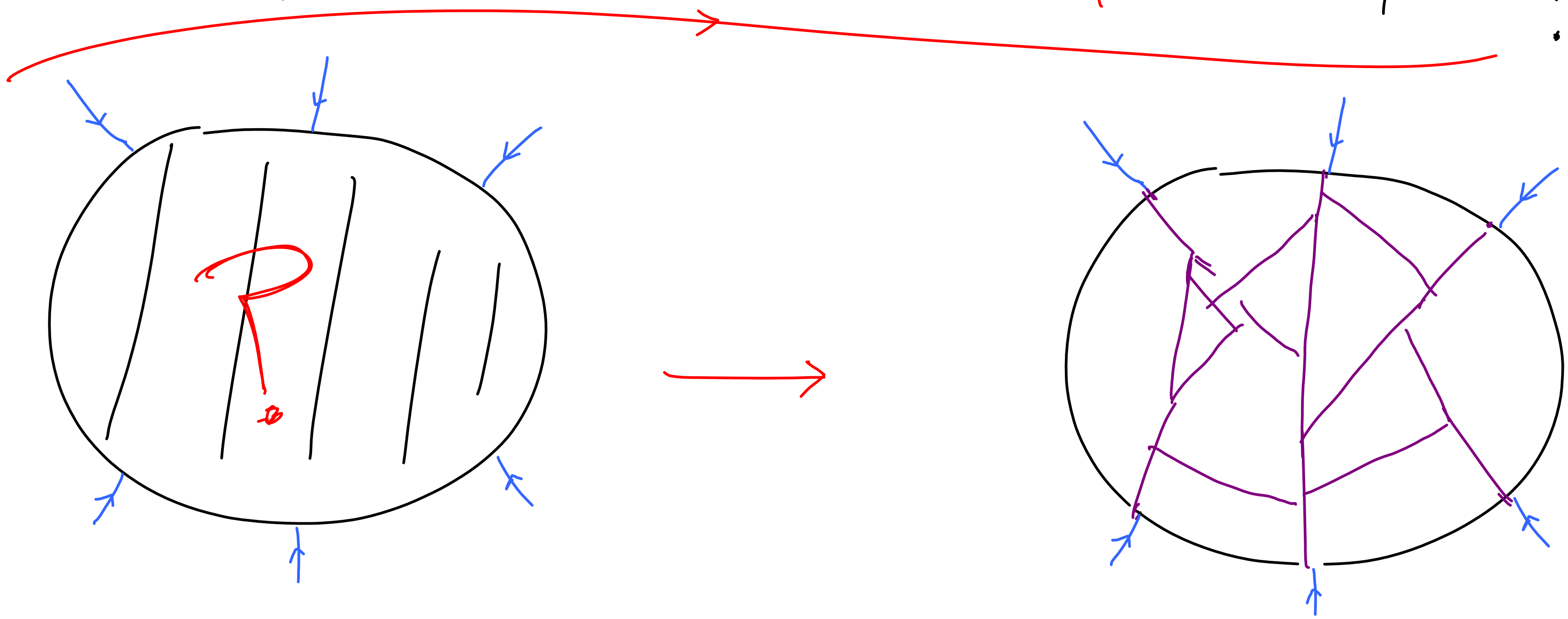
(to appear sometime in 2022...)

What is the Q to which A is the Answer?

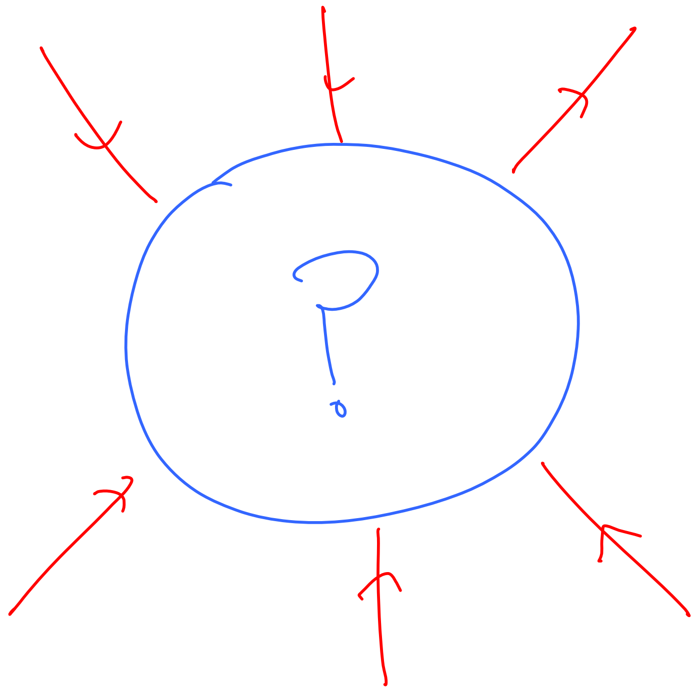


+ see how "something else" gives rise to the properties of Amplitudes we ordinarily ascribe to ST+QM!

What is the Q to which A is the Answer?



Local, Unitary Evolution in Spacetime



Emerging Picture:

Combinatorics

$\Updownarrow$

Positive Geometries

$\Updownarrow$

Canonical Forms

$\Updownarrow$

Functions / Symbols

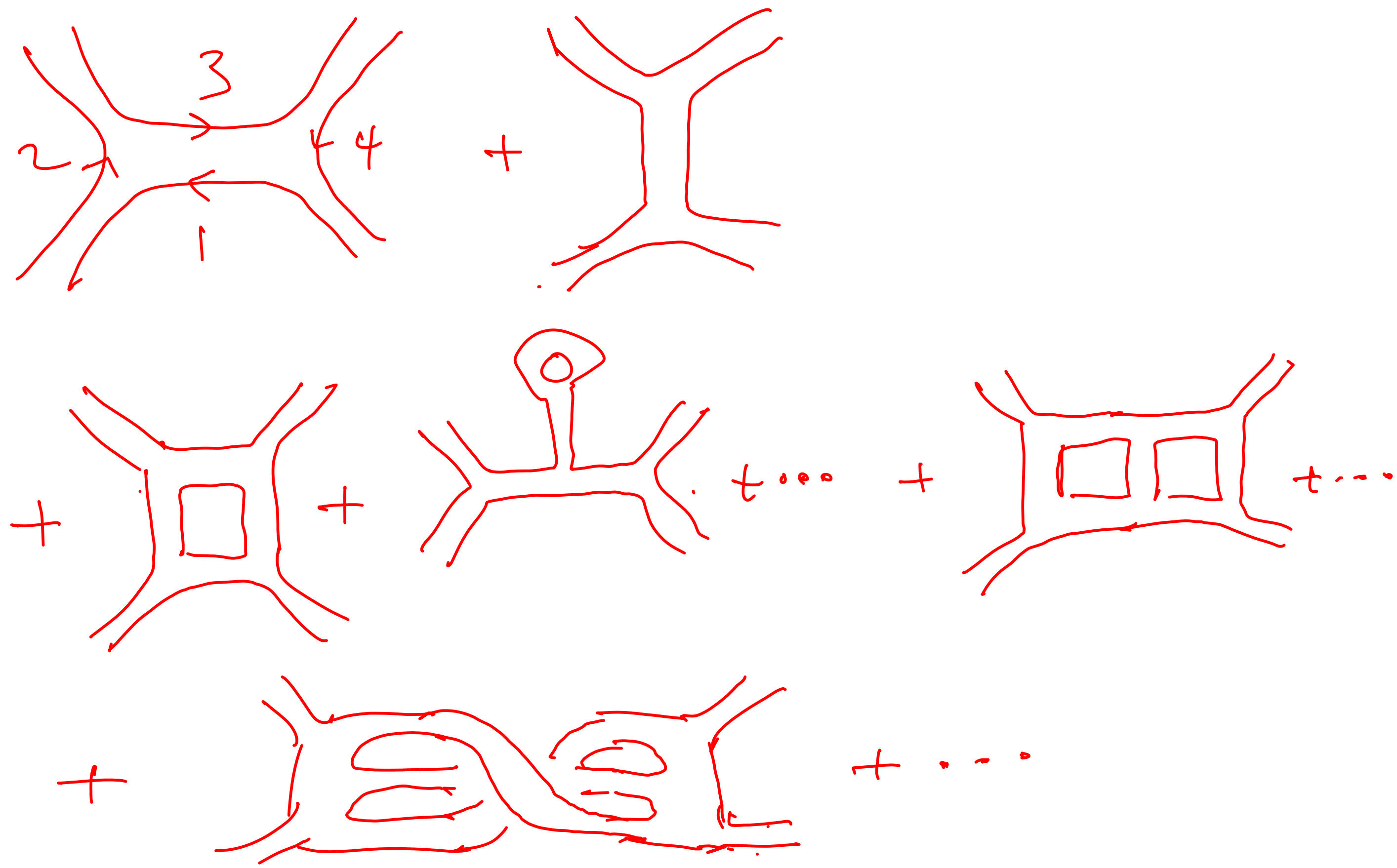
$\Updownarrow$

Non-Pert-Amplitudes

* Amplituhedra, Associahedra, Cosmological Polytopes, ...
* Describes REAL WORLD PHYSICS in reasonable approximation, exposes hidden symmetries + new mathematical structures
* Pure magic in physics right under our noses!

Simplest Theory of Colored Scalars

$$\mathcal{L} = \mathrm{tr}(\partial\phi)^2 + g\,\mathrm{tr}\,\phi^3$$



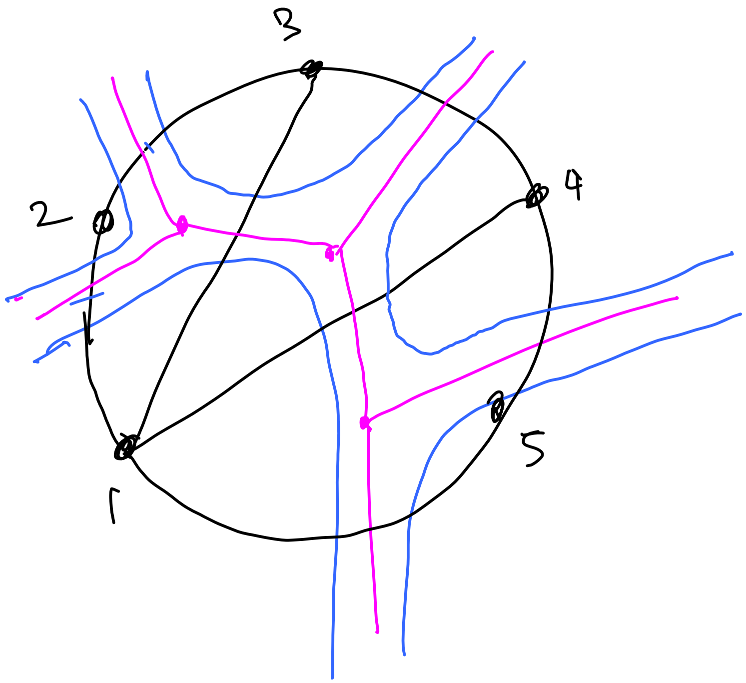
All Tree + 1-loop Amplitudes:

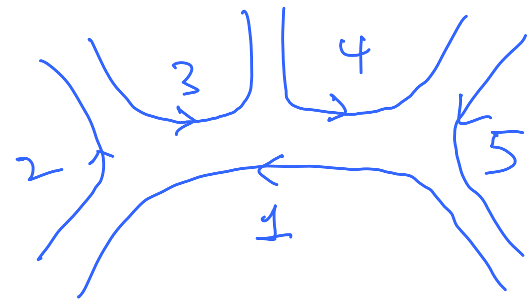
$$\mathcal{A}_n^{\text{tree}} = \int_{S^{n-4}} \frac{\langle t\, d^{n-4} t \rangle}{\left[\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} 2\, p_{i-1} \cdot p_{j-1}\, f_{ij} + \sum_{j=3}^{n-1} t_j \left(X_{1j} + m^2 \right) \right]^{n-3}}$$

$$\mathcal{A}_n^{\text{1-loop}} = \int_{\substack{S^{n-1},\\ \sum_i t_i \geq 0}} \frac{\langle t\, d^{n-1} t \rangle}{\left(\sum_i \alpha_i \right)^{D/2} \left[\sum_{i,j} \alpha_i \alpha_j X_{ij} + \sum_k \alpha_k \left(m^2 \sum_i \alpha_i + \sum f_{ij}\, 2 p_i \cdot p_j + \sum_i T f_{i,i+n} + B f_{i+1,i+n} \right) \right]^{n-D/2}}$$

Where $X_{ij} = (p_i + \cdots + p_{j-1})^2$; $f_{ij} = \max(0, t_j, t_j + t_{j-1}, \ldots, t_j + \cdots + t_{i+2})$; $\alpha_i = f_{i,i+n} - f_{i,i+n+1}$

Color Diagrams ⟷ Triangulation of Surface.





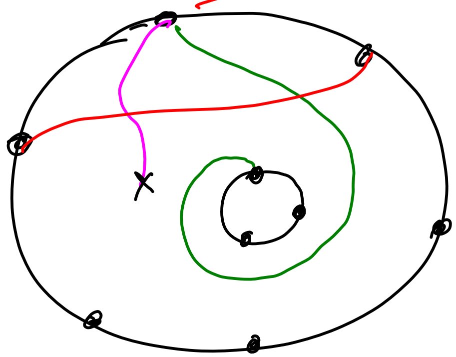
"Fat Graph".

ONE FAT GRAPH DEFINES

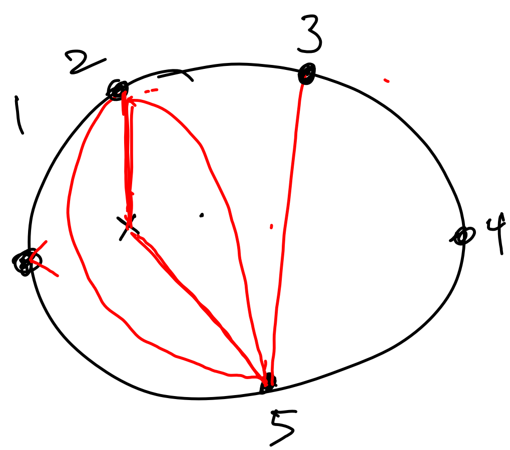
AN (ORIENTABLE) SURFACE,

BY GIVING A TRIANGULATION

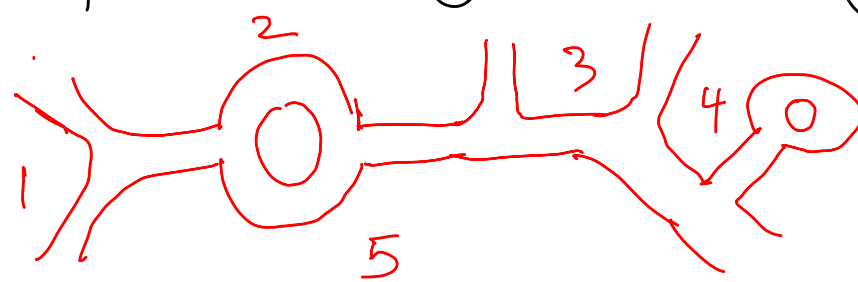
Kinematic Space $^{(0)}$



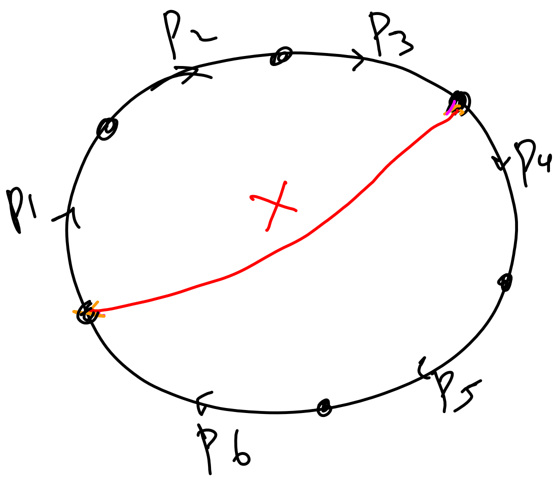
Propagators $\longleftrightarrow$ chords on Surface.



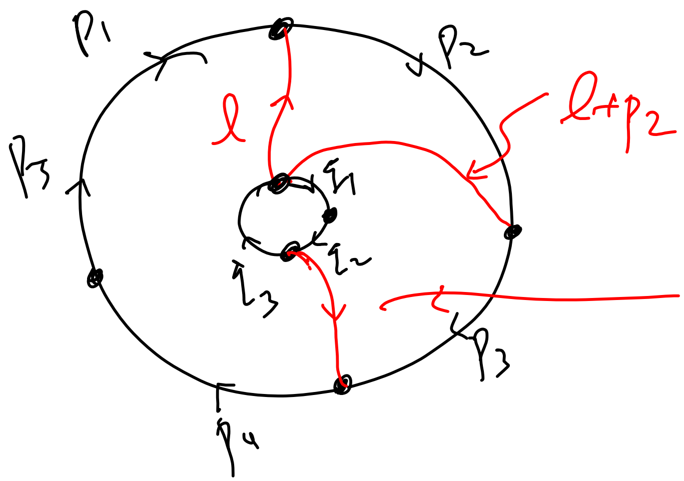
Feynman Diagram $\longleftrightarrow$ Maximal set of non-intersecting curves = Triangulation.



Momenta $\longleftrightarrow$ Homology

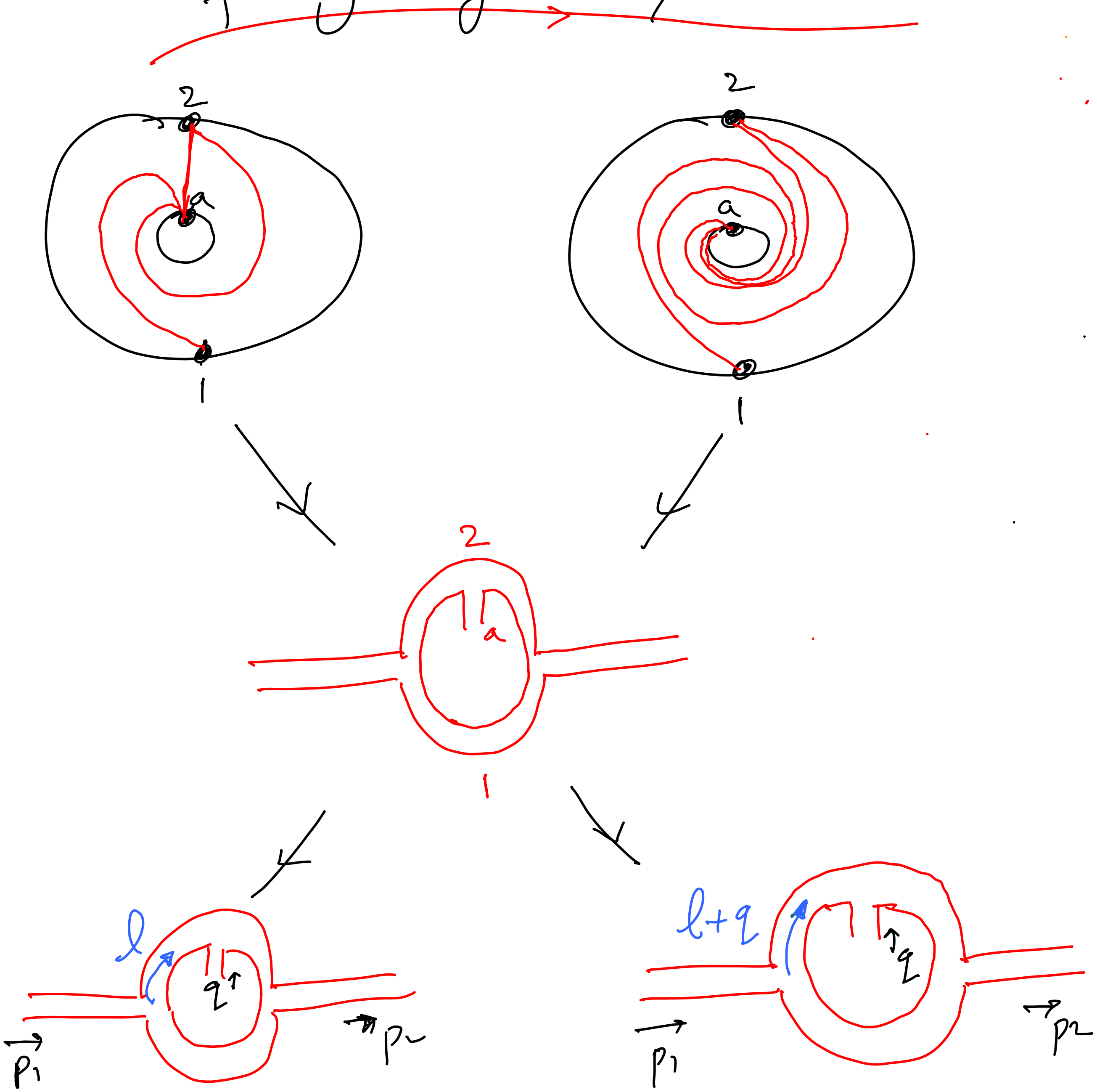


$$p_X^2 = (p_1 + p_2 + p_3)^2$$

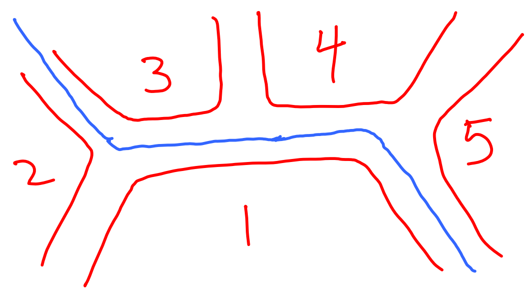


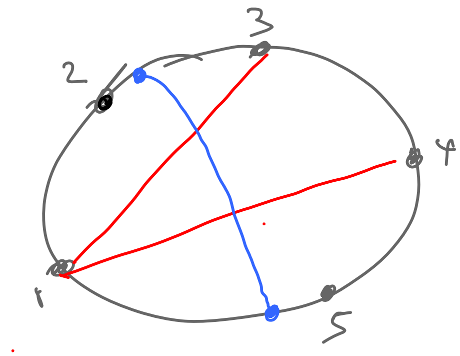
$$(l + p_2 + p_3 + q_1 + q_2)$$

Infinitely Many Curves / MCG



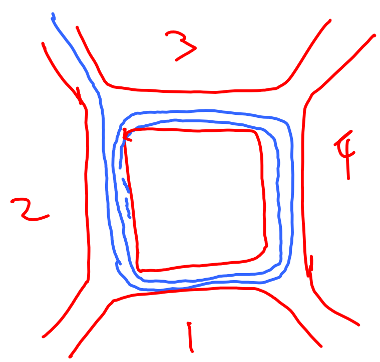
Kinematic Space: Trips Through Fat Graph.

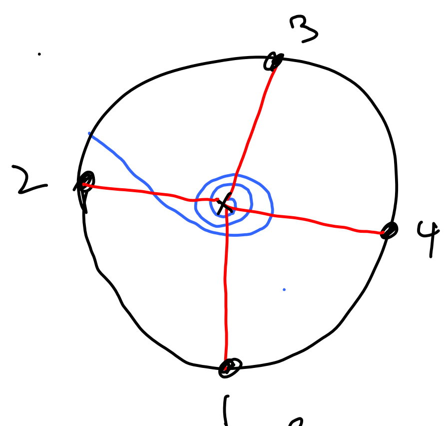


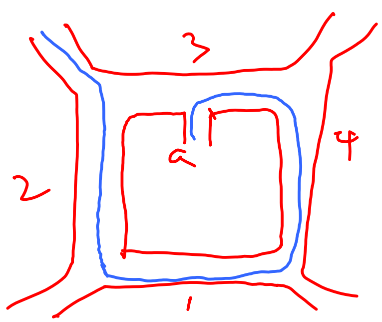


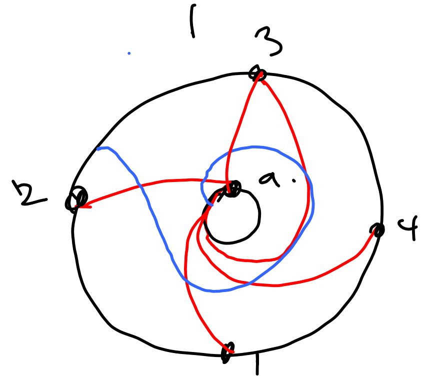
"Lamination"

"Curve"





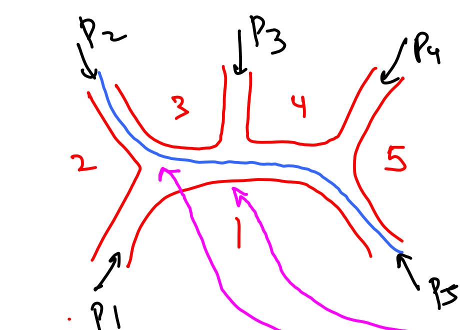




Momenta Read Off From Trip

Assign momenta [including loop momenta] to fat graph.

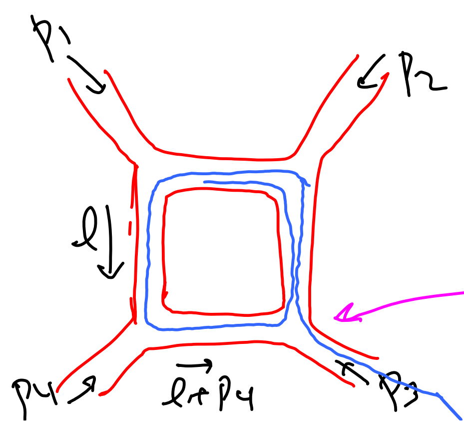
$$P[\text{Curve}] = P_{\text{start}} + \sum_{\text{right turns}} P_{\substack{\text{entering}\\ \text{from left}}}$$



$$P = \overset{\text{start}}{P_2} + 0 + P_3$$

Left turn.

Right turn



$$P = \overset{\text{start}}{P_3} + (l + P_4) + \underbrace{0 + 0 + 0 + \dots}_{\text{All lefts afterwards}}$$

Right turn

ST + QM.

Why should we consider triangulations of the surface (= F.D. = ST processes), and sum over all of them (= QM)?

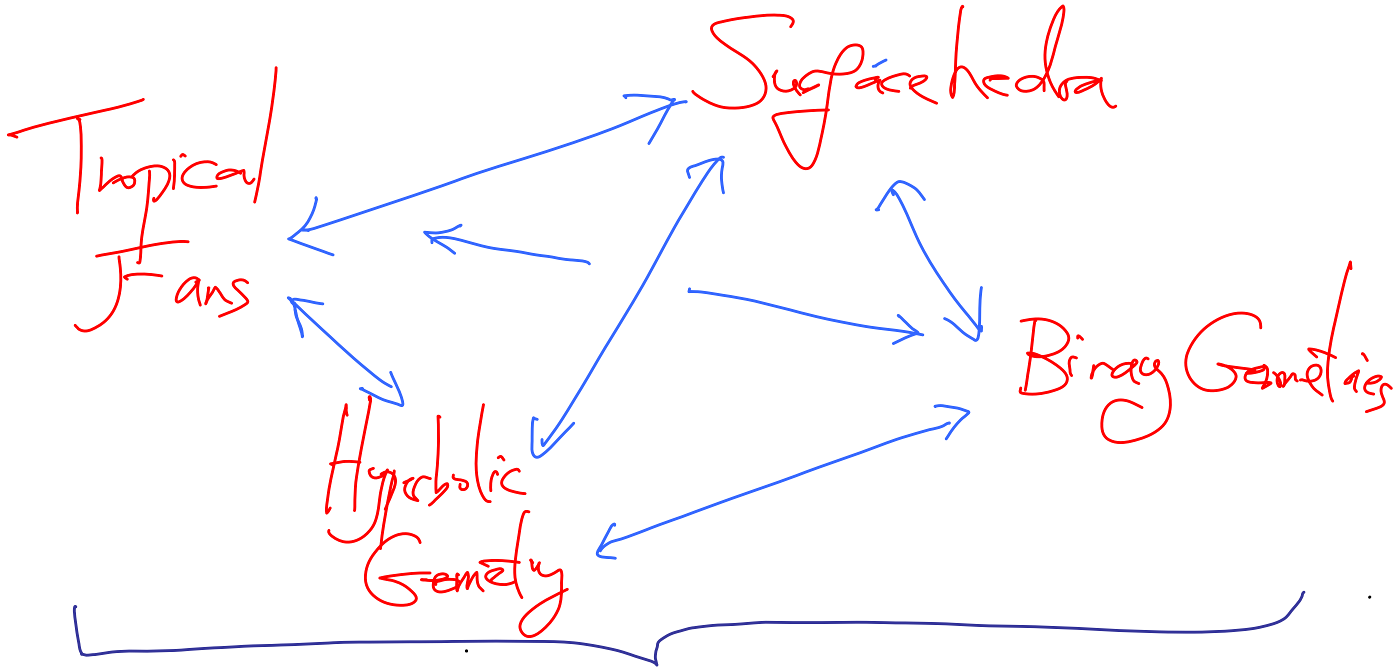
Answer is lying in plain sight when we think concretely about labels for curves / Kin. space!

* This will give us new [of course much more efficient] ways of computing amplitudes — with no reference to the diagrams at all — exposing hidden symmetries

* This picture — fundamentally about particles with the 2D surface just about combinatorics of FD — hands us the generalization particles → "strings", with no reference to a worldsheet, CFT, Vertex Ops...

Fundamentally Combinatorial description of Particles + "Strings"

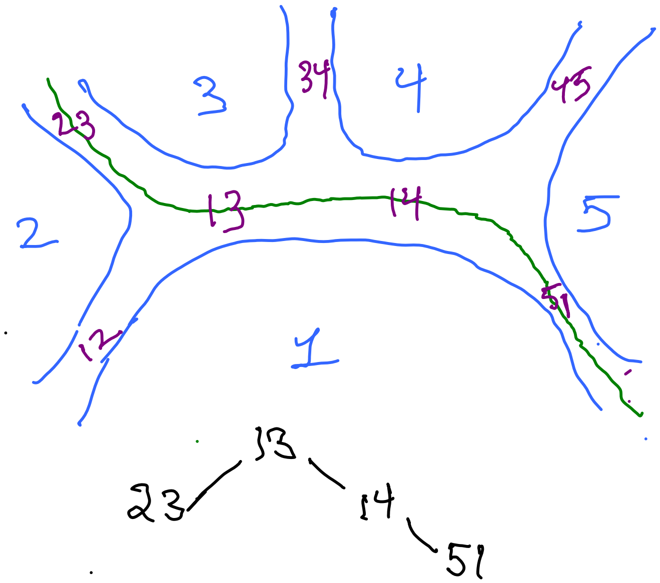
There are many inter-related views of these ideas:



COUNTING PROBLEM IN KINEMATIC SPACE

[CATEGORIES OF QUIVER REPRESENTATIONS]

Curves as Words



"Google Navigate" of Trip Through Fat Graph.

13 : 12 13 34

14 : 12 13 14 45

24 : 23 13 14 45

25 : 23 13 14 51

35 : 34 14 51

12 : 12 23

23 : 23 13 34

34 : 34 14 45

45 : 45 51

51 : 51 14 13 12

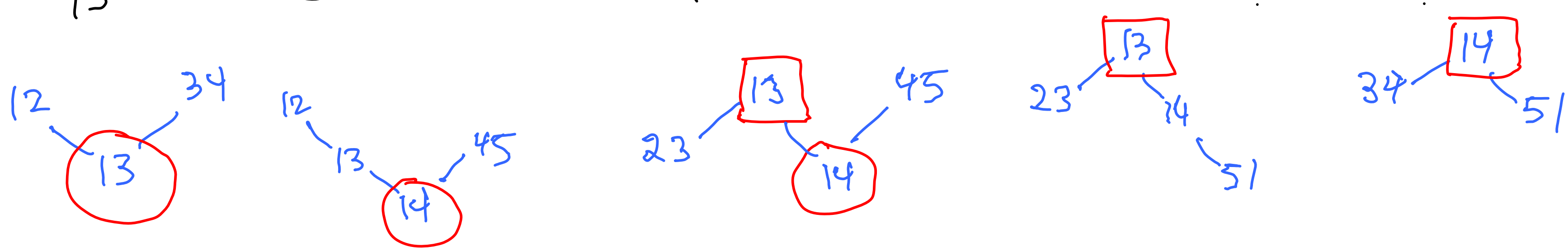
"Geometric Vectors"

$L_{13,24,12,23,34,45,51}$;
Basis

$$L_{24} = L_{14} - L_{13} + L_{23}$$
$$L_{25} = -L_{13} + L_{23} + L_{51}$$
$$L_{35} = -L_{14} + L_{34} + L_{51}$$

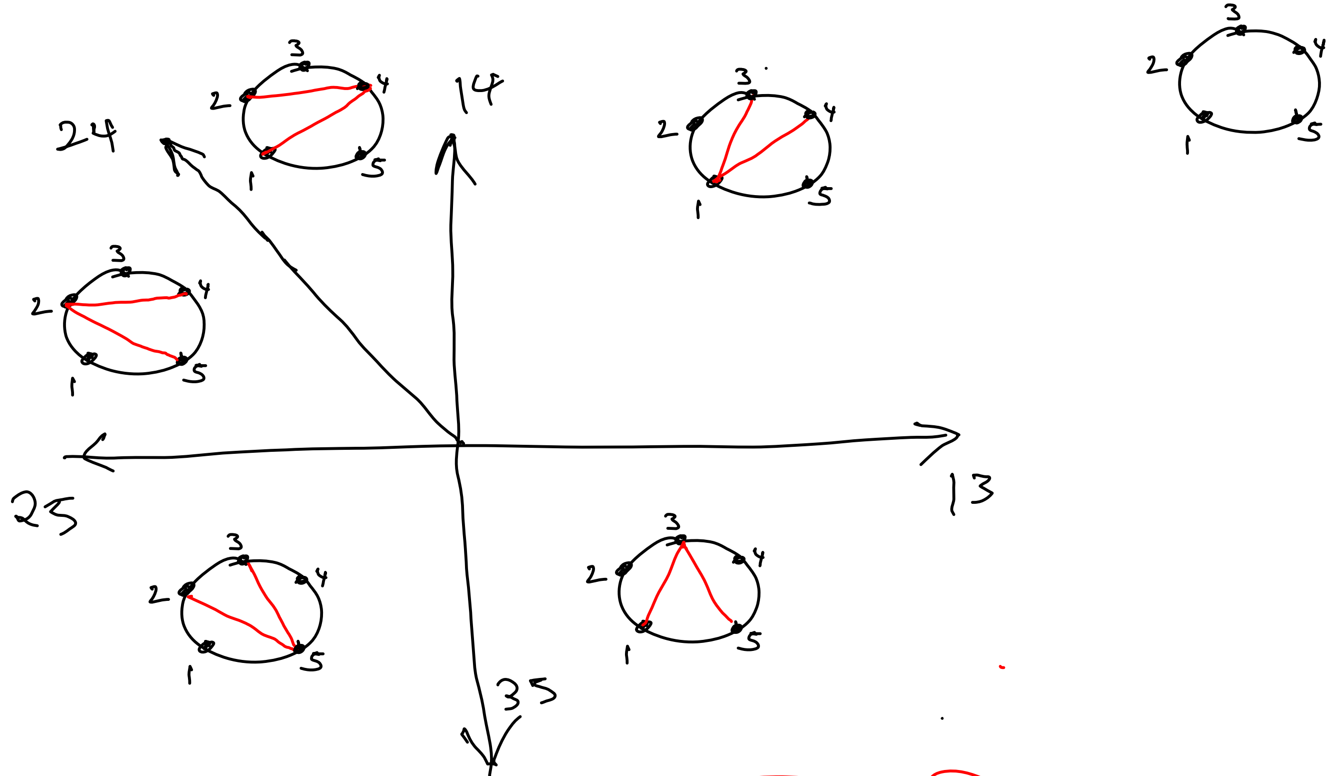
"Go On-Shell" = Project through $L_{12,23,\ldots,51} \longrightarrow$ "g-vectors"

$\vec{g}_{13}$ $\quad$ $\vec{g}_{14}$ $\quad$ $\vec{g}_{24} = \vec{g}_{14} - \vec{g}_{13}$ $\quad$ $\vec{g}_{25} = -\vec{g}_{13}$ $\quad$ $\vec{g}_{35} = -\vec{g}_{14}$



$\vec{g}$: Peaks + Valleys of word.

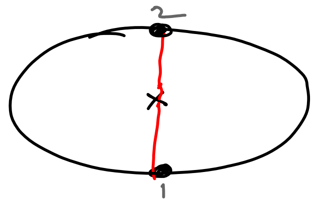
Complete Fan!

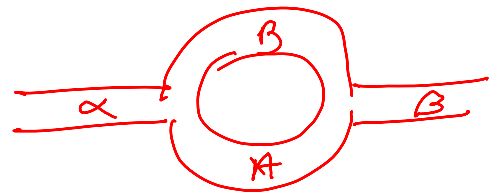


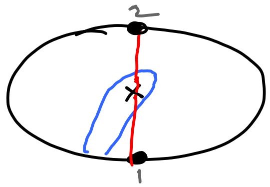
* Cones = Triang = Feyn. Diagrams,

Completely Cover Entire Space

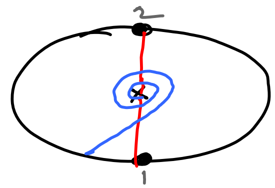
$\boxed{\text{Spacetime + Quantum Mechanics.}}$



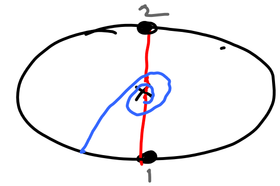




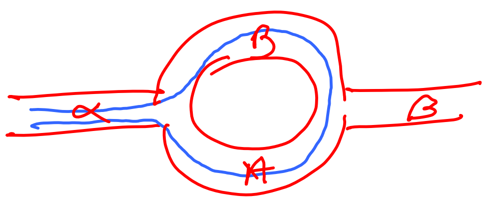
II^0

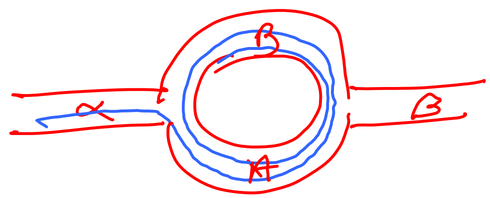


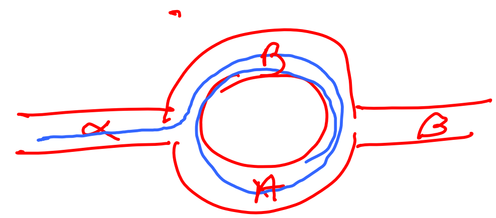
I^+



I^-







α A B α

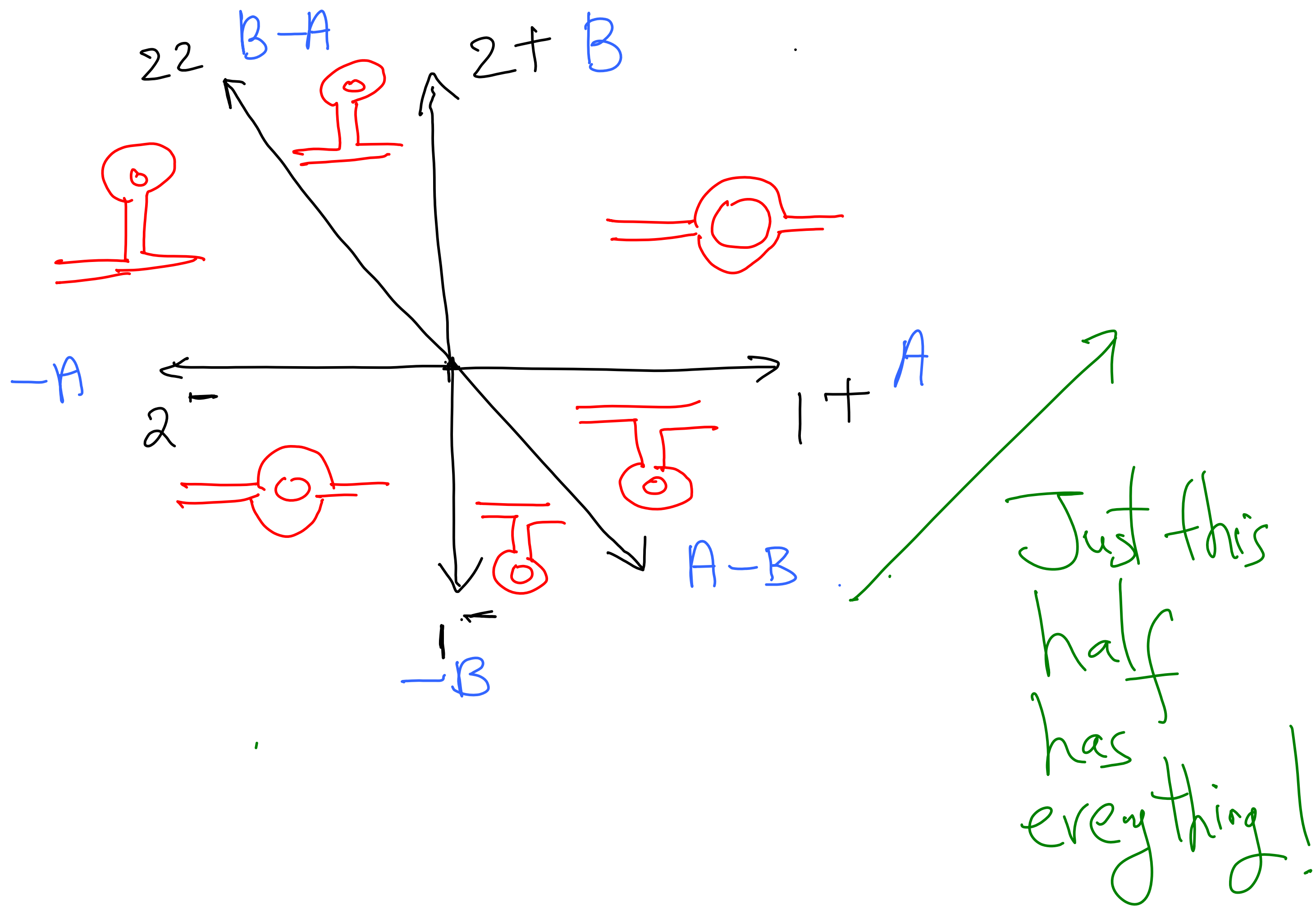
α A B A B ...

α B A B ...

$A - B$

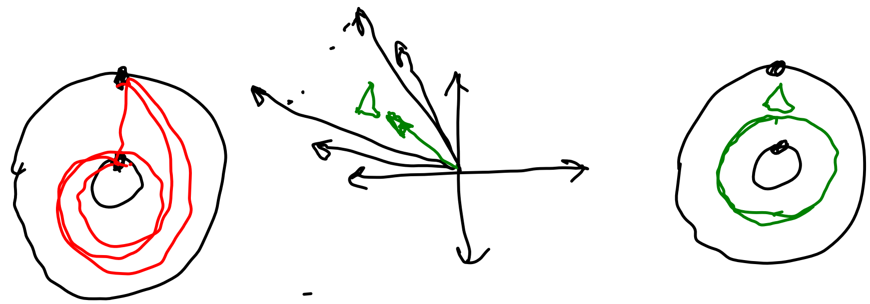
A

$-B$


22
B−A
2+
B
−A
2−
1+
A
1−
−B
A−B
Just this half has everything!

Fan Generalities

* In general infinite:



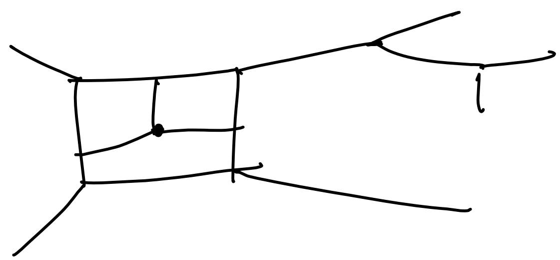
* All cones/diagrams "fit together" ⟹ hidden symmetry:

$$\Omega = \sum_{\text{cones } C} \prod_{x \text{ in } C} d\log X_1 \cdots d\log X_n \cdot \operatorname{sgn} |\vec{g}_{x_1} \cdots \vec{g}_{x_n}|$$ is "projectively invariant"

* Also $|\vec{g}_{x_1} \cdots \vec{g}_{x_n}| = \pm 1$! So, setting $X_i = \vec{g}_i \cdot \vec{x} + d_i$,

$$\Omega\big|_H = d^d x_i \cdot \text{Integrand}$$

Reminder on Schwinger Parametrization



$$\frac{1}{p_e^2+m^2} = \int_0^\infty d\alpha_e \, e^{-\alpha_e(p_e^2+m^2)}$$

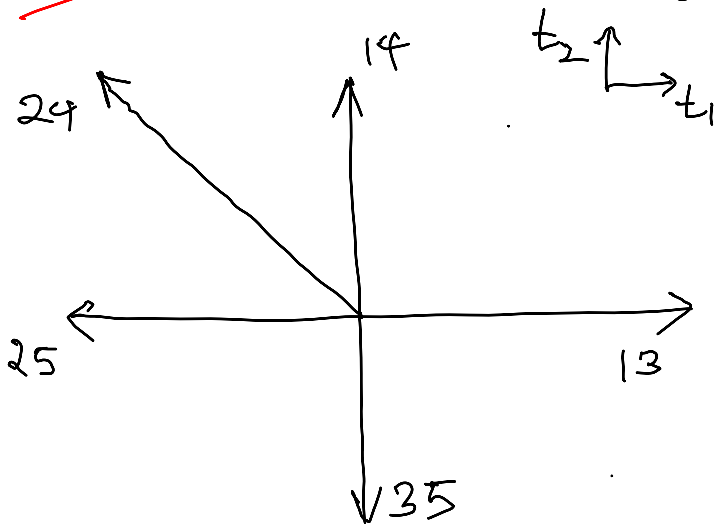
Loop integration is Gaussian:

$$I_G = \int d^E\alpha_E \frac{1}{\mathcal{U}_G(\alpha)^{D/2}} e^{-\mathcal{F}(\alpha)/\mathcal{U}(\alpha)}$$

$$\mathcal{U}_G = \sum_{\substack{\text{all ways}\\ \text{of cutting } G\\ \text{to a tree}}} \prod \alpha \ , \quad \mathcal{F}_G = \sum_{\substack{\text{all ways}\\ \text{cutting } G\\ \text{to two trees}}} \left(P_{T_1}^2\right) \prod \alpha$$

Graph Symanzik Polynomials

Complete Fan $\longrightarrow$ Unified Schwinger Representation.



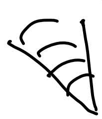
$$-u^t_X[g_Y] = \delta_{XY}.$$

(Piecewise-Linear on Fan)

$$\boxed{\text{Then } \alpha'^2 \int_{-\infty}^{\infty} dt_1\, dt_2 \prod_X e^{\alpha' X u_X} = \frac{1}{X_{13}X_{14}} + \text{cyclic} = \text{Amp}}$$

$$\left\{ \text{e.g. } \alpha'^2 \int dt_1\, dt_2\, e^{-\alpha'(u_{14}X_{14}+u_{24}X_{24})} \right.$$

$$\longrightarrow \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \alpha_{14}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha_{24}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



$$\left. = \alpha'^2 \int_0^{\infty} d\alpha_{14}\, d\alpha_{24}\, e^{-\alpha'(\alpha_{14}X_{14}+\alpha_{24}X_{24})} = \frac{1}{X_{14}\, X_{24}} \right\}$$

Unified Schwinger for Any Surface.

Put $\hat{U}_X = e^{u_X^t}$, $\quad -u_X^t[g_Y] = \delta_{XY}$

Integrand $= \alpha'^n \int d^n t \prod_X \hat{O}_X^{\alpha' X}$.

As usual w/ Schwinger param — loop integrals Gaussian:

$$\boxed{A = \int_{-\infty}^{\infty} \frac{d^N t}{MCG} \frac{1}{U^{D/2}} e^{-F/U} = \int_{S^{N-1}} \frac{\langle t d^{N-1} t \rangle}{MCG} \frac{1}{U^{\frac{D}{2}(L+1)-N}} \frac{1}{F^{N-\frac{D}{2}L}}}$$

U, F : "Surface Symanzik Polynomials"

Surface Symanzik Polynomials

For each [hom. class, ...] curve X, assign [schw. parameter] α_X.

* For every choice of L α_{X_i} that cuts surface to disk:

$$P_{X_i} = M_i^j \ell_j + P_i^{bndy},$$

$$\mathcal{U}_S = \sum_{\text{all cutting}} \prod_i \alpha_{X_i} (\det M)^2$$

* For every choice of $(L+1)$ α_{X_i} cutting to two disks

$$\mathcal{F}_S = \sum_{\text{all cutting}} \prod_i \alpha_{X_i} \cdot (\det M)^2 \cdot \left(\sum P_X\right)^2$$

any L of $L+1$ Any one disk.

... But naively, finding $-u_X^t(g_Y) = \delta_{XY}$

is just as hard as enumerating all

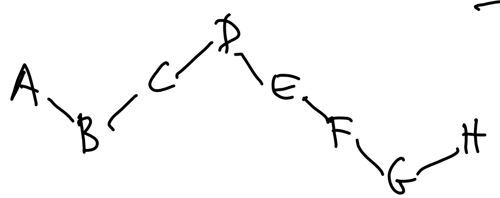
Feynman Diagrams! True of "generic fam"...

We will find u_X^t directly from

a counting problem associated

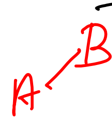
with the word for X !!

"F-Polynomials" for Words

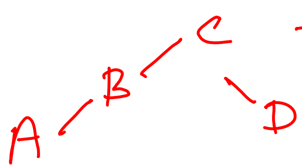
$F\Big[$ 

$\Big] =$ "Relationships with Baggage" Generating Function.

* Can pick anyone, but if you pick them, must pick everyone in their "past lightcone"

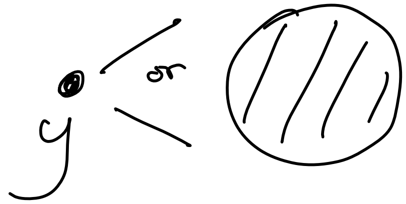
$F[A] = 1 + A$ $F\Big[$ 

$\Big] = 1 + A + BA,$

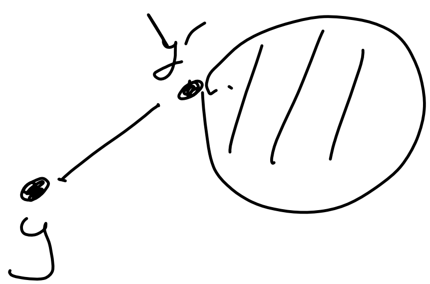
$F\Big[$ 

$\Big] = 1 + A + D + AD + BA + BAD + CBAD$

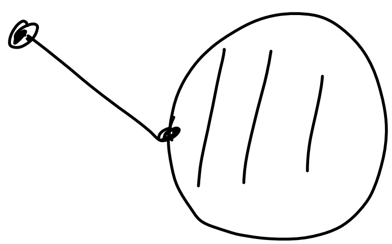
Recursion for $F[W]$



$$F = F^{Yes} + F^{No}$$



$$\begin{pmatrix} F^Y \\ F^N \end{pmatrix} = \begin{bmatrix} y f^Y + y f^N \\ f^N \end{bmatrix} = \underbrace{\begin{pmatrix} y & y \\ 0 & 1 \end{pmatrix}}_{M^{\uparrow}(y)} \begin{pmatrix} f^Y \\ f^N \end{pmatrix}$$

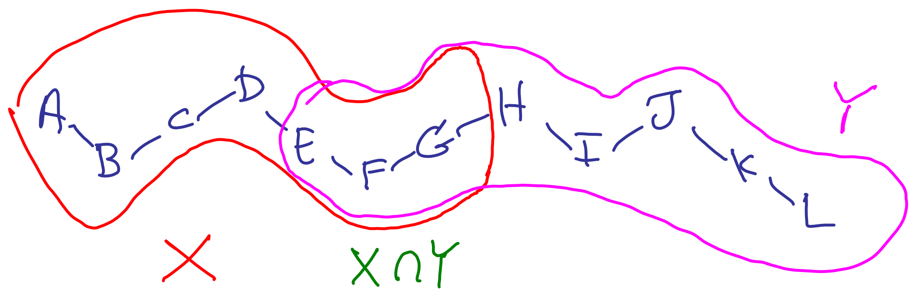


$$\begin{pmatrix} F^Y \\ F^N \end{pmatrix} = \begin{bmatrix} y f^Y \\ f^N + f^Y \end{bmatrix} = \underbrace{\begin{pmatrix} y & 0 \\ 1 & 1 \end{pmatrix}}_{M^{\downarrow}(y)} \begin{pmatrix} f^Y \\ f^N \end{pmatrix}$$

$$M[\text{Word}] = M^{\bullet}[y_1]\, M^{\bullet}[y_2] \cdots M^{\bullet}[y_{n-1}]\, M^{\uparrow}[y_n] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

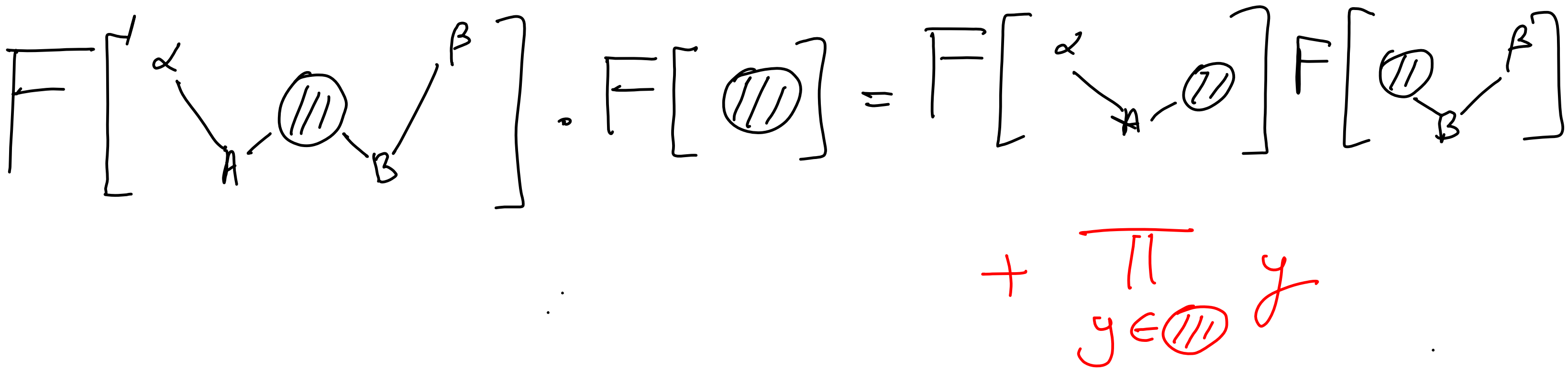
$$F = F^Y + F^N = B + D$$

Word Identities

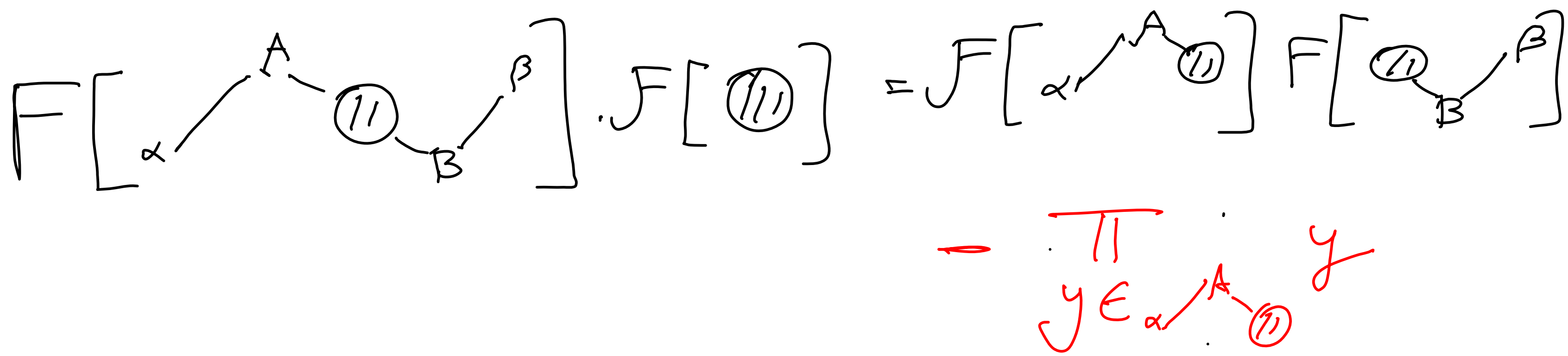


$$F_X \cdot F_Y = F_{X \cup Y} \cdot F_{X \cap Y} + \prod_{y \in X \cap Y} y \cdot Q(y).$$

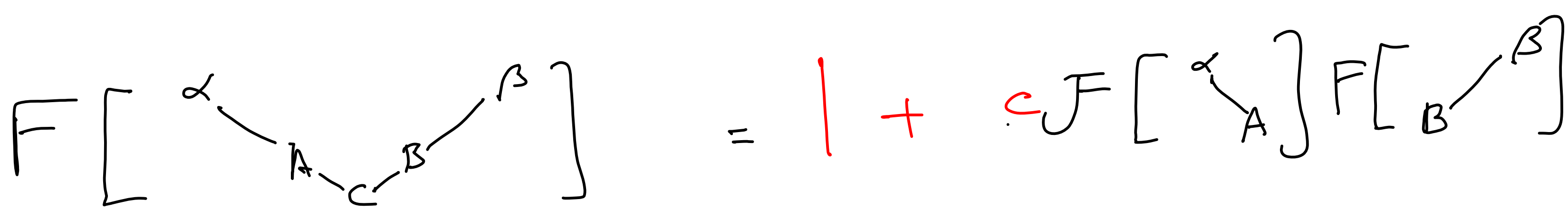
... But $Q(y)$ is not in general simple....



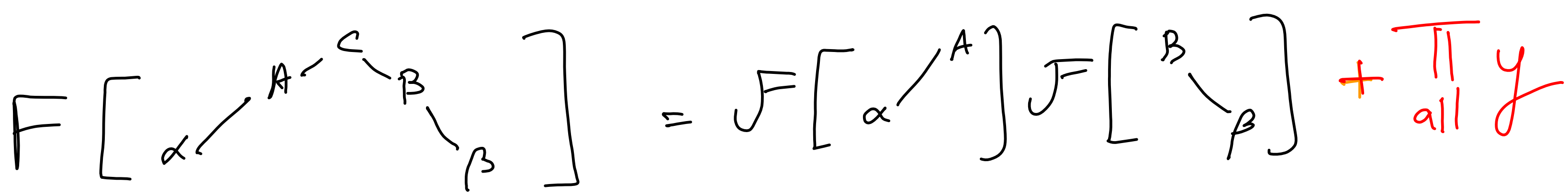
F
α
A
B
β
F
=
F
F
+ ∏ y
y ∈



F
α
A
B
β
F
=
F
F
− ∏ y
y ∈

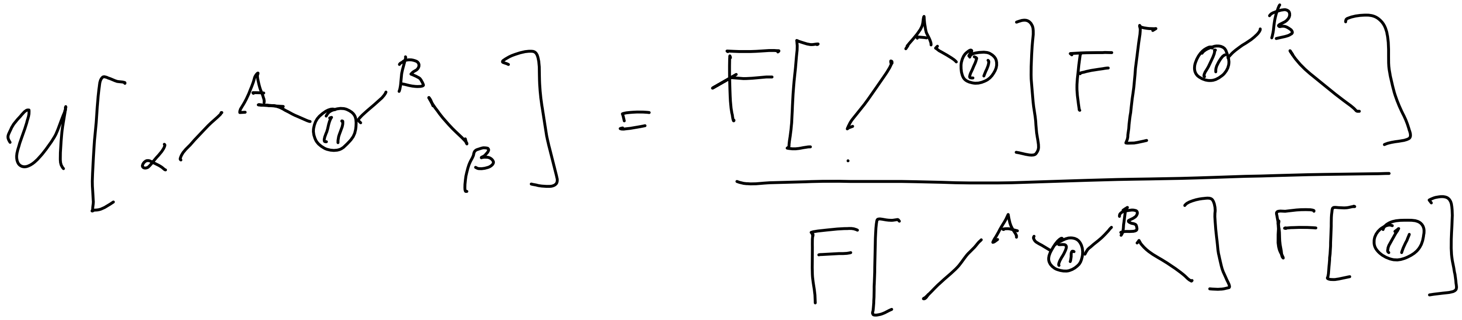


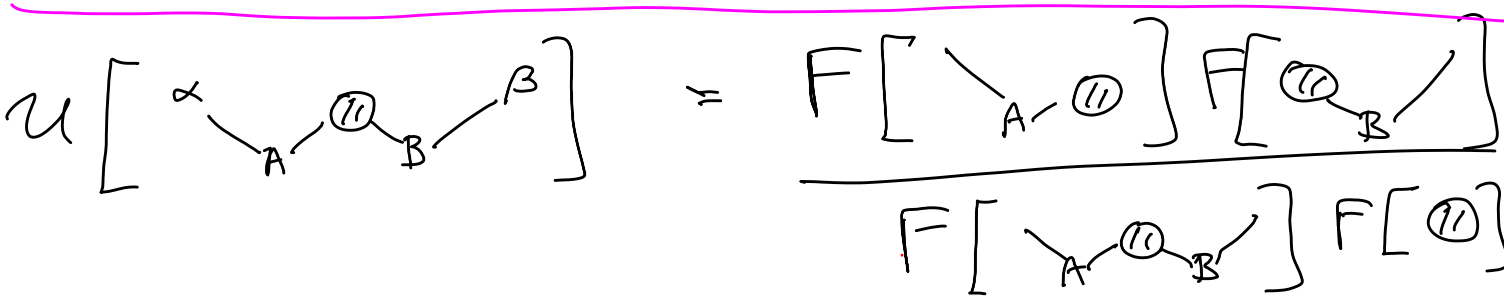
F
α
A
C
B
β
= 1 + c F
F

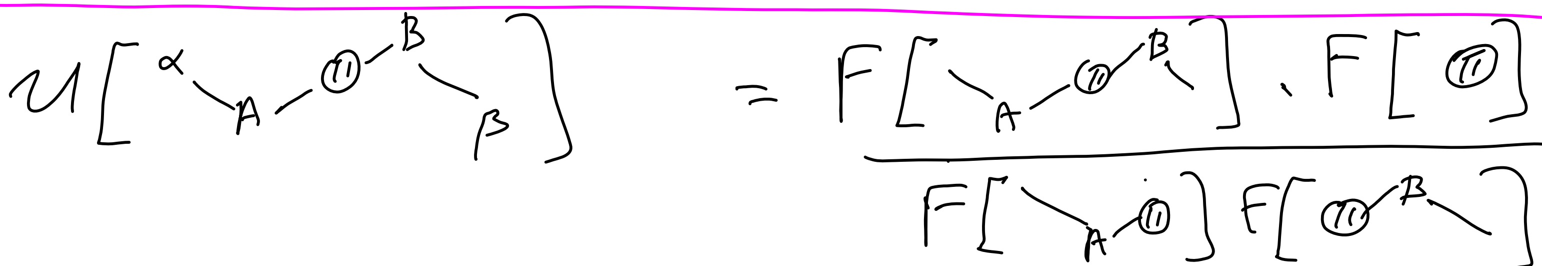


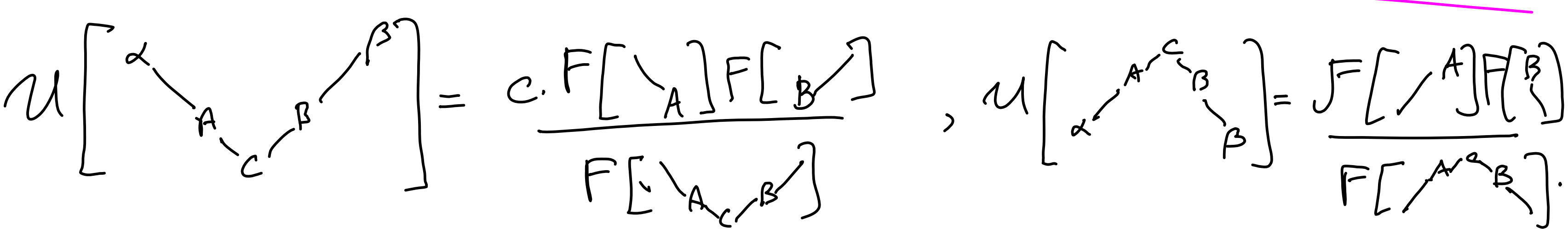
F
α
A
C
B
β
=
F
F
+ ∏ y
all

$$0 < \mathcal{U}[\text{Word}] \leq 1$$







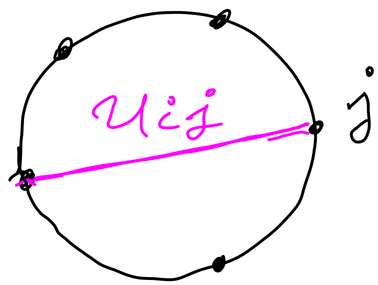


Binary Geometry :

$u_x \to 0$
All incomp.
$u_Y \to 1$!

$$\boxed{u_X + \prod_Y u_Y^{\#(X,Y)} = 1}$$

Solve u eqns → Directly [∞ product]
→ Hyperbolic geometry
→ COUNTING PROBLEM



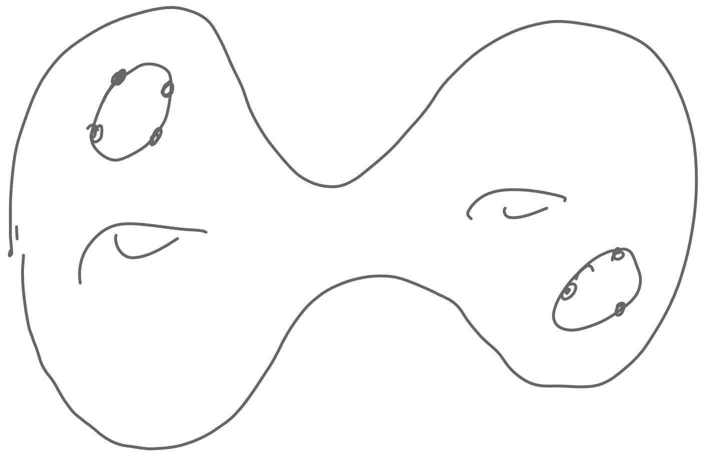
$$u_{ij} + \prod_{\substack{kl \\ \text{crossing} \\ ij}} u_{kl} = 1.$$

- Remarkably give a variety of correct dim!
- Positivity: $u_{ij} \geq 0 \longrightarrow u_{ij} \leq 1$

If $u_{ij} \to 0$, all u's for crossing chords $\longrightarrow 1$!

"Binary" geometry of compatibility combinatorics

Binary Realization For All Surfaces



$$u_X + \prod_Y u_Y^{\operatorname{int}\#(X,Y)} = 1$$

* "Global" description of compactification of Teichmuller space
  (c.f. "local" description of Fock–Goncharov)

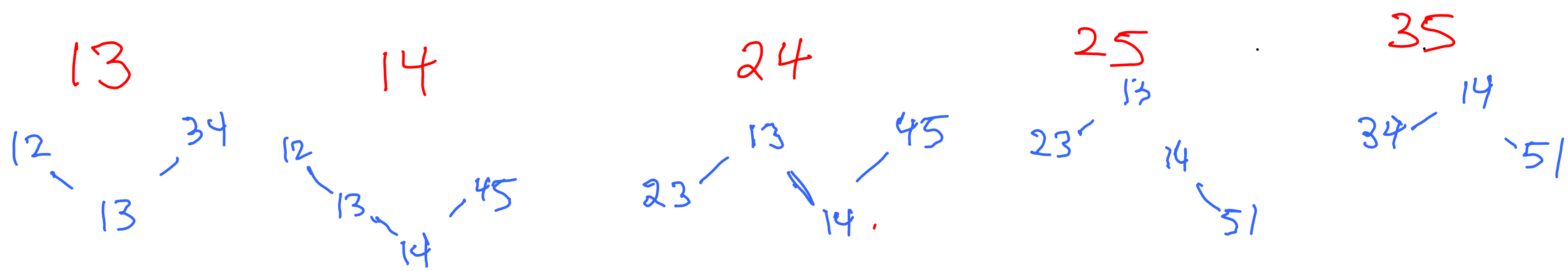
$$\mathrm{Tropical}\ u_X = u_X^t\,!$$

$u_X[y_i]$, put $y_i = e^{-t_i}$. Go to $|\vec{t}| \to \infty$

$$u_X[y_i \to e^{-t_i}] \longrightarrow e^{(\mathrm{Trop}\, u_X)(t)}$$

$$\boxed{u_X^t(\vec{t}) \equiv \mathrm{Trop}\, u_X(\vec{t})\,!}$$

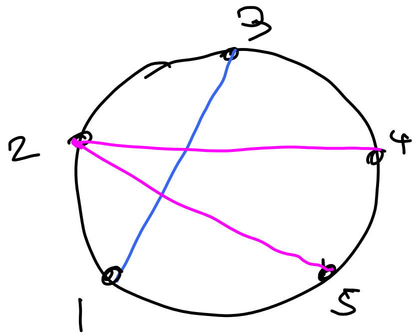
Ex: 5 pt tree



$$u_{13} = \frac{y_{13}\, F[\cdot]\, F[\cdot]}{F[13]\, F[\cdot]} = \frac{y_{13}}{1+y_{13}}, \qquad u_{14} = \frac{y_{14}\, F[13]\, F[\cdot]}{F\left[{}^{13}{}_{\backslash 14}\right] F[\cdot]} = \frac{y_{14}(1+y_{13})}{1+y_{14}+y_{13}y_{14}}$$

$$u_{24} = \frac{F\left[{}^{13}{}_{\backslash 14}\right] F[\cdot]}{F[13]\, F[14]} = \frac{(1+y_{14}+y_{14}y_{13})}{(1+y_{14})(1+y_{13})},$$

$$u_{25} = \frac{F[\cdot]\, F[14]}{F\left[{}^{13}{}_{\backslash 14}\right] F[\cdot]} = \frac{(1+y_{14})}{1+y_{14}+y_{14}y_{13}}, \qquad u_{35} = \frac{F[\cdot]\, F[\cdot]}{F[14]\, F[\cdot]} = \frac{1}{1+y_{14}}$$



$$u_{13} + u_{24}u_{25} = 1 \quad + \text{cyclic.}$$

$$u_{13} = \frac{y_{13}}{1+y_{13}}$$

$$u_{14} = \frac{(1+y_{13})\,y_{14}}{1+y_{14}+y_{13}y_{14}}$$

$$u_{24} = \frac{1+y_{14}+y_{14}y_{13}}{(1+y_{14})(1+y_{13})}$$

$$u_{25} = \frac{(1+y_{14})}{(1+y_{14}+y_{14}y_{13})}$$

$$u_{35} = \frac{1}{(1+y_{14})}$$

+ note when $y_{13}, y_{14} > 0$

$$0 < u_{13} \leq 1$$

Tropical Limit

Now, let's put $y_{13} = e^{-t_{13}}$, $y_{14} = e^{-t_{14}}$

As $|\vec{t}| \longrightarrow \infty$,

$$\underbrace{\left(1 + y_{14} + y_{14} y_{13}\right)}_{F} \longrightarrow \exp\left[\underbrace{\max(0, -t_{14}, -t_{14} - t_{13})}_{\mathrm{Trop}\,F}\right]$$

And so $\quad u_X \longrightarrow \exp\left[\mathrm{Trop}\, u_X\right] \ldots$

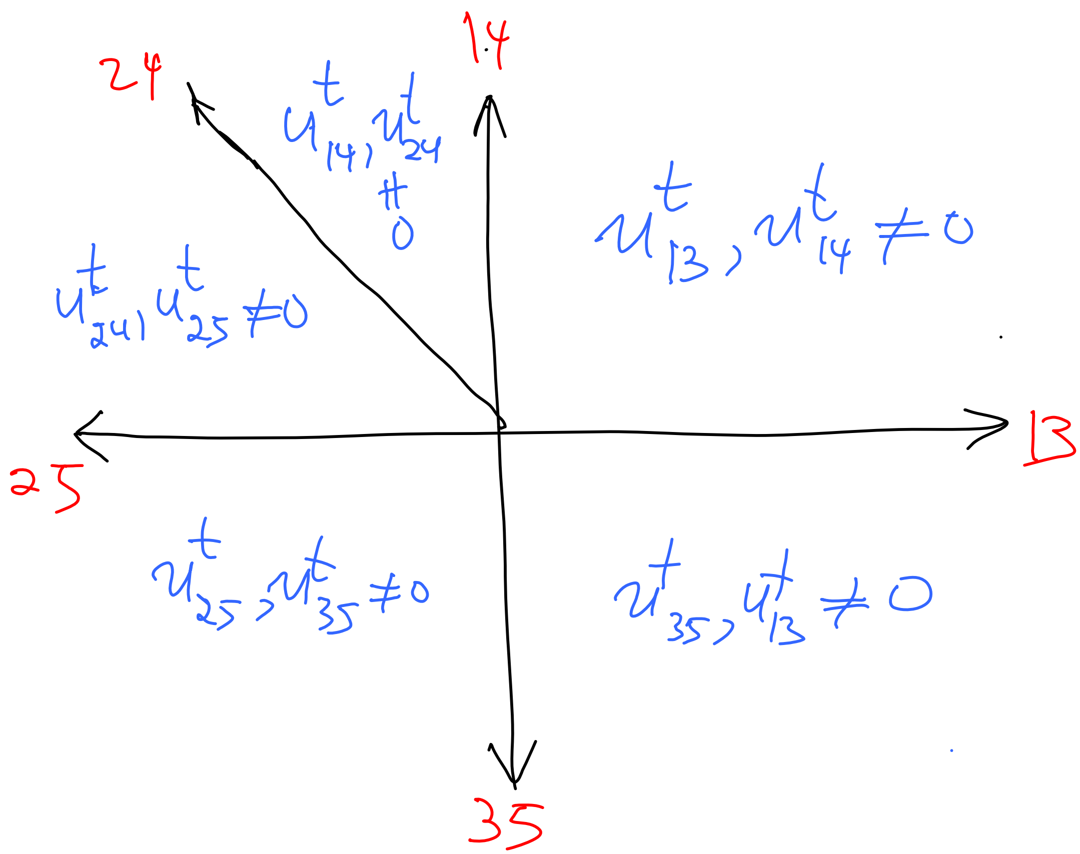
$$u_{13} = \frac{y_{13}}{1+y_{13}} \longrightarrow u^t_{13} = -t_{13} - \max(0, -t_{13})$$

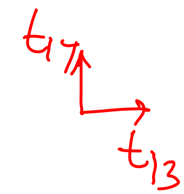
$$u_{14} = \frac{(1+y_{13})\,y_{14}}{1+y_{14}+y_{13}y_{14}} \longrightarrow u^t_{14} = -t_{14} + \max(0, -t_{13}) - \max(0, -t_{14}, -t_{14}-t_{13})$$

$$u_{24} = \frac{1+y_{14}+y_{14}y_{13}}{(1+y_{14})(1+y_{13})} \longrightarrow u^t_{24} = \max(0, -t_{14}, -t_{14}-t_{13}) - \max(0, -t_{14}) - \max(0, -t_{13})$$

$$u_{25} = \frac{(1+y_{14})}{(1+y_{14}+y_{14}y_{13})} \longrightarrow u^t_{25} = \max(0, -t_{14}) - \max(0, -t_{14}, -t_{14}-t_{13})$$

$$u_{35} = \frac{1}{(1+y_{14})} \longrightarrow u^t_{35} = -\max(0, -t_{14})$$

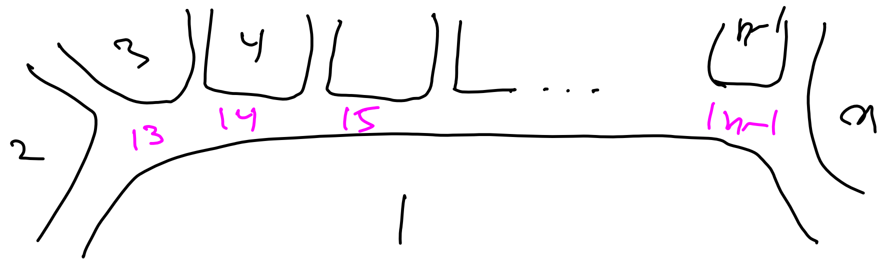




$$\boxed{-u^t_x[g_y] = \delta_{xy}}$$

Unified Schw. Param.

Trees for general n



$1j$: $12 - 13 - 14 - \cdots - 1j - jj{+}1$

jn : $jj{+}1 - 1j{+}1 - \cdots - 1n{-}1 - n1$

ij : $ii{+}1 - 1i{+}1 - 1i{+}2 - 1j - jj{+}1$

Only $F[W]$ is $F[\,1j - 1j{+}1 \cdots 1k\,]$,

$$\operatorname{trop} F = \max(0, -t_j, -t_j - t_{j+1}, \ldots, -t_j - \cdots - t_k)$$

$\#F$'s $\sim n^2$, $\#$Cones/Diag $\sim 4^n$!!

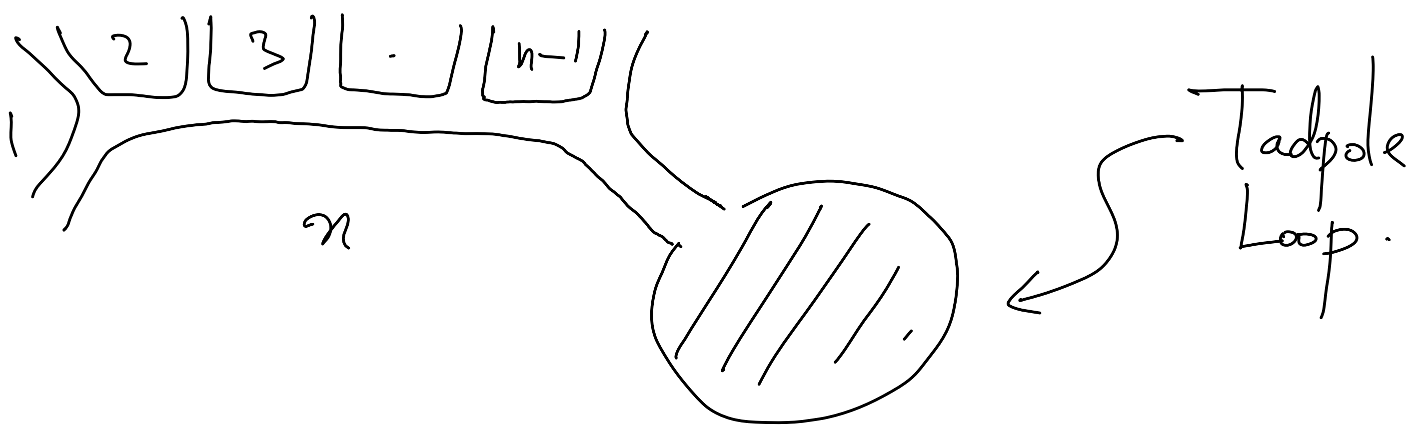
All-Loop Amplitude

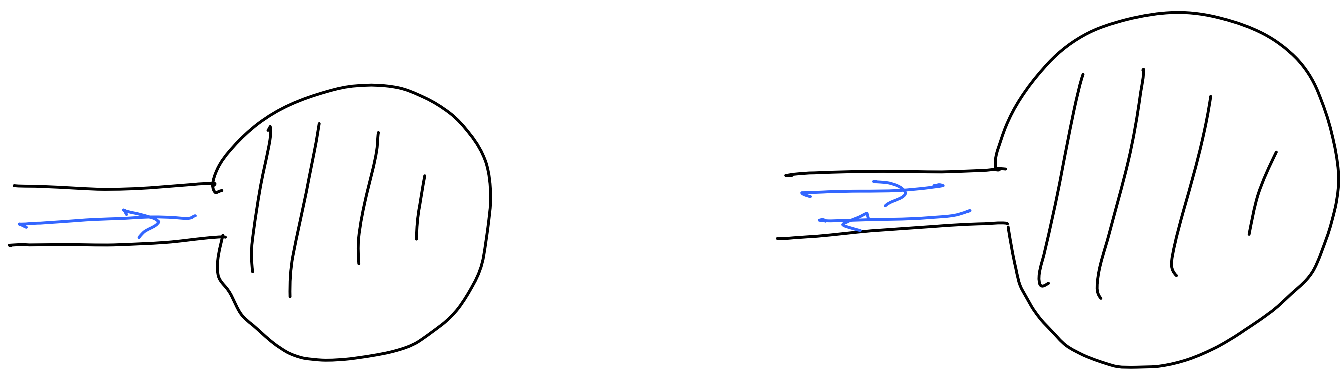
$$\mathcal{A} = \alpha'^{N} \int \frac{dt_1 \ldots dt_N \; d^D\ell_1 \ldots d^D\ell_L}{MCG} \prod_{\substack{\text{All Curves} \\ X}} e^{\alpha' \, \mathrm{Trop}\, u_X(t) \left[P_X^2 + m^2\right]}$$

$$= \int_{S^{N-1}} \frac{\langle t \, d^{N-1} t \rangle}{MCG} \frac{1}{\mathcal{U}^{\frac{D}{2}(L+1)-N}(u_X^t)} \frac{1}{\mathcal{F}(u_X^t)^{N-\frac{D}{2}}}$$

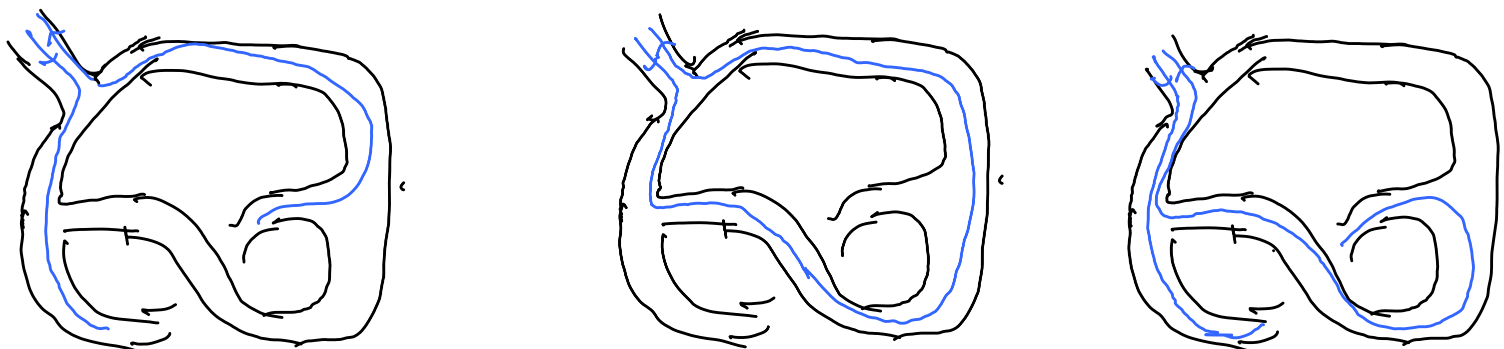
... Just need "Fundamental Domain" ...

All-Loop Tree-Tadpole "Factorization"





All-Loop Tree-Tadpole "Factorization"



Only Curves Needed For Fund. Domain

Particle Amplitudes

$$\mathcal{A} = \alpha'^{N} \int \frac{dt_1 \cdots dt_N \; d^D\ell_1 \cdots d^D\ell_L}{\mathrm{MCG}}$$

$$\prod_{\substack{\text{All Curves} \\ X}} e^{\alpha' \, \mathrm{Trop}\, u_X(t) \left[P_X^2 + m^2 \right]}$$

String Amplitudes!

$$\mathcal{A} = \alpha'^{N} \int \frac{dt_1 \cdots dt_N \; d^D\ell_1 \cdots d^D\ell_L}{MCG}$$

$$\prod_{\text{Really All Curves } X} \mathcal{U}_X^{\left[p_X^2 + m^2\right]}$$

$$\prod_{j\text{-wound closed curves } \Delta_j} \mathcal{U}_{\Delta_j}^{j^2}$$

(including self-int. curves! Don't affect FT limit)

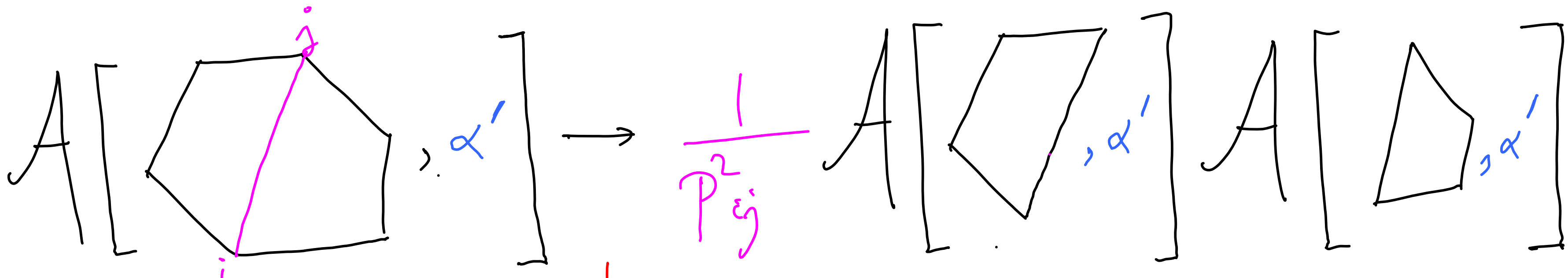
This preserves all correct (external) factoriz.

even @ finite α!, thanks to

Binary Property: singularities only

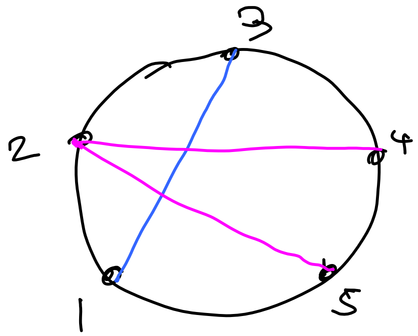
as $u_X \to 0$, but incompatible

$u_Y \to 1$ so drop out!



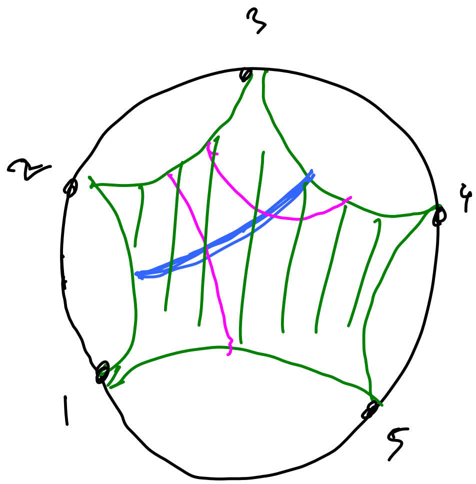
localizes to $u_{ij} \to 0$

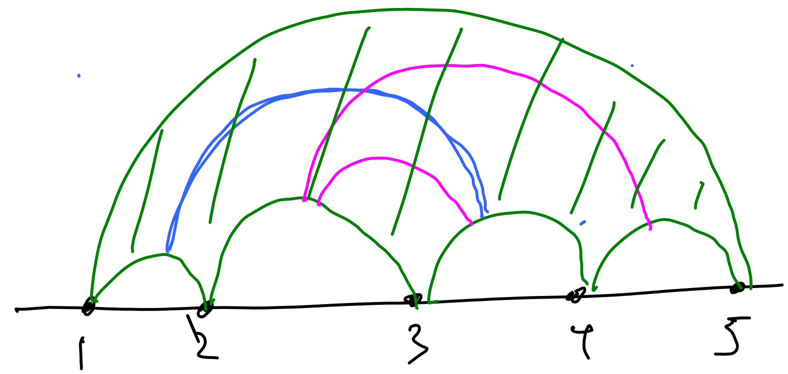
<u>Factorizes</u> @ finite α' because all incomp. $u_{kl} \to 1$



$u_{13} + u_{24}u_{25} = 1$ + cyclic.

Can Check: solved by

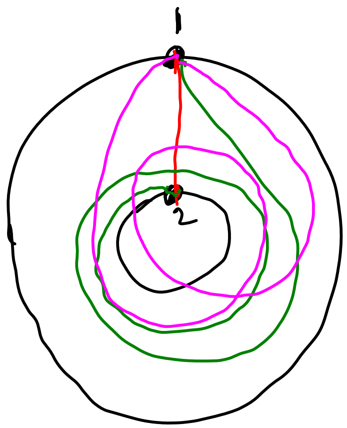


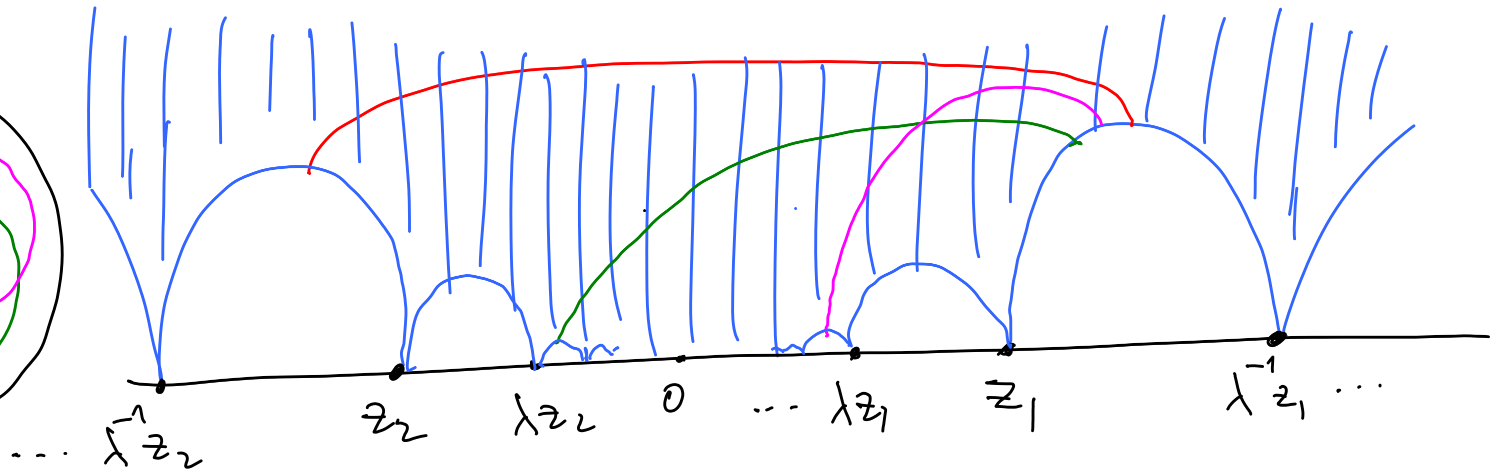


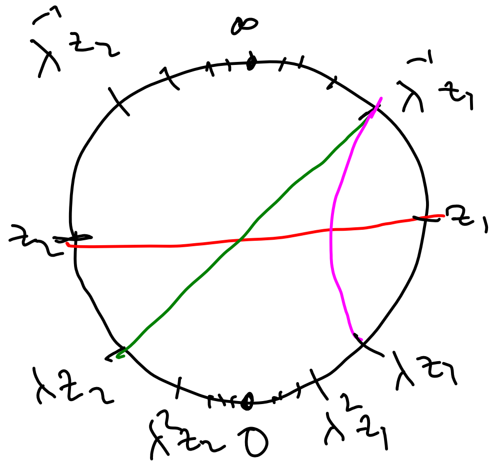
$z_1 < z_2 \cdots < z_5$

$$u_{ij} = \frac{(z_{i-1} - z_j)(z_i - z_{j-1})}{(z_{i-1} - z_{j-1})(z_i - z_j)}$$

Connection to Hyperbolic geometry + worldsheet







"∞-gon"

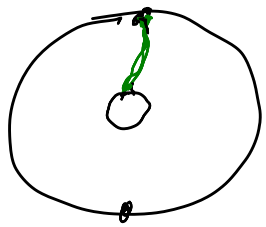
$$z_j = \lambda^j z_1, \; z_j = \lambda^j(-z_2)$$

$$u_{ab} = u_{a+1\, b+1}$$

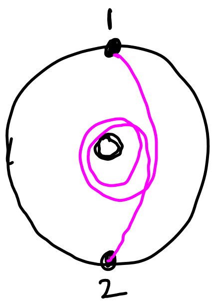
Binary equations for disk + identifications

$\Longrightarrow$ "maximal" binary equations for annulus!

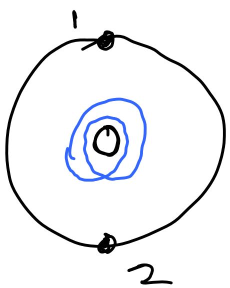
$$u_X + \prod_Y u_Y^{\text{int}\#[X,Y]} = 1$$



$$A = \int d^D\ell \quad \omega_{dlog} \quad u_{Y_1}^{\frac{1}{2}\ell^2 - 1} \quad u_{Y_2}^{\frac{1}{2}(\ell+k_1)^2 - 1}$$

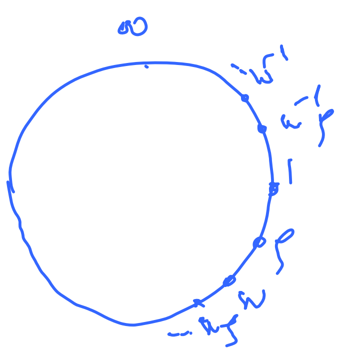


$$\prod_{n=1}^{\infty} u_{12,n}^{\frac{1}{2}k_1^2 - 1} \quad u_{21,n}^{\frac{1}{2}k_2^2 - 1} \quad \prod_{n=0}^{\infty} u_{11,n}^{\frac{1}{2}0 - 1} \quad u_{22,n}^{\frac{1}{2}0 - 1} \left(\prod_{j=1}^{\infty} u_{j}^{j^2} \right)^{A}$$



$$= \int \frac{dw\, d\rho}{w^2 (1-\rho)^2} \left(\frac{-2\pi}{\log w} \right)^{D/2} e^{-\frac{\log^2 \rho}{\log w}}$$

$$\underbrace{\text{Jacobi } \theta \rightarrow \left[\prod_{j} \frac{(1-w^j\rho)(1-w^j/\rho)}{(1-w^j)^2} \right]^2 \left(\prod_{j=1}^{\infty} (1-w^j) \right)^{A} \leftarrow \text{Ded. } \eta}_{\text{Enforce log sing. as } w \to 0}$$



$$\Longrightarrow \quad A = -24, \quad D = 26$$

Bosonic String

Some Immediate Questions/Generalizations

* Amplitude is clearly counting something — what is it?

* General theories — is there "tropical fan magic" in Numerators?

* Simplification as n particles $\longrightarrow \infty$!

$$
\begin{tikzcd}
\text{Associahedron} \arrow[r, phantom, "\subset"] \arrow[d] & \text{Tree Amplituhedron} \arrow[d] \\
\text{Surfacehedron all orders in } 1/N \arrow[r, phantom, "\subset"] & \text{Non-planar Amplituhedron?}
\end{tikzcd}
$$

$\mathcal{N}=4$ Fantasy

Gluons $\longleftrightarrow$ Amplituhedron $\longleftrightarrow$ Dual Amplituhedron (Unified Schwinger Rep) $\longleftrightarrow$ String in AdS

Here

$\phi^I_a \longleftrightarrow$ Surfacehedra $\longleftrightarrow$ Dual Surfacehedra (Unified Schwinger Rep) $\longleftrightarrow$ ~~Topical~~ String.