From conformal correlators to analytic S-matrices: CFT_1 / QFT_2

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Introduction

Idea

Extract non-perturbative information of S-matrices from <u>flat space limit</u> of conformal correlators

QFT in AdS_{d+1}	$\longleftrightarrow$	Conformal theory$_d$
	$m_i^2 R^2 = \Delta_i(\Delta_i - d)$	
flat space $R \to \infty$		$\Delta_i \to \infty$ $\left(\frac{m_i}{m} = \lim\limits_{\Delta_i \to \infty} \frac{\Delta_i}{\Delta} \right)$

[Paulos, Penedones, Toledo, van Rees, Vieira '16]

Motivation

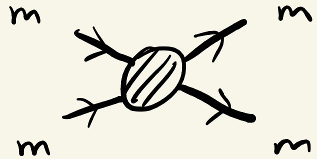
- Interplay S-matrix & Conformal Bootstrap
- Understand analytic properties of scattering amplitudes

Introduction

Here:

1D 4 point function $\longleftrightarrow$ 2D $2\to 2$ scattering

$\langle \phi\phi\phi\phi \rangle$

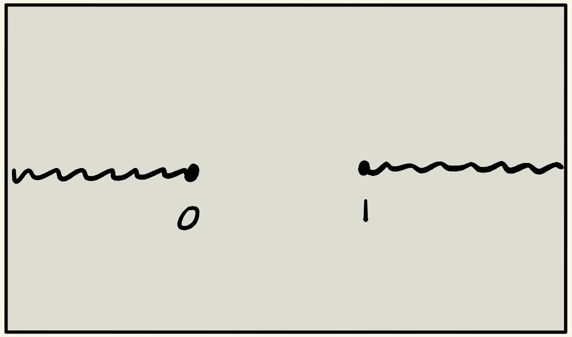


$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = \frac{g(z)}{x_{13}^{2\Delta_\phi}\, x_{24}^{2\Delta_\phi}}$$

$$S(s)\,\delta^2(p_1-p_3)\,\delta^2(p_2-p_4)$$

cross ratio $z = \dfrac{x_{12}x_{34}}{x_{13}x_{24}}$

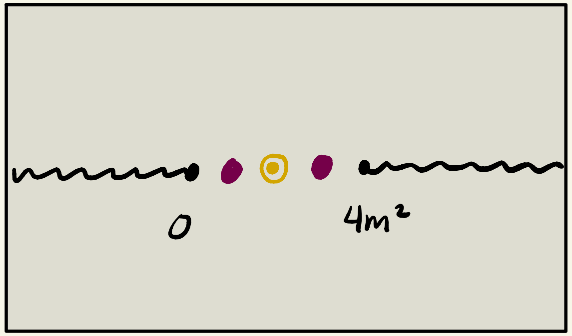
Center of mass energy $s = (p_1+p_2)^2$



$g(z)$

flat space lim

$(\Delta_\phi \to \infty)$



$S(s)$

Set-up

- Starting point:

  Position space map of [Komatsu, Paulos, van Rees, Zhao '20]

$$\lim_{\Delta_\phi \to \infty} g(z) = S\left(s = 4m^2(1-z)\right) \qquad (z \in \mathcal{D})$$

- Limit not well-defined for all z $\to$ Analytic continuation

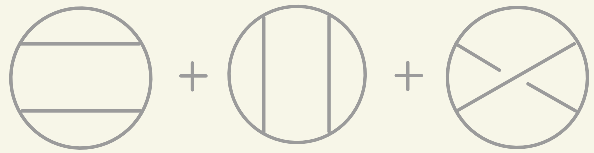
$$F\left[g(z)\right] = S\left(s = 4m^2(1-z)\right)$$

"flat space limit" $\quad F$:

1. $\lim\limits_{\Delta, \Delta_\phi \to \infty}$, $z \in \mathcal{D}$
2. Analytic continuation $z \in \mathbb{C}$

Generalized Free Fields

- $$g^{\pm}(z) = \pm 1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}$$



$\left(\begin{array}{l} + \text{ Boson} \\ - \text{ Fermion} \end{array}\right)$

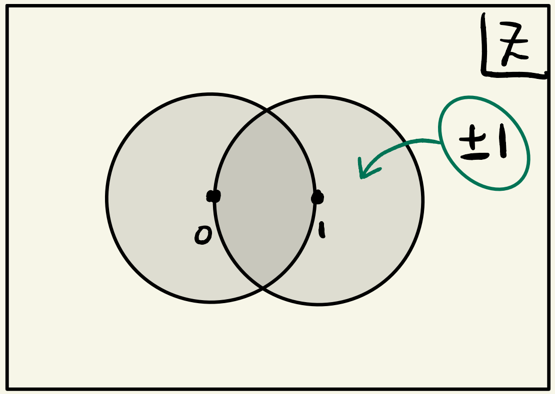
$$= \sum_{\Delta \in \Delta_n^{\pm}} a_\Delta^{\text{free}} \, G_\Delta^{\Delta_\phi}(z)$$

$z^{\Delta - 2\Delta_\phi} \, {}_2F_1(\Delta, \Delta, 2\Delta, z)$ (1D) Conformal Blocks

$\lambda^2_{\phi\phi\mathcal{O}_\Delta} = \dfrac{2\Gamma(\Delta)^2}{\Gamma(2\Delta-1)} \dfrac{\Gamma(\Delta+2\Delta_\phi-1)}{\Gamma(2\Delta_\phi)^2\,\Gamma(\Delta-2\Delta_\phi+1)}$ GFF OPE coeffs.

$\Delta_n^{\pm} = 2\Delta_\phi + 2n + \left\{\begin{array}{l} 0 \\ 1 \end{array}\right.$

1. $$\lim_{\Delta_\phi \to \infty} g^{\pm}(z) = \begin{cases} \pm 1, & |z|>1,\ |1-z|>1 \\ \infty, & \text{otherwise} \end{cases}$$



2. $$\mathcal{F}\left[g^{\pm}(z)\right] = \pm 1 = S^{\text{free}}$$

More general correlators?

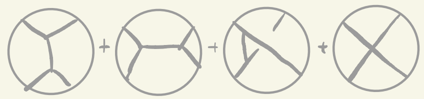
Main Tools

Polyakov blocks

Extremal functionals

Bounds on $g(z)$, OPE coeffs.

Polyakov blocks

 $= G_\Delta + \sum_n \left[\alpha_n G_n + \beta_n \partial G_n\right]$

[Gopakumar, Kaviraj, Sen, Sinha '16]

$$g(z) = \sum_\Delta a_\Delta G_\Delta^{\Delta_\phi}(z) = \sum_\Delta a_\Delta P_\Delta^{\Delta_\phi}(z)$$

- $P_\Delta^{\Delta_\phi}(w)$
  - Crossing sym manifest $P_\Delta(w) = P_\Delta(1-w)$
  - Relation to "MASTER FUNCTIONALS" Ω_w

$$P_\Delta^{\Delta_\phi}(w) = G_\Delta^{\Delta_\phi}(w) - \Omega_w^{\Delta_\phi}(\Delta)$$

Extremal functionals

- Crossing equation $\quad g(z) = g(1-z) \Rightarrow \sum_\Delta a_\Delta \underbrace{F_\Delta^{\Delta_\phi}(z)}_{G_\Delta^{\Delta_\phi}(z) - G_\Delta^{\Delta_\phi}(1-z)} = 0$

  crossing vector

- Act w/ linear functionals

$$\omega(\Delta) \equiv \omega\left[F_\Delta^{\Delta_\phi}(z)\right] \in \mathbb{R}$$

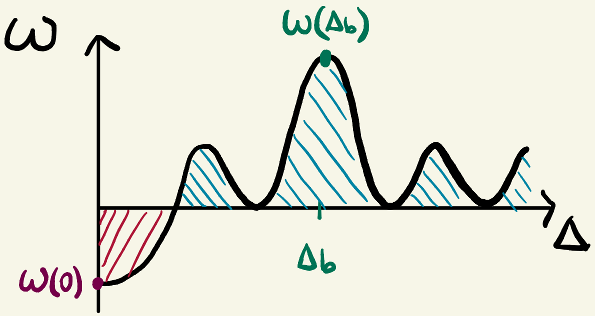
[Mazáč '16; Mazáč, Paulos '18]

here: $$\omega(\Delta) = \int_1^\infty h(z)\, \mathrm{Disc}\left[F_\Delta^{\Delta_\phi}(z)\right] dz = \frac{1}{2}\int_{1/2}^{1/2+i\infty} f(z)\, F_\Delta^{\Delta_\phi}(z)\, dz + \int_{1/2}^{1} g(z)\, F_\Delta^{\Delta_\phi}(z)\, dz$$

$$\sum_\Delta a_\Delta \omega(\Delta) = 0 \Rightarrow \text{Constrain CFT data } \{\Delta, a_\Delta\}$$

↳ $\lambda_{\phi\phi\mathcal{O}_\Delta}^2 \geq 0$

- $\omega(\Delta) \geq 0 \;\; \forall \;\; \Delta \geq \Delta^*$



$$\sum_{\Delta < \Delta^*} a_\Delta \omega(\Delta) + \sum_{\Delta \geq \Delta^*} a_\Delta \omega(\Delta) = 0$$

→ rule out $\sum_{\Delta \geq \Delta^*} a_\Delta \omega(\Delta) \neq 0$

→ bounds $a_{\Delta_b} \leq \dfrac{-\omega(0)}{\omega(\Delta_b)}$

- Different functionals naturally lead to different bounds.

MASTER functionals $\Omega_w(\Delta)$

- Bound correlator at point w [Paulos '20]

$$\Omega_w(\Delta) \geq G_\Delta^{\Delta_\phi}(w) \quad \Rightarrow \quad \boxed{g^-(w) \leq \underbrace{\bar{g}(w)}_{g(w) - \sum_{\Delta \leq 2\Delta_\phi} a_\Delta P_\Delta(w)} \leq g^+(w)}$$

subtracted correlator

$(0<w<1)$

- MASTER kernels $f_w(z)$, $g_w(z)$

$$g_w(z) = \hat{g}_w(z) + \delta(z-w)$$

$$\hat{g}_w(z) = (1-z)^{2\Delta_\phi - 2} f_w\left(\frac{1}{1-z}\right)$$

Explicit kernel at $\Delta_\phi \to \infty$

$$\boxed{f_w(z) = \sqrt{\frac{w(1-w)}{z(z-1)}} \; \frac{2z-1}{(z-w)(z-1+w)}}$$

- Relation to Polyakov blocks

$$\boxed{P_\Delta(w) = G_\Delta(w) - \Omega_w(\Delta)}$$

$$P_\Delta(w) \overset{\Delta > 2\Delta_\phi}{\propto} \sin^2\left[\frac{\pi}{2}(\Delta - 2\Delta_\phi)\right] \int_0^1 dz \, \hat{g}_w(z) G_\Delta^{\Delta_\phi}(z)$$

$\longrightarrow 0$ for $\Delta = 2\Delta_\phi + 2n$

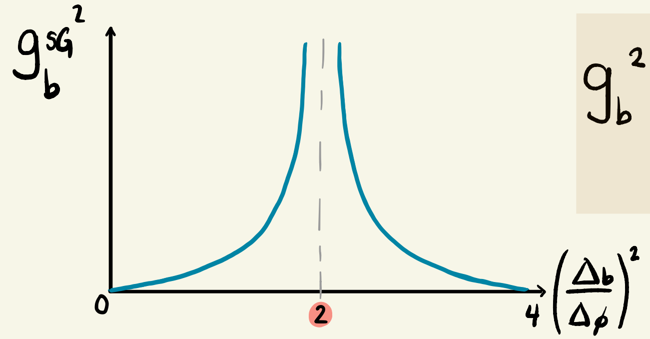
- Dispersion relation

$$\boxed{\bar{g}(w) = -\int_0^1 dz \, \hat{g}_w(z) \, \mathrm{dDisc}\, \bar{g}(z)}$$

$$\mathrm{dDisc}\, g(z) = g(z) - (1-z)^{2\Delta_\phi} \mathrm{Re}\left[g\left(\frac{z}{z-1}\right)\right]$$

Bounds on OPE coeffs.
$(\Delta_\phi \to \infty)$

- Bound OPE $\Delta_b < 2\Delta_\phi$ $\Rightarrow$ sine-Gordon $\omega^{sG}(\Delta)$ [Mazáč, Paulos '18]



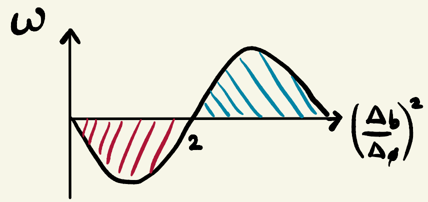
$$g_b^2 = \lim_{\substack{\Delta_\phi \to \infty \\ \text{fixed } \Delta_b/\Delta_\phi}} \frac{m_b}{\Delta_\phi} \left(\frac{a_{\Delta_b}}{a^{\text{free}}_{\Delta_b}} \right) \le g_b^{sG\,2}$$

No bound available when

$$\left(\frac{\Delta_b}{\Delta_\phi}\right)^2 = \left(\frac{m_b}{m}\right)^2 = 2 \quad \text{OR}$$

Crossing sym. pairs

$$\{ m_b^2,\ m_{b'}^2 = 4m^2 - m_b^2 \}$$



$\Rightarrow$ one way to ensure bounds: gap $\Delta_0 > \sqrt{2}\,\Delta_\phi$

- Bound (average) OPE $\Delta > 2\Delta_\phi$

$$g^-(w) \le \bar{g}(w) \le g^+(w) \Longrightarrow \lim_{\Delta_\phi \to \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a^{\text{free}}_\Delta} \right) \mathcal{N}(\Delta, 1) = 1 \qquad (1 > 4)$$

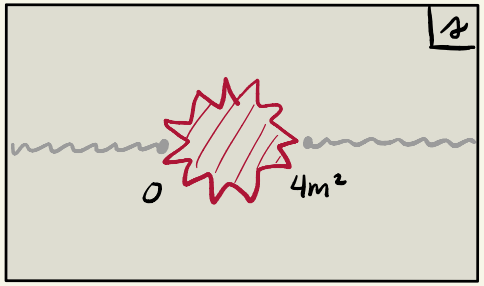
$(1-z)^{2\Delta_\phi} G_\Delta^{\Delta_\phi}(z) \sim \mathcal{N}\left(\Delta, \frac{4}{1-z}\right)$, Gaussian centered @ $\Delta = \sqrt{1}\,\Delta_\phi$
width $\sim \sqrt{\Delta_\phi}$

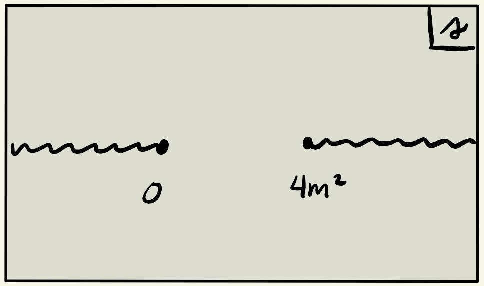
Results

Main idea

- Flat space lim separates into two parts below/above $2\Delta_\phi$

$$S(s=4(1-w)) = \underbrace{\mathcal{F}\Big[\sum_{0\le\Delta\le 2\Delta_\phi} a_\Delta P_\Delta(w)\Big]}_{\Downarrow} + \underbrace{\lim_{\Delta_\phi\to\infty}\Big[\bar{g}(w) = \sum_{\Delta>2\Delta_\phi} a_\Delta P_\Delta(w)\Big]}_{\Downarrow}$$





Non-analyticities!

Bound states , anomalous thresholds

Analytic in cut plane

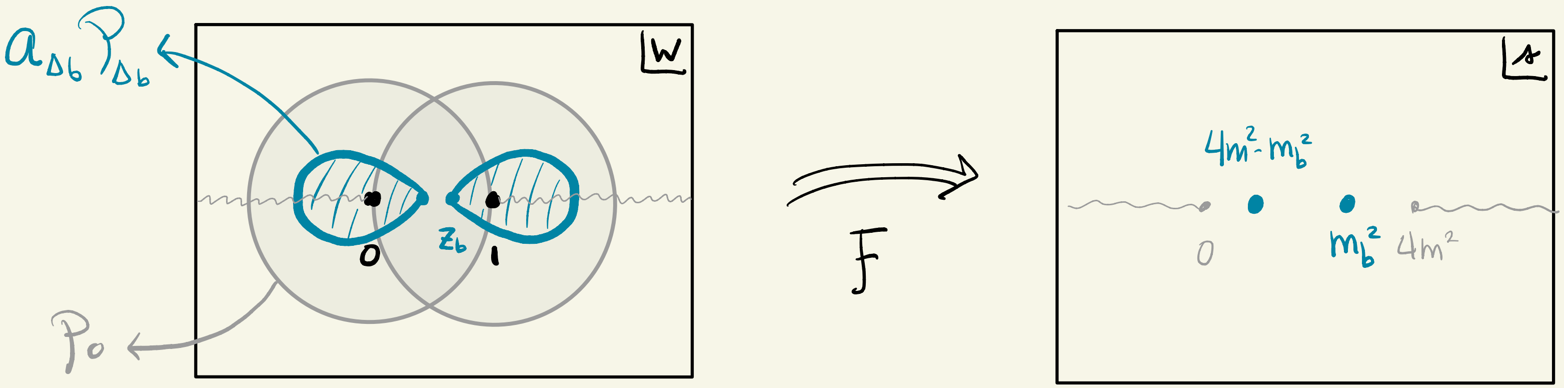
$$\propto \int_{4m^2}^{\infty} ds'\, \rho(s')\, K(s',s)$$

- Identity $\Delta=0$: $P_0(w) = g^+(w)$, $\mathcal{F}[P_0(w)] = 1$

$\mathcal{F}\left[a_{\Delta_b} P_{\Delta_b}(w)\right]$

• Single block w/ $0<\Delta_b<2\Delta_\phi$

$$\lim_{\Delta_\phi\to\infty} a_{\Delta_b} P_{\Delta_b}(w) = \underbrace{\lim_{\Delta_\phi\to\infty} a_{\Delta_b}\int_1^\infty dz\, f_w(z)\, G^{\Delta_\phi}_{\Delta_b}(1-z)}_{\sim \frac{a_{\Delta_b}}{a^{free}_{\Delta_b}}\, f_w\left(z_b=\frac{\Delta_b^2}{4\Delta_\phi^2}\right) \ \text{saddle}} + \lim_{\Delta_\phi\to\infty} \text{``}\, a_{\Delta_b} G^{\Delta_\phi}_{\Delta_b}(1-w)\,\text{''}$$



$$s=4(1-w)$$

$$m_b^2=4z_b=4\left(\frac{\Delta_b}{\Delta_\phi}\right)^2$$

$$g_b^2 \propto \frac{a_{\Delta_b}}{a^{free}_{\Delta_b}}$$

$\Longrightarrow$ Bound state pole!

$$\mathcal{F}\left[a_{\Delta_b} P_{\Delta_b}(w)\right] = \sqrt{\frac{s(4-s)}{m_b^2(4-m_b^2)}}\left[\frac{g_b^2}{s-m_b^2}-\frac{g_b^2}{s-(4-m_b^2)}\right] \subset S(s=4(1-w))$$

Dispersion relation

$$g(w) = \sum_\Delta a_\Delta P_\Delta(w) = P_0^{\Delta_\phi}(w) + \sum_{\Delta_0 \leq \Delta < 2\Delta_\phi} a_\Delta P_\Delta^{\Delta_\phi}(w) + \sum_{\Delta > 2\Delta_\phi} a_\Delta P_\Delta^{\Delta_\phi}(w)$$

$$\overset{\mathcal{F}}{\Longrightarrow} \quad \boxed{S(s) = 1 + \int_{s_0}^{4m^2} ds' \, \tilde{\rho}(s') \tilde{K}(s', s) + \int_{4m^2}^{\infty} ds' \, \rho(s') K(s', s)}$$

$$\tilde{\rho}\left(s' = \left(\frac{\Delta}{\Delta_\phi}\right)^2\right) = \left\langle \frac{a_\Delta}{a_\Delta^{\text{free}}} \right\rangle, \quad \tilde{K}(s', s) = \frac{2}{\pi} \frac{\sqrt{s(4-s)}}{\sqrt{s'(4-s')}} \frac{s'-2}{(s'-s)(s'-4+s)}$$

- Analyticity properties in terms of CFT data.
- Crossing ✓

[checked in many examples in perturbation theory]

Unitarity.

Phase shift + bound $a_{\Delta > 2\Delta_\phi}$

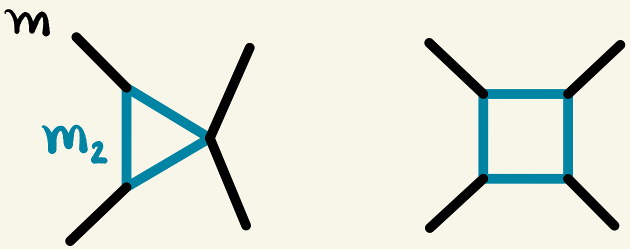
$$S(s > 4m^2) = \mathcal{F}\left[g(z<0) = \sum_\Delta a_\Delta \frac{G_\Delta^{\Delta_\phi}\left(\frac{z}{z-1}\right)}{(1-z)^{2\Delta_\phi}} e^{-i\pi(\Delta - 2\Delta_\phi)} \right] = \lim_{\Delta_\phi \to \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}}\right) \mathcal{N}(\Delta, s) \, e^{-i\pi(\Delta - 2\Delta_\phi)}$$

$$\Longrightarrow \quad \boxed{|S(s > 4m^2)| \leq 1}$$

$$\lim_{\Delta_\phi \to \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}}\right) \mathcal{N}(\Delta, s) = 1$$

Anomalous thresholds

- Landau diagrams $\Longrightarrow$ extra singularities (poles in 2D)



$$m_2^2 \le \frac{1}{2} m^2$$

- Claim: anomalous thresholds $\Longleftrightarrow$ unbounded OPE

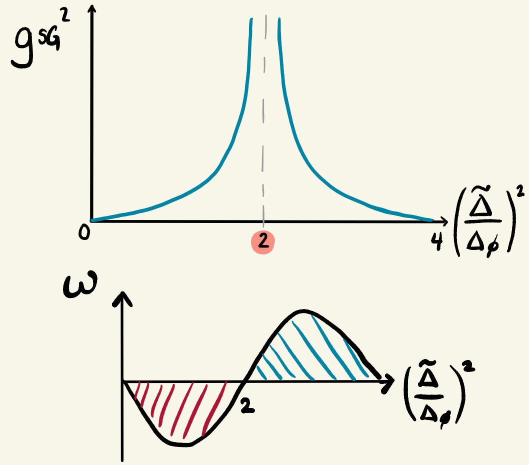
$$\sum_{\Delta_0 \le \tilde{\Delta} \le 2\Delta_\phi} a_{\tilde{\Delta}} P_{\tilde{\Delta}}(w) \quad \text{unbounded if}$$

$\longrightarrow$ $\left(\frac{\tilde{\Delta}}{\Delta_\phi}\right)^2 = \left(\frac{\tilde{m}}{m}\right)^2 = 2$

$\longrightarrow$ Crossing sym. pairs $\left\{ \tilde{m}^2, \tilde{m}'^2 = 4m^2 - \tilde{m}^2 \right\}$

$$\Downarrow \mathcal{F}$$

$$\int_{s_0}^{4m^2} d\tilde{s} \; \tilde{\rho}(\tilde{s}) \, K(\tilde{s}, s)$$



- $m_2^2 \le \frac{1}{2} m^2$, Two-particle states: $(2m_2)^2 \le 2 m^2 \Rightarrow$ unbounded

Anomalous thresholds

- For $\Delta_0 \leq \Delta^*$ can add solutions to crossing with $\mathcal{F}[g^B(w)] = 0$ and $a_\Delta > 0$.

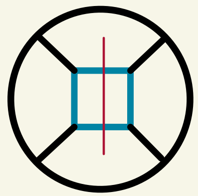
e.g. $g^B_\alpha(w) = \dfrac{1}{[w(1-w)]^{(2-\alpha)\Delta_\phi}}$, for $\alpha \leq 4/3$ $(\Delta_0 \leq 4/3\,\Delta_\phi)$

[Hogervorst, van Rees '17]
[Antunes, Costa, Penedones, Salgarkar, van Rees '21]

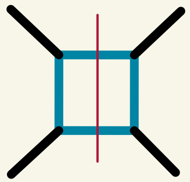
$\Rightarrow$ $g + \lambda g^B$ arbitrarily large and positive OPE coeffs.

BOX

$$\tilde{\rho}_{CFT}(s) = \lim_{\Delta_\phi \to \infty} \frac{a^{\text{box}}_{[22]_n}}{a^{\text{free}}_{[22]_n}}$$

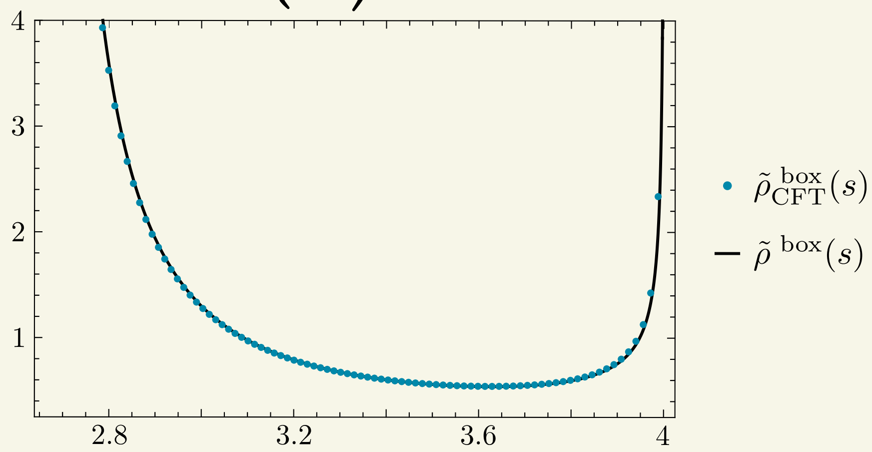


$\underset{\Delta_\phi \to \infty}{\Longrightarrow}$

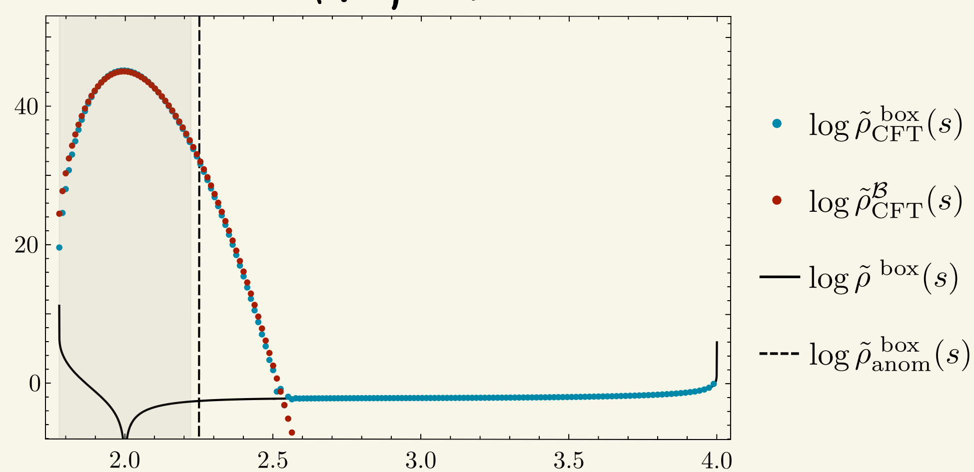


$\rho(s) \propto \mathrm{Disc}\, T(s)$

safe: $\left(\frac{m_2}{m}\right)^2 = \frac{2}{3} > \frac{1}{2}$ $(\Delta_1 = 500)$



anom: $\left(\frac{m_2}{m}\right)^2 = \frac{4}{9} < \frac{1}{2}$ $(2\Delta_2 = 4/3\,\Delta_\phi)$



Final Remarks

$$\left.\begin{array}{l}\text{Analytic}\\ \text{crossing}\\ \text{unitary}\end{array}\right\}\ \underset{2D}{\text{S-matrices}}\ \text{from}\ \underset{1D}{\text{CFT}}\ \text{correlators}$$

Connection to dual S-matrix Bootstrap, extremality...

Future:

- Clarify Anomalous thresholds
- Mixed/Higher point functions, higher D
- Integrable S-matrices from CFT

Thank you!

$\mathcal{F}[a_{\Delta_b} P_{\Delta_b}(w)]$

- Large Δ_ϕ CBs

$$G^{\Delta_\phi}_{\Delta_b}(1-z) \sim \text{factors}\left[2^{2m_b}\frac{(1-\sqrt{z})^{m_b-2}}{(1+\sqrt{z})^{m_b+2}}\right]^{\Delta_\phi}$$

saddle @ $z = z_b = \frac{1}{4}\left(\frac{\Delta_b}{\Delta_\phi}\right)^2 = m_b^2/4$

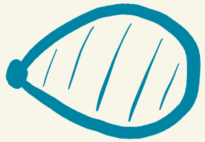
- Bound OPE coeffs

$$g_b^2 \sim \frac{a_{\Delta_b}}{a^{\text{free}}_{\Delta_b}} \leq g_b^{sG\,2}$$

$$a_{\Delta_b} \sim g_b^2\, a^{\text{free}}_{\Delta_b} = g_b^2\left[2^{2m_b}\frac{(2-m_b)^{m_b-2}}{(2+m_b)^{m_b+2}}\right]^{-\Delta_\phi}$$

- $$\lim_{\Delta_\phi\to\infty} a_{\Delta_b} G^{\Delta_\phi}_{\Delta_b}(1-z) = \begin{cases} 0 \,, & [\;]<1 \\ \infty \,, & [\;]>1 \end{cases}$$

$$\left[\frac{\left((1-\sqrt{z})(2+m_b)\right)^{m_b-2}}{\left((1+\sqrt{z})(2-m_b)\right)^{m_b+2}}\right] = 1 \Rightarrow$$



- Finite piece

$$P_\Delta(w) = G_\Delta(w) - \Omega_w(\Delta)$$

$$f_w(z) = \sqrt{\frac{w(1-w)}{z(z-1)}}\;\frac{2z-1}{(z-w)(z-1+w)}$$

$$g_b^2 \sim \frac{a_{\Delta_b}}{a^{\text{free}}_{\Delta_b}} \leq g_b^{sG\,2}$$

$$a_{\Delta_b} P_{\Delta_b}(w) \supset a_{\Delta_b}\int_1^\infty dz\, f_w(z)\, G^{\Delta_\phi}_{\Delta_b}(1-z) \sim \frac{a_{\Delta_b}}{a^{\text{free}}_{\Delta_b}} f_w(z_b) = g_b^2\sqrt{\frac{w(1-w)}{z_b(1-z_b)}}\;\frac{z_b-1/2}{(z_b-w)(z_b-1+w)}$$

$$= \mathcal{F}\left[a_{\Delta_b} P_{\Delta_b}(w)\right]$$

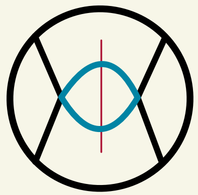
Example : Bubble

Generalized Free Bosons ϕ_1, ϕ_2 + $g_{1122}\,\phi_1\phi_1\phi_2\phi_2$

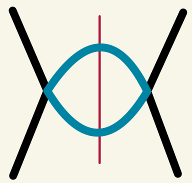
• $\langle\phi_1\phi_1\phi_1\phi_1\rangle$ at $\mathcal{O}(g_{1122}^2)$: $g^{\text{bubble}}(w) = \sum\limits_{\substack{\Delta \\ [11]_n,\,[22]_n}} a_\Delta^{\text{bubble}} P_\Delta(w) = \sum\limits_n a^{\text{bubble}}_{[22]_n} P_{[22]_n}(w)$

$P_{[11]_n} = 0$ $\qquad$ $2\Delta_2 + 2n$

$$\tilde{\rho}_{CFT}(s) = \lim_{\Delta_\phi\to\infty} \underbrace{\frac{a^{\text{bubble}}_{[22]_n}}{a^{\text{free}}_{[22]_n}}}$$



$\xRightarrow[\Delta_\phi\to\infty]{}$



$$\rho(s) \propto \frac{\overbrace{\mathrm{Disc}\,T(s)}^{\text{from Cutkosky}}}{\sqrt{s(4-s)}}$$

compute from simpler Witten diagrams like 

closed form: $a^{\text{bubble}}_{[22]_n} \propto \left(\dfrac{\Gamma(\Delta_1+\Delta_2+n-\frac{1}{2})\,\Gamma^4\,\Gamma\,\Gamma}{\Gamma^2\,\Gamma^2\,\Gamma\,\Gamma}\right)^2$

$\sqrt{s} = \dfrac{[22]_n}{\Delta_1}$, $m_2 = \dfrac{\Delta_2}{\Delta_1}$

$$\tilde{\rho}^{\text{bubble}}_{CFT}(s) = \lim_{\Delta_\phi\to\infty} \frac{a^{\text{bubble}}_{[22]_n}}{a^{\text{free},''}_{[22]_n}} = \frac{1}{4}\,\frac{1}{\sqrt{s(4-s)}\,\sqrt{s(s-4m_2^2)}} = \tilde{\rho}^{\text{bubble}}_{FLAT}(s)$$

Dual S-mat B. & functionals

S-matrix Dual Problem

$$\max\left[F_p^{S\text{-mat}} = \int_{s_0}^{4} \tilde{c}(s)\,\tilde{\rho}(s) + \int_{4}^{\infty} c(s)\,\rho(s) \right]$$

$$F_p^{S\text{-mat}} \le F_p^{S\text{-mat}} + \int_{s_0}^{4} \tilde{k}(s)\,\tilde{\rho}(s) + \int_{4}^{\infty} \Big[|k(s)| - \mathrm{Re}\big(k(s)\,S(s)\big) \Big]$$

$$\boxed{F_p^{S\text{-mat}} \le \int_{4}^{\infty} \Big[|k(s)| - \mathrm{Re}\big(k(s)\big) \Big] = F_D^{S\text{-mat}}}$$

$$k(s) \longleftrightarrow f(z)$$

Conformal Bootstrap

$$\max\left[F_p^{CFT} = \sum_{\Delta \ge \Delta_0} a_\Delta\, \mu(\Delta) \right]$$

$$F_p^{CFT} \le F_p^{CFT} + \sum_{\Delta \ge \Delta_0} a_\Delta \big[\Omega(\Delta) - \mu_\Omega(\Delta) \big] \le -\Omega(0) = F_D^{CFT}$$

$$\boxed{F_p^{CFT} \le \int_{-\infty}^{0} \Big[|f(z)| - \mathrm{Re}\, f(z) \Big] = F_D^{CFT}}$$

- $\Omega(\Delta) \ge \mu_\Omega(\Delta) \quad (\Delta \ge \Delta_0)$
- $\sum_\Delta a_\Delta\, \Omega(\Delta) = 0$
- $\Omega(\Delta) = \frac{1}{2}\int_{1/2}^{1/2+i\infty} f(z)\, F_\Delta^{\Delta_\phi}(z)\, dz + \int_{1/2}^{1} g(z)\, F_\Delta^{\Delta_\phi}(z)\, dz \;;\quad g(z) = (1-z)^{2\Delta_\phi - 2} \left| f\!\left(\frac{1}{1-z}\right) \right| + \delta g(z)$