Fun with twistor-space geometries:

from SYM to ABJM & symbology of Feynman integrals

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Based on 2204.08297 with Chia-Kai Kuo and Zhenjie Li, Yaoqi Zhang

2207.13482 with Jiahao Liu, Yichao Tang, Qinglin Yang; see also Q. Yang 2203.16112; SH, Z. Li, R. Ma, Z. Wu, Q. Yang & Y. Zhang, 2206.04609

Amplitudes in QFT have continued offering us amazing surprises for decades, especially since twistor-string revolution [Witten 03]

A lot inspired by rethinking amplitudes using twistors variables [Penrose,...], *e.g.* BCF(W), CSW *etc.*; more recently positive Grassmannian & the amplituhedron [Arkani-Hamed et al] for all-loop integrands in $\mathcal{N}=4$ SYM & beyond

Two recent adventures with geometries in (momentum-) twistor space:

- An unexpected relation between $(n=4)$ amplituhedron & all-loop integrands in $\mathcal{N}=4$ SYM [Arkani-Hamed, Trnka 13] & ABJM theory & enormous reduction of *negative geometries* [Arkani-Hamed, Henn, Trnka 21; Chicherin's talk]

- A new, powerful method for the symbology for Feynman integrals (in dim. reg.): intersecting lines in twistor space for leading singularities (for DCI integrals [Q. Yang 22]) $\implies$ explains *cluster algebras* [Chicherin et al 20, He et al 21]

Motivations

Rich mathematical structures revealed by amplituhedron [c.f. talks by Ferro, Heslop...]

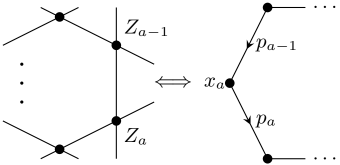
- higher-loop less explored (even for $n=4$): simplified models $\to$ solve integrands, integrals & even non-perturbatively?
- beyond $\mathcal{N}=4$ SYM: all-loop integrands in other theories? [Nima's talk]
- ABJM: simplest guess *i.e.* dim. reduced amplituhedron works (for $n=4$)!

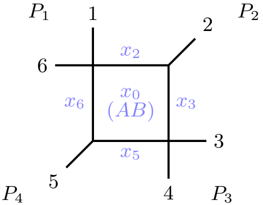
MPL symbology [Goncharov et al...] (cluster algebras [Golden et al]...) & bootstrap for $\mathcal{N}=4$ SYM $(n=6,7)$ [Dixon et al; Drummond et al...] $\to$ Feynman integrals

- twistors good for parametrizing/rationalizing $D=4$ kinematics & symbol letters: non-DCI, external masses, non-planar
- (extended) cluster algebras for some all-loop, DCI integrals; deep connections with tropical Gr (also for $n \geq 8$) [Drummond et al; Arkani-Hamed et al;...]
- $D=4-2\epsilon$ integrals? symbology (including LS) from twistor geometries $\to$ more structures in canonical differential eqs (for UT basis)? [Henn; ...]

Momentum twistors [Hodges]: "light rays" of dual spacetime, inspired by duality of $\mathcal{N} = 4$ SYM planar amplitudes with Wilson loops [Alday et al; Brandhuber et al; ...]

- $Z^I = (\lambda^\alpha, \mu^{\dot\alpha} := x^{\alpha,\dot\alpha}\lambda_\alpha)$: manifest dual conformal symmetry [Drummond et al]
- null polygon: $\lambda_a \tilde\lambda_a = p_a = x_{a+1} - x_a \leftrightarrow \{Z_1, \cdots, Z_n\}$ for n edges; $x_a := (Z_{a-1}, Z_a)$ is a line in twistor space





- massive momentum = 2 massless ones: 2me box $(P_1^2, P_3^2 \neq 0)$ depends on 4 lines x_2, x_3, x_5, x_6, *e.g.* $P_1^2 = \frac{\langle 5612 \rangle}{\langle 56 I_\infty \rangle \langle 12 I_\infty \rangle}$
- (dual) loop momentum $x_0 \leftrightarrow$ a line (AB) in twistor space

Part I: Reduced amplituhedron for ABJM

The n-point L-loop amplituhedron: $Z_{a=1,\cdots,n}$ for external kinematics and $(AB)_{i=1,\cdots,L}$ for loop momenta

For $n=4$ (only $k=0$): a $4L$-dim geometry in $(AB)_i$ space (Z's fixed):

$$\langle (AB)_i 12\rangle > 0, \quad \langle (AB)_i 23\rangle > 0, \quad \langle (AB)_i 34\rangle > 0, \quad \langle (AB)_i 14\rangle > 0,$$
$$\langle (AB)_i 13\rangle < 0, \quad \langle (AB)_i 24\rangle < 0$$

as well as mutual positivity: $\langle (AB)_i (AB)_j\rangle > 0$ [Arkani-Hamed, Trnka 13]

External & loop momenta in $D=3$: twistor-space lines with symplectic conditions (in momentum space: $\lambda = \tilde{\lambda}$) [Elvang et al 14]:

$$\mathbf{\Omega}_{IJ} Z_a^I Z_{a+1}^J = \mathbf{\Omega}_{IJ} A_i^I B_i^J = 0\,, \quad \text{with } \mathbf{\Omega} = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix}.$$

($a=1,2,\cdots,n$ and $i=1,\cdots,L$) $\to$ reduced amplituhedron

Focusing on $n=4$: a $3L$-dim geometry in constrained $(AB)_i$ for $D=3$

With parametrization $Z_{A_i} = Z_1 + x_i Z_2 - w_i Z_4, \quad Z_{B_i} = y_i Z_2 + Z_3 + z_i Z_4$
$\Longrightarrow$ def. of $n = 4$ reduced amplituhedron:

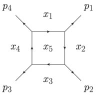
$$\forall i : x_i, y_i, z_i, w_i > 0, \quad x_i z_i + y_i w_i = 1,$$
$$\forall i, j : (x_i - x_j)(z_i - z_j) + (y_i - y_j)(w_i - w_j) < 0$$

First look at $L = 1$: the *canonical form* in $D = 4 =$ box integral

$$\Omega_1^{(D=4)} = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

Dim. reduction $\rightarrow D = 3$ box with ϵ num. $=$ one-loop ABJM integrand [Chen, Huang 11]:

$$\Omega_1 = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w}\delta(xz + yw - 1) = \frac{d^3(AB)\langle 1234\rangle^{3/2}(\langle AB13\rangle\langle AB24\rangle)^{1/2}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

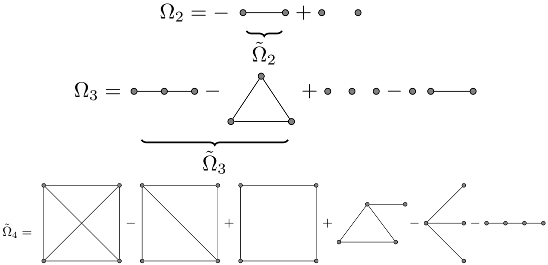


$$= \int d^3 x_5 \frac{\epsilon(5,1,2,3,4)}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}, \qquad \epsilon(i,j,k,l,m) \equiv \epsilon_{\mu\nu\rho\sigma\tau} x_i^\mu x_j^\nu x_k^\rho x_l^\sigma x_m^\tau$$

Negative geometries $\rightarrow$ bipartite geometries

Decomposition into sum of negative geometries: 

The sum of connected graphs gives logarithm of amplitudes [Arkani-Hamed et al], *e.g.*

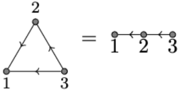


Huge simplifications when reduced to $D = 3$: only bipartite graphs (with "arrows") survive! *e.g.* $\tilde{\Omega}_3$ only chain (no triangle), $\tilde{\Omega}_4$ only trees & the box

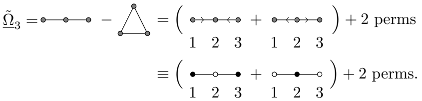
Transitive reduction: $D=3$ mutual negativity $\implies$ (time) ordering, no closed loop $\to$ all non-bipartite (directed) graphs cancel (theorem [He et al 22]);

$$\sum_{\mathscr{G}}(-)^E \mathcal{A}_{\mathscr{G}} \overset{\text{time}}{\underset{\text{order}}{=\!=}} \sum_{\text{directed acyclic } G} (-)^E \mathcal{A}_G \overset{\text{trans.}}{\underset{\text{red.}}{=\!=}} \sum_{\text{bipartite } g} (-)^E \mathcal{A}_g$$

e.g.



$\implies$



A tiny fraction ($\to 0$ as $L\to\infty$) of graphs remain (relatively simple ones):

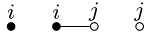
L	top. of G	top. of g	directed acyclic graphs	bipartite g
2	1	1	2	1
3	2	1	18	3
4	6	3	446	19
5	21	5	26430	195
6	112	17	3596762	3031
7	853	44	1111506858	67263

singularity structure: black (white) node has only poles $y_i w_i$ ($x_i z_i = 1 - y_i w_i$) !

Shorthand notation: *e.g.* $\underline{\Omega}_1 = \frac{c\epsilon_1}{s_1 t_1}$ (strip off $d^3\ell$)

$$\ell_i \equiv (AB)_i, \qquad c \equiv \langle 1234 \rangle, \qquad \epsilon_i \equiv (c\langle \ell_i 13\rangle\langle \ell_i 24\rangle)^{1/2};$$
$$s_i \equiv \langle \ell_i 12\rangle\langle \ell_i 34\rangle \sim y_i w_i, \qquad t_i \equiv \langle \ell_i 23\rangle\langle \ell_i 14\rangle \sim z_i w_i, \qquad D_{ij} \equiv -\langle \ell_i \ell_j\rangle.$$

Canonical forms for bipartite geometries: fix numerators given the poles

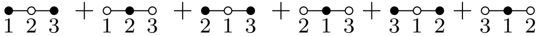


$$\frac{1}{s_i} \qquad \frac{1}{D_{ij}} \qquad \frac{1}{t_j}.$$

The log of amps for $L = 2, 3$: only (non-planar) ladders!

$\tilde{\underline{\Omega}}_2 =$ 

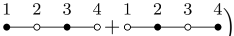
$$= \frac{2c^2}{D_{12}}\left(\frac{1}{s_1 t_2} + \frac{1}{t_1 s_2}\right) = -2\frac{\langle 1234\rangle^2}{\langle \ell_1 12\rangle\,\langle \ell_1 34\rangle\,\langle \ell_1 \ell_2\rangle\,\langle \ell_2 23\rangle\,\langle \ell_2 14\rangle} + (\ell_1 \leftrightarrow \ell_2)$$

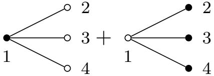
$\tilde{\underline{\Omega}}_3 =$ 

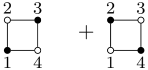
$$= \frac{4c^2\epsilon_2}{s_1 t_2 s_3 D_{12} D_{23}} + (s \leftrightarrow t) + 2 \text{ perms.},$$

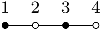
nicely confirm $n = 4$ 3-loop ABJM integrand conjectured in [Bianchi et al 11]

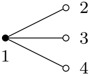
$L = 4$: only chain, star, box graphs $\implies \tilde{\Omega}_4 = -C - S + B$:

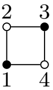
$C = \Big($  $\Big) + 11$ perms

$S = \Big($  $\Big) + 3$ perms

$B = \Big($  $\Big) + 2$ perms ,

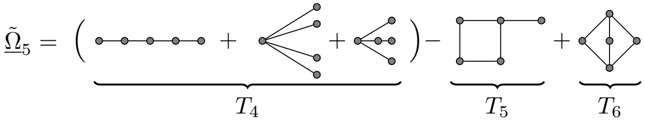
 $= 8c^2 \dfrac{\epsilon_2\epsilon_3}{D_{12}D_{23}D_{34}s_1t_2s_3t_4}$

 $= 8c^3 \dfrac{t_1}{D_{12}D_{13}D_{14}s_1t_2t_3t_4}$

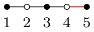
 $= 4\dfrac{4\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - c(\epsilon_1\epsilon_3 N^t_{24} + \epsilon_2\epsilon_4 N^s_{13}) - c^2 N^{\text{cyc}}_{1,2,3,4}}{D_{12}D_{23}D_{34}D_{41}s_1t_2s_3t_4}$

$$N^s_{ij} := \langle \ell_i 12\rangle\langle \ell_j 34\rangle + \langle \ell_j 12\rangle\langle \ell_i 34\rangle$$
$$N^t_{ij} := \langle \ell_i 14\rangle\langle \ell_j 23\rangle + \langle \ell_j 14\rangle\langle \ell_i 23\rangle$$
$$N^{\text{cyc}}_{i,j,k,l} := \langle \ell_i 12\rangle\langle \ell_j 34\rangle\langle \ell_k 12\rangle\langle \ell_l 34\rangle + \text{cyc}(1,2,3,4)$$

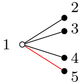
$L=5$: contributions from 5 topologies (4, 5, 6 edges):



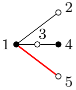
- The trees extremely simple (inverse soft construction for general trees)



$$= \frac{8c^3\epsilon_2}{s_1t_2s_3t_4D_{12}D_{23}D_{34}} \times \frac{2\epsilon_4}{s_5D_{4,5}}$$

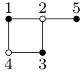


$$= \frac{8c^3s_1}{t_1s_2s_3s_4D_{12}D_{13}D_{14}} \times \frac{2\epsilon_1}{s_5D_{15}}$$



$$= \frac{8c^3\epsilon_3\epsilon_1}{s_1t_2t_3s_4D_{12}D_{13}D_{34}} \times \frac{2t_1}{\epsilon_1t_5D_{15}}$$

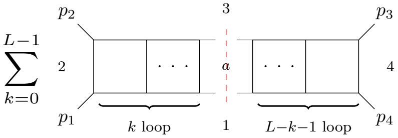
- More non-trivial forms for T_5 (box + edge) and T_6 ($K_{2,3}$ graph) *e.g.*



$$= 8c\frac{4\epsilon_1\epsilon_3\epsilon_4s_2 - \epsilon_1\epsilon_2\epsilon_3N^t_{24} - c(-\epsilon_1t_2N^t_{34} - \epsilon_3t_2N^t_{14} + \epsilon_4s_2N^s_{13} + \epsilon_2N^{\rm cyc}_{1,2,3,4})}{s_1t_2s_3t_4s_5D_{12}D_{23}D_{34}D_{41}D_{25}}$$

Our construction makes symmetries manifest, *e.g.* parity: even/odd ϵ for even/odd L; importantly, various all-loop ABJM cuts derived from geometry!

- Soft cut: *e.g.* $\langle \ell_i 12\rangle = \langle \ell_i 23\rangle = \langle \ell_i 34\rangle = 0$ or $y_i = z_i = w_i = 0 \implies$ manifest mutual positivity $D_{i,j} > 0$ for any j, residue $= (L-1)$-loop
- Vanishing cut: any cut isolating odd-point amplitude, *e.g.* $w_i = w_j = D_{i,j} = 0$ (triple cut) $\implies D_{i,j} \leq 0$, the residue vanishes; similarly, five-point cut $w_i = y_j = D_{i,j} = 0$ vanishes
- Double (unitarity) cut: $\langle \ell 14\rangle = \langle \ell 23\rangle = 0$

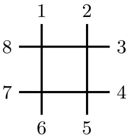


Explicitly checked up to $L = 5$. Can we prove it from the geometry?

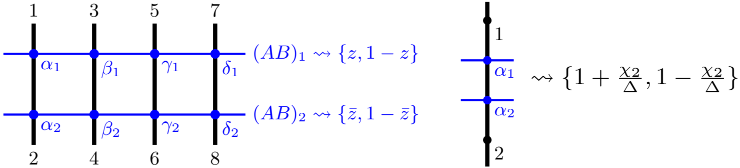
Part II: Symbology of Feynman integrals from twistor geometries

Cutting propagators $\leftrightarrow$ intersecting lines in twistor space: Schubert problems

One-loop boxes: 2 solutions for quadruple cut; compute cross-ratios on any line with ≥ 4 points, *e.g.* DCI four-mass box: leading singularity (LS) $\propto \Delta^{-1}$:



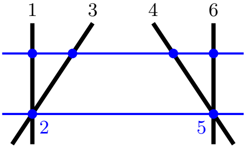
$$\int \frac{\mathrm{d}^4A\,\mathrm{d}^4B}{\mathrm{vol}\,GL(2)\,\cancel{\langle ABI_\infty\rangle^4}}\,\frac{\cancel{\langle\lambda_A\lambda_B\rangle^4}\langle\lambda_1\lambda_2\rangle\langle\lambda_3\lambda_4\rangle\langle\lambda_5\lambda_6\rangle\langle\lambda_7\lambda_8\rangle}{\langle AB12\rangle\langle AB34\rangle\langle AB56\rangle\langle AB78\rangle}$$



$$z\bar{z} = \mathbf{u} = \frac{\langle 1234\rangle\langle 5678\rangle}{\langle 1256\rangle\langle 3478\rangle},\ (1-z)(1-\bar{z}) = \mathbf{v} = \frac{\langle 1278\rangle\langle 3456\rangle}{\langle 1256\rangle\langle 3478\rangle},\ \Delta = \sqrt{(1-\mathbf{u}-\mathbf{v})^2 - 4\mathbf{u}\mathbf{v}}$$

In $\mathcal{N}=4$ SYM: generate A_3, E_6 DCI alphabets for $n=6,7$ [Nima's talk 20]; $9+9$ algebraic & ~ 200 rational letters for $n=8$; numerous DCI integrals [Q. Yang 22]

2me box with $(12), (23), (45), (56)$: reproduce DCI letters of finite-part



$$(AB)_1 = (\bar{2} \cap \bar{5}) = \{\frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle}, 1 - \frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle}\}$$

$$(AB)_2 = (25) \rightsquigarrow \{\ \}$$

$$\begin{array}{c} (12)\cap\bar{5} \\ (56)\cap\bar{2} \end{array} \begin{pmatrix} \overset{(12)\cap\bar{5}}{1} & \overset{(23)\cap\bar{5}}{\langle 2356\rangle} & \overset{(45)\cap\bar{2}}{-\langle 1235\rangle} & \overset{(56)\cap\bar{2}}{0} \\ 0 & -\langle 2456\rangle & \langle 1245\rangle & 1 \end{pmatrix} \implies \frac{(14)(23)}{(13)(24)} = \frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle} = 1 - \frac{(12)(34)}{(13)(24)}$$

For m points on a line $\to \frac{m(m-3)}{2}$ cross-ratios, forming A_{m-3} with u eqs [c.f. ABHY 18]:

$$\mathcal{U} = \frac{(ab)(cd)}{(ac)(bd)}, \ \mathcal{V} = \frac{(ad)(bc)}{(ac)(bd)}, \quad \mathcal{U} + \mathcal{V} = 1$$

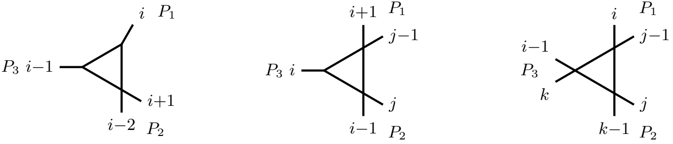
for any four points $\{a, b, c, d\} \to A_1 \subset A_{m-3}$ sub-algebra; equivalently $\log \mathcal{U}$ reads

$$L([ad],[bc]) := \int_a^d \mathrm{d}\log\frac{(xc)}{(xb)} = L([bc],[ad]), \quad L([ab],[cd]) = L([cd],[ab]) = \log \mathcal{V}$$

Replace the $d\log$ form by $\Omega = \frac{dx}{y}$ on elliptic curve of *e.g.* $n = 10$ double-box [Bourjaily et al] $\implies$ "elliptic symbol letters" [Q. Yang, C. Zhang; c.f. Wilhelm's talk]

Feynman integrals in $D = 4 - 2\epsilon$ depend on I_∞ (infinity line), so do general integrals in $D = 4$, *e.g.* triangles → building blocks for 1-loop integrals

$$\int \frac{\mathrm{d}^4 A\, \mathrm{d}^4 B}{\mathrm{vol}\, GL(2)\ \langle ABI_\infty\rangle^{\cancel{4}}} \frac{\cancel{\langle ABI_\infty\rangle^3} \langle i{-}1iI_\infty\rangle\langle j{-}1jI_\infty\rangle\langle k{-}1kI_\infty\rangle}{\langle ABi{-}1i\rangle\langle ABj{-}1j\rangle\langle ABk{-}1k\rangle}$$



$$\begin{cases} [i{-}2i{-}1i] := I_\infty \cap \overline{i{-}1} \\ [i{-}1ii{+}1] := I_\infty \cap \bar{i} \end{cases}$$

$$\begin{cases} [ij{-}1j] \\ [i{-}1ii{+}1] \end{cases}$$

$$\alpha_\pm(ijk) = P + Q\frac{\cdots + \Delta_{ijk}}{\cdots}$$

No cross-ratios, since no dimensionless var.

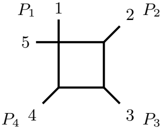
$$(AB)_1 \rightsquigarrow \{\tfrac{m_1^2}{m_2^2}, 1 - \tfrac{m_1^2}{m_2^2}\}$$

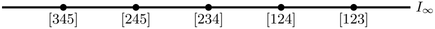
$$(AB)_2 \rightsquigarrow \{\ \}$$

$$(AB)_1 \rightsquigarrow \{z, 1 - z\}$$

$$(AB)_2 \rightsquigarrow \{\bar{z}, 1 - \bar{z}\}$$

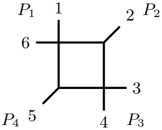
Proposal: symbol letters from intersections on lines of (non-DCI) Schubert problems, *e.g.* cutting 4 factors including $\langle ABI_\infty\rangle$ → intersections on I_∞

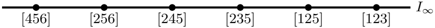




A_2

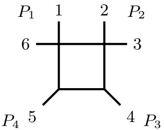
$\{m_1^2, s, t, m_1^2{-}s{-}t, m_1^2{-}s, m_1^2{-}t\}$

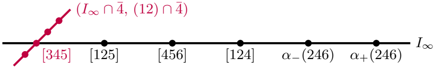




A_3

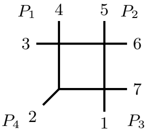
$\{m_1^2, m_3^2, s, t, m_1^2{-}s, m_3^2{-}s, m_1^2{-}t, m_3^2{-}t, m_1^2{+}m_3^2{-}s{-}t, m_1^2 m_3^2{-}st\}$

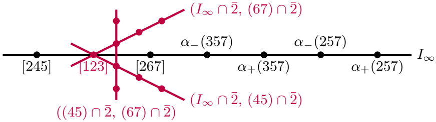




$A_3 \cup A_1 \subset C_3$

$\{s, t, m_1^2, m_2^2, m_1^2 - t, m_2^2 - t, \Delta_{2,4,6}, \frac{z}{\bar{z}}, \frac{1-z}{1-\bar{z}}, \frac{sz(1-\bar{z})+t}{s\bar{z}(1-z)+t},$
$m_1^2 m_2^2 - m_1^2 t - m_2^2 t + st + t^2\}$



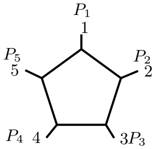


$A_4 \cup (A_1)^3 \subset C_4$

alphabets see also [Chicherin, Henn, Papathanasiou 21]

General one-loop integrals: pentagons

Pentagons sufficient for one-loop integrals, *e.g.* 0-mass case (union of 1-mass box cases + parity-odd letters from mixing different boxes)

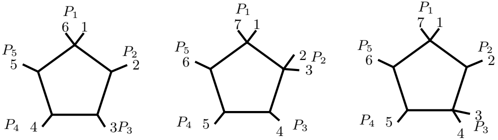


$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}P_1^\mu P_2^\nu P_3^\rho P_4^\sigma = \sqrt{\det(2P_i\cdot P_j)} \xrightarrow{\text{parity}} -\mathrm{tr}_5$$

10 points on I_∞ (only 4-dim) $\Longrightarrow$ 24 independent cross-ratios

parity-inv. subspace $\begin{cases} W_1 = s_{12},\ W_{11} = s_{12} - s_{45},\ W_{16} = s_{12} + s_{23} - s_{45}, \\ W_{26} = \frac{s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12} + \mathrm{tr}_5}{s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12} - \mathrm{tr}_5}, \end{cases}$ +cyclic.

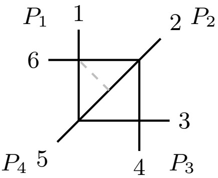
$\rightarrow$ correct alphabet except for tr_5 itself (LS of $D=6$ pentagon)

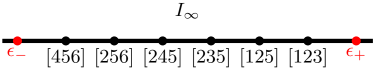


Similarly one-loop pentagon with 1 and 2 masses: 29, 40 (2me) and 45 (2mh), agree with alphabets from DE (1 or 2 extra two-loop letters)

Equiv. to DCI ones with a point to I_∞: *e.g.* 1m (2me) box kinematics $\simeq A_2(A_3)$ sub-algebra of $\mathrm{Gr}(4,7) \sim E_6$ ($\mathrm{Gr}(4,8)$) [w. Li, Yang 21]

Higher loops and bootstrap





- 2me box kinematics: only LS of slashed-box contributes new intersections on I_∞; minor$(\epsilon_+, \epsilon_-) \propto \Delta_{nc} = \sqrt{(s+t)^2 - 4m_1^2 m_3^2}$: new, LS square root;
- cross-ratios of the form $\frac{(X_{1,\epsilon_+})(X_{2,\epsilon_-})}{(X_{1,\epsilon_-})(X_{2,\epsilon_+})}$ $\to$ algebraic letters with Δ_{nc}; mix with 1-loop Schubert $\to$ 3 (indep.) algebraic letters [Chicherin et al 18, Abreu et al 20]

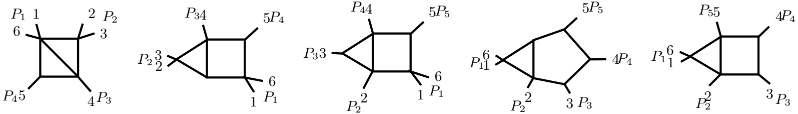
$$L_1 = \frac{s+t+\Delta_{nc}}{s+t-\Delta_{nc}}, \quad L_2 = \frac{s-t+\Delta_{nc}}{s-t-\Delta_{nc}}, \quad L_3 = \frac{-2m_1^2+s+t+\Delta_{nc}}{-2m_1^2+s+t-\Delta_{nc}}.$$

- $9+3$ alphabet, known as extended A_3, obtained in [w. Li, Yang 21]

Bootstrap $\ell = 2, 3$ (finite) integrals with ES/adjacency [Caron-Huot et al; Drummond et al]: exactly L_i are last entries (derivatives A_3 functions, "truncated" DE)

$$\mathrm{d}I^{(\ell)} = \sum_{i=1}^{3} F_i^{(2\ell-1)}(A_3)\, \mathrm{d}\log L_i, \quad \text{for } \ell = 2, 3$$

1m pentagon: 4-pt 2me+2mh subtopologies + genuine 5-pt integrals $\implies$ all relevant (+ irrelevant) letters of DE (except for tr_5)



Such cross-ratios in twistor space generalize the notion of LS (a special case) by mixing different LS; hint for such cross-ratios in IBP & canonical DE?

Generate more two-loop alphabets *e.g.* for 3m box kinematics, agree with DE alphabet [Dlapa et al 21]; 4m alphabets recently computed w. 74 MIs, 68 letters (11 LS square roots, 34 "mixed" algebraic letters *etc.*) [SH, Li, Ma, Wu, Yang, Zhang 22]

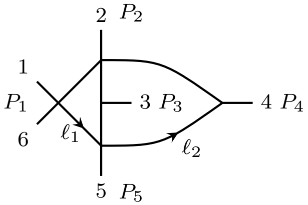
Possible to predict letters for *e.g.* 2m pentagon or even hexagon kinematics [c.f. Henn's talk] (lots of MIs, requires more works), but need selections

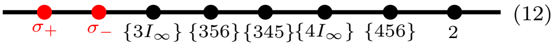
Easy to get different orderings, *e.g.* $3! = 6$ orderings of $A_2 \to C_2$ alphabet [Chicherin et al; Dixon et al] $\to n = 3$ form factors, valid through $L = 8$ [c.f. Dixon's talk]

Non-planar integrals: inspired by [Bern et al 18...], possible to write LS using twistors;

e.g. cutting hexabox integrals with $\ell_1^2 = (y_1 - x_1)^2$, $\ell_2^2 = (y_2 - x_5)^2$, ..., $(y_1 - y_2)^2$ and $(y_1 - y_2 + x_{3,4})^2 \implies \langle AB12\rangle = ... = \langle CD45\rangle = \langle ABCD\rangle = 0$

& $\langle ABI_\infty\rangle\langle CD\ \bar{3} \cap (3I_\infty)\rangle - (AB \leftrightarrow CD) = 0$





Inverse LS $\propto \Sigma_5^{(1)}$ [Abreu et al 21]: produce *e.g.* on (12) two new intersections; by mixing with one-loop points, $\frac{(\sigma_1,X_1)(\sigma_2,X_2)}{(\sigma_1,X_2)(\sigma_2,X_1)}$ give all 5 algebraic letters

Summary & outlook

- ABJM amplituhedron: higher loops? higher points (trees with orthogonal Gr. [w. Kuo, Zhang; Huang et al 21])? relations to Wilson loops?
- Integration $\to \Gamma_{\rm cusp}$ [in progress]; resum & non-perturbative info.?
- Why such a relation? fixed by symmetry? integrability & AdS/CFT?
- A new method for symbology of Feynman integrals in $D = 4 - 2\epsilon$: applications to precision frontier (alphabet for $n = 6$ *etc.*)?
- "symbology" for functions beyond MPL [in progress] (twistor geometries useful for their elliptic curves & Calabi-Yau manifold [Vergu, Volk 20])
- Eventually derived from DE? relations with other methods: Landau analysis [Denen et al 16 ...], diagrammatic coaction [Abreu et al 17,...], *etc.*

Thank you for your attention!