Generalised asymptotics in the self-dual sector

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Asymptotic symmetries and fall-offs

- asymptotic symmetries $\rightarrow$ infrared properties of radiative fields (e.g. amplitude factorisation)
- Yang-Mills: at null infinity

$$r^2 = (\vec{x})^2, \quad u = t - r, \quad z = \frac{x^1 + ix^2}{x^3 + r}$$

focus on the free data with fall-off

$$A_\mu \qquad \rightarrow \qquad A_z = A_z^0(u, z, \bar{z}) + \frac{A_z^{(-1)}(u, z, \bar{z})}{r} + \dots$$

- leading soft behaviour $\rightarrow$ asymptotic symmetry with parameters that are $\mathcal{O}(r^0)$, i.e. $\Lambda^{(0)} = \Lambda^{(0)}(z, \bar{z})$:

$$\delta A_z^{(0)} = D_z \Lambda^{(0)}$$

Asymptotic symmetries and fall-offs

- subleading soft-theorems have symmetry origin [Low,Lysov, Pasterski,Strominger,Casali,Mitra,He,...]

$$lim_{\omega\to\infty}(1+\omega\partial_\omega)\mathcal{M}_{n+1}(k_1,...,k_n;\omega\hat{q}) = S^{(1)}\mathcal{M}_n(k_1,...,k_n)$$

  with $S^{(1)}$ some differential operator acting on the momenta.
- proposed $\mathcal{O}(r)$ gauge parameters [Campiglia,Laddha]

$$\Lambda^{(1)} = r\varepsilon(z,\bar{z}).$$

- These violate fall-off!

$$\delta A_z^{(0)} \stackrel{?}{=} D_z\Lambda^{(1)}$$

- gravity

$$g_{zz} = \mathcal{O}(r),\ \ g_{\bar{z}\bar{z}} = \mathcal{O}(r)$$

  superrotations

$$\delta g_{\bar{z}\bar{z}} \stackrel{?}{=} 2r^2\gamma_{z\bar{z}}\partial_{\bar{z}}Y^z + \mathcal{O}(r)$$

  BMS restricted to $Y^z = 1, z, z^2$, but can generalise to look at subleading effects[Barnich,Troessaert,Campiglia,Laddha,Donnay,Pasterski,Puhm,Kapec,Lysov,Cachazo,...].

Asymptotic symmetries and fall-offs

- fall-off violation

$$\delta A_z^{(0)} \stackrel{?}{=} D_z \Lambda^{(1)}$$

- how to construct the phase space such that these transformations make sense ? The naïve guess of allowing

$$A_z = r A_z^{(1)} + A_z^{(0)} + \dots$$

leads to divergences when computing charges.

- Issue was resolved [Campiglia,Peraza] for $\mathcal{O}(r)$ – linearised phase space which supports these transformations
- **Can we extend to all oders in the fields and all orders in r (i.e. $\Lambda = \dots r^n \Lambda^{(n)} + \dots$)?**
- Start with a simplified set-up: self-dual sector.

Self-dual YM

- light-cone coordinates
  $U = \frac{X^0 - X^3}{\sqrt{2}}, \; V = \frac{X^0 + X^3}{\sqrt{2}}, \; Z = \frac{X^1 + iX^2}{\sqrt{2}}, \; \bar{Z} = \frac{X^1 - iX^2}{\sqrt{2}}.$
  Notation:
  $$x^i := (U, \bar{Z}), \quad y^\alpha := (V, Z).$$
  which splits space-time into two $2d$ subspaces. The Minkowski metric is then
  $$ds^2 = 2\eta_{i\alpha} dx^i dy^\alpha = -2dUdV + 2dZd\bar{Z}.$$
  and we introduce the anti-symmetric "area element"
  $$\Pi_{\alpha\beta} dy^\alpha \wedge dy^\beta = dV \wedge dZ - dZ \wedge dV$$
- self-dual condition
  $$\tilde{F}_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} = iF_{\mu\nu}$$
  in light-cone gauge $\mathcal{A}_U = 0$:
  $$\mathcal{A}_i = 0, \quad \mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \Phi$$

Self-dual YM

- light-cone gauge $\mathcal{A}_U = 0$:

$$\mathcal{A}_i = 0, \quad \mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \Phi$$

  has residual symmetry

$$\delta_\Lambda \mathcal{A}_\mu = D_\mu \Lambda \quad \text{with} \quad \partial_i \Lambda = 0 \implies \Lambda = \Lambda(y)$$

  whch preserves Lorenz gauge $D^\mu \mathcal{A}_\mu = 0$.

- The standard fall-off at null infinity

$$A_z = A_z^0(u, z, \bar{z}) + \frac{A_z^{(-1)}(u, z, \bar{z})}{r} + \ldots$$

  corresponds to

$$\mathcal{A}_\alpha = \left\{ \mathcal{A}_V = \mathcal{O}(V^0) + \ldots \, , \ \mathcal{A}_Z = \mathcal{O}(V^0) + \ldots \right\}$$

  in a V-expansion. *We will think in terms of V instead of r from now on.*

Radiative phase-space

- Fall-offs
$$\mathcal{A}_\alpha = \left\{ \mathcal{A}_V = \mathcal{O}(V^0) + \ldots \, , \; \mathcal{A}_Z = \mathcal{O}(V^0) + \ldots \right\}$$
we define our radiative phase-space as
$$\Gamma^{\text{rad}}_{\text{YM}} = \{ \mathcal{A}_\mu : \mathcal{A}_i = 0, \mathcal{A}_Z = \sum_{n=0}^{+\infty} \mathcal{A}_Z^{(-n)} V^{-n} \}$$
subject to constraints coming from e.o.m. and the self-duality condition.
- The alowed fall-off in the trasformation parameter is then
$$\Lambda = \sum_{n=-\infty}^{0} \Lambda^{(n)} V^n ,$$

Linear extended phase space [Campiglia,Peraza'21]

- Standard fall-off $\Lambda = \sum_{n=-\infty}^{0} \Lambda^{(n)} V^n$,
- Subleading effects are a consequence of

$$\Lambda = \sum_{n=-\infty}^{1} \Lambda^{(n)} V^n,$$

Of course this is incompatible with $\mathcal{A}_Z = \sum_{n=0}^{+\infty} \mathcal{A}_Z^{(-n)} V^{-n}$

- Define linear extended phase space

$$\Gamma_{\text{lin,YM}}^{\text{ext}} := \{\tilde{\mathcal{A}}_Z = \mathcal{A}_Z + D_Z \Psi\}$$

The extended gauge field is then

$$\tilde{\mathcal{A}}_\alpha = \mathcal{A}_\alpha + D_\alpha \Psi$$

We have introduced a field Ψ with fall-off

$$\Psi = V \Psi^{(1)}$$

We take $\Psi^{(n)} = \Psi^{(n)}(Z)$, thus they have the same coordinate dependence as the $\Lambda^{(n)}$.

Linear extended phase space

- We have defined
$$\tilde{\mathcal{A}}_Z = \mathcal{A}_Z + D_Z \Psi, \quad \Psi = V \Psi^{(1)}$$
- The consistency condition is that the extended space field $\tilde{\mathcal{A}}_\alpha$ transforms as a gauge field in the extended space, i.e
$$\delta_\Lambda \tilde{\mathcal{A}}_Z = \tilde{D}_Z \Lambda,$$
where $\tilde{D}$ is the covariant derivative associated to $\tilde{\mathcal{A}}$.
- The natural variation for $\mathcal{A}_Z^{(0)}$ is
$$\delta_\Lambda A_Z^{(0)} = D_Z^{(0)} \Lambda^{(0)} := \partial_Z \Lambda^{(0)} + i[\mathcal{A}_Z^{(0)}, \Lambda^{(0)}],$$
which allows us to read off
$$\delta_\Lambda \Psi^{(1)} = \Lambda^{(1)} - i[\Psi^{(1)}, \Lambda^{(0)}].$$

Charges

- Construct charges from these symmetries (compatible with subleading soft gluon factorization, reduce to standard ones when $\Lambda^{(1)} = 0$)
- Temporarily going back to Bondi coordinates

$$Q^1_{\Lambda^{(1)}} = \int d^2x Tr(\Lambda^{(1)}\pi)$$

with

$$\pi(x) = -\frac{1}{2}\int_{-\infty}^{+\infty} duu\partial_u D_a^-(D^a F_{ru} + D_b F^{ba}), \quad D^- = D(u \to -\infty)$$

and

$$Q^0_{\Lambda^{(0)}} = Q^{0,rad}_{\Lambda^{(0)}} + Q^1_{[\Psi,\Lambda^{(0)}]}$$

satisfying

$$\{Q^0_{\Lambda^{(0)}}, Q^1_{\Lambda^{(1)}}\} = Q^1_{[\Lambda^{(0)},\Lambda^{(1)}]}$$

- Trying to naively extend the parameter beyond $\mathcal{O}(V)$ fails in this approach...

Full extended phase space YM[SN,Peraza to appear]

- We would now like to extend the gauge parameter to all orders in V

$$\Lambda = \sum_{n=-\infty}^{+\infty} \Lambda^{(n)} V^n,$$

- The defiition of the linear extended phase space ($\tilde{\mathcal{A}}_Z = \mathcal{A}_Z + D_Z\Psi$) is reminscent of the *Stückelberg procedure* which reinstates boken local symmetries in the action:
  - Perform transformation: $A_z \to \mathcal{A}_Z + D_Z\Lambda$
  - promote parameter to a new field $\Lambda \to \Psi$
- then going to all orders in V requires going to all orders in Ψ:

$$\hat{\mathcal{A}}_\alpha = e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi} + ie^{i\Psi}\partial_\alpha e^{-i\Psi}$$

Full extended phase space YM

- Focus on $\alpha = Z$:

$$\hat{\mathcal{A}}_Z = e^{i\Psi}\mathcal{A}_Z e^{-i\Psi} + ie^{i\Psi}\partial_Z e^{-i\Psi}$$

- Consistency condition

$$\delta_\Lambda \hat{\mathcal{A}}_Z = \hat{D}_Z \Lambda.$$

then gives

$$\mathcal{O}_{-i\Psi}(\delta_\Lambda \Psi) = \Lambda - e^{i\Psi}\Lambda^{(0)}e^{-i\Psi}$$

with

$$\mathcal{O}_X := \frac{1 - e^{-ad_X}}{ad_X}, \quad ad_X(Y) = [X, Y]$$

working to **all orders** in ψ and V !

- Invert $\mathcal{O}_{-i\Psi}$ to get

$$\delta_\Lambda \Psi = \Lambda_- - \tfrac{i}{2}[\Psi, \Lambda_+] - \frac{1}{12}[\Psi, [\Psi, \Lambda_-]] + \ldots$$

with $\Lambda_\pm = \Lambda \pm \Lambda^{(0)}$

Self-dual gravity

- Defined via
$$\tilde{R}_{\mu\nu\rho}{}^{\sigma} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\eta\lambda} R_{\eta\lambda\rho}{}^{\sigma} = iR_{\mu\nu\rho}{}^{\sigma}$$
- Split the metric as (exact) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ Then in light-cone gauge
$$h_{i\mu} = 0, \quad h_{\alpha\beta} = \Pi_{\alpha}^{\ i}\Pi_{\beta}^{\ j}\partial_i\partial_j\phi,$$
automatically satisfies a transverse-traceless gauge condition
$$\partial^{\mu} h_{\mu\nu} = 0, \quad \eta^{\mu\nu} h_{\mu\nu} = 0$$
Light-cone gauge preserved by diffeo
$$\delta_{\xi} h_{\mu\nu} := \mathcal{L}_{\xi}\eta_{\mu\nu} + \mathcal{L}_{\xi} h_{\mu\nu}$$
with $\xi_i = 0, \quad \xi_{\alpha} = b_{\alpha}(y)$. automatically preserves transverse-traceless as well.
- double copy [Monteiro,O'Connell,...]

Double copy for symmetries [Campiglia,SN'21]

- Statting with

$$\delta_\xi h_{\mu\nu} := \mathcal{L}_\xi \left(\eta_{\mu\nu} + h_{\mu\nu}\right), \quad \text{where} \quad \xi_i = 0, \quad \xi_\alpha = b_\alpha(y)$$

define the "Hamiltonian"

$$\lambda = 2\Omega_i{}^\alpha x^i b_\alpha, \quad \text{with} \quad \Omega_i{}^\alpha \Pi_\alpha^j = \delta_i^j$$

then

$$\delta_\lambda h_{\alpha\beta} = \Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\{\lambda, h_{\alpha\beta}\} = \Pi_{(\alpha}^{\ i}\partial_i \left(\partial_{\beta)}\lambda - \tfrac{1}{2}\Pi_{\beta)}^{\ j}\partial_j\{\lambda,\phi\}\right)$$

Poisson bracket:

$$\{f, g\} := \Pi^{ij}\partial_i f \partial_j g$$

This arises from the YM transformation

$$\delta_\Lambda \mathcal{A}_\alpha = \partial_\alpha\Lambda + i\Pi_\alpha^{\ i}\partial_i[\Lambda, \Phi]$$

via the rules

$$\Phi \to \phi, \qquad -i[\ ,\ ] \to \tfrac{1}{2}\{\ ,\ \}, \qquad \Lambda \to \lambda$$

standard fall-off for λ

$$\lambda = \sum_{n=-\infty}^{0} V^n \lambda^{(n)}$$

Gravity linear extended phase space

- Graviton fall-off in V:

$$h_{\alpha\beta} = \sum_{n=0}^{+\infty} \frac{h_{\alpha\beta}^{(-n)}}{V^n},$$

- want parameter with

$$\lambda = \sum_{n=-\infty}^{1} V^n \lambda^{(n)}$$

- Linear extended phase space (focusing on $(\alpha\beta) = (ZZ)$)

$$\Gamma^{\text{ext}}_{\text{lin,grav}} := \{\tilde{h}_{ZZ} = h_{ZZ} + \Pi^i_Z \partial_i \partial_Z \psi - \frac{1}{2}\{\psi, h_{ZZ}\}\}$$

with

$$\psi = V\psi^{(1)}$$

and ψ has the same form as the diffeo Hamiltonian

$$\psi^{(1)}(x^i, Z) = 2\Omega_i{}^\alpha x^i \beta^{(1)}_\alpha(Z).$$

Gravity linear extended phase space

- Extended phace space

$$\Gamma^{\text{ext}}_{\text{lin,grav}} := \{\tilde{h}_{ZZ} = h_{ZZ} + \Pi^i_Z \partial_i \partial_Z \psi - \frac{1}{2}\{\psi, h_{ZZ}\}\}$$

consistency condition

$$\delta_\lambda \tilde{h}_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2}\{\lambda, \tilde{h}_{\alpha\beta}\},$$

leads to

$$\delta_\lambda h^0_{ZZ} = \Pi^i_Z \partial_i \partial_Z \lambda^{(0)} - \frac{1}{2}\{\lambda^{(0)}, h^{(0)}_{ZZ}\}$$

and

$$\delta_\lambda \psi^{(1)} = \lambda^{(1)} + \tfrac{1}{2}\{\psi^{(1)}, \lambda^{(0)}\}.$$

- Double copy follows from symmetries relation:

$$\boldsymbol{\Psi}^{(1)} \to \psi^{(1)}$$

Full gravity extension

- Consider diffeo transformation to all orders

$$g'_{\mu\nu} = \left(e^{-\mathcal{L}_\xi} g\right)_{\mu\nu}$$

with the Lie derivative $(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho}$.

- In our case

$$g'_{\alpha\beta} = -\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda + e^{-\frac{1}{2}\mathfrak{ad}_\lambda} g_{\alpha\beta}, \qquad \mathfrak{ad}_\theta := \{\theta, \cdot\}$$

- Then we can allow for a parameter

$$\lambda = \sum_{n=-\infty}^{+\infty} V^n \lambda^{(n)}$$

by extending the phase space

$$\Gamma^{\text{ext}}_{\text{full,grav}} := \{\hat{g}_{\alpha\beta} = -\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\psi + e^{-\frac{1}{2}\mathfrak{ad}_\psi} g_{\alpha\beta}\}$$

with ψ now going to all orders in V: $\psi = \sum_{n=-\infty}^{+\infty} V^n \psi^{(n)}$

Full gravity extension

- extended the phase space

$$\Gamma^{\text{ext}}_{\text{full,grav}} := \{\hat{g}_{\alpha\beta} = -\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\psi + e^{-\frac{1}{2}\mathfrak{a}\mathfrak{d}\psi}g_{\alpha\beta}\}$$

definition above still preserves the splitting

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}$$

with $\hat{h}_{i\mu} = 0$ and

$$\hat{h}_{\alpha\beta} = -\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\psi + e^{-\frac{1}{2}\mathfrak{a}\mathfrak{d}\psi}h_{\alpha\beta}$$

Double copy to all orders

- First rewrite

$$\hat{\mathcal{A}}_\alpha = e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi} + ie^{i\Psi}\partial_\alpha e^{-i\Psi} = e^{iad_\Psi}\Pi_\alpha^{\ i}\partial_i\Phi + \mathcal{O}_{-i\Psi}(\partial_\alpha\Psi)$$

- Propose double copy extends trivially:

$$\Phi \to \phi, \qquad -iad \to \tfrac{1}{2}\mathfrak{a}\mathfrak{d}, \qquad \Psi^{(n)} \to \psi^{(n)}$$

then the double copy of $\hat{\mathcal{A}}_\alpha$ is

$$\hat{H}_{\alpha\beta} = \Pi_{(\alpha}^{\ i}\partial_i\left(e^{-\frac{1}{2}\mathfrak{a}\mathfrak{d}_\psi}\Pi_{\beta}^{\ i}\partial_i\phi + \mathcal{W}_{\frac{1}{2}\psi}(\partial_\beta\psi)\right) = e^{-\frac{1}{2}\mathfrak{a}\mathfrak{d}_\psi}h_{\alpha\beta} + \Pi_{(\alpha}^{\ i}\partial_i\left(\mathcal{W}_{\frac{1}{2}\psi}(\partial_\beta\psi)\right)$$

where

$$\mathcal{O}_{-i\Psi} := \frac{1-e^{-ad_{-i\Psi}}}{ad_{-i\Psi}} \quad \to \quad \mathcal{W}_{\frac{1}{2}\psi} := \frac{1-e^{-\mathfrak{a}\mathfrak{d}_{\frac{1}{2}\Psi}}}{\mathfrak{a}\mathfrak{d}_{\frac{1}{2}\Psi}}$$

- The we can show that

$$\hat{H}_{\alpha\beta} = \hat{h}_{\alpha\beta} = -\Pi^{i}_{(\alpha}\partial_i\partial_{\beta)}\psi + e^{-\frac{1}{2}\mathfrak{a}\mathfrak{d}_\psi}h_{\alpha\beta}$$

so we have established the double copy for the full extended phase space !

Double copy to all orders

- The above can be interpreted as double copy for (a subset of) local symmetries to all orders .i.e.

$$\mathcal{A}'_\alpha = e^{i\Lambda}\mathcal{A}_\alpha e^{-i\Lambda} + ie^{i\Lambda}\partial_\alpha e^{-i\Lambda}$$

with $\Lambda = \Lambda(y)$ double copies to

$$h'_{\mu\nu} = \left(e^{-\mathcal{L}_\xi} g\right)_{\mu\nu} - \eta_{\mu\nu}$$

with $\xi_i = 0, \quad \xi_\alpha = b_\alpha(y)$ under

$$\Phi \to \phi, \qquad -iad \to \tfrac{1}{2}\mathfrak{a}\mathfrak{d}, \qquad \Lambda \to \lambda$$

where $\lambda = 2\Omega_i{}^\alpha x^i b_\alpha$.

Conclusions and future directions

- Extend to full YM (light-cone gauge helps!), full gravity
- Relation to infinite tower of symmetries in self-dual sector
- Relation to $w_{1+\infty}$ algebras

Thank You !