Dispersion Relations in Conformal Field Theories

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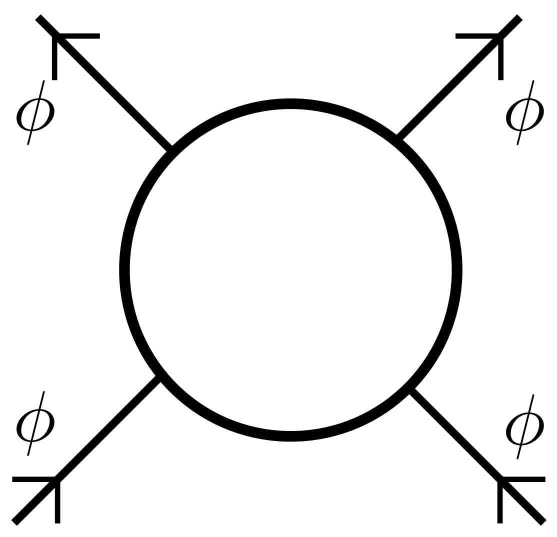
Dispersion Relations for the S-Matrix

Idea: Reconstruct the amplitude from its imaginary part

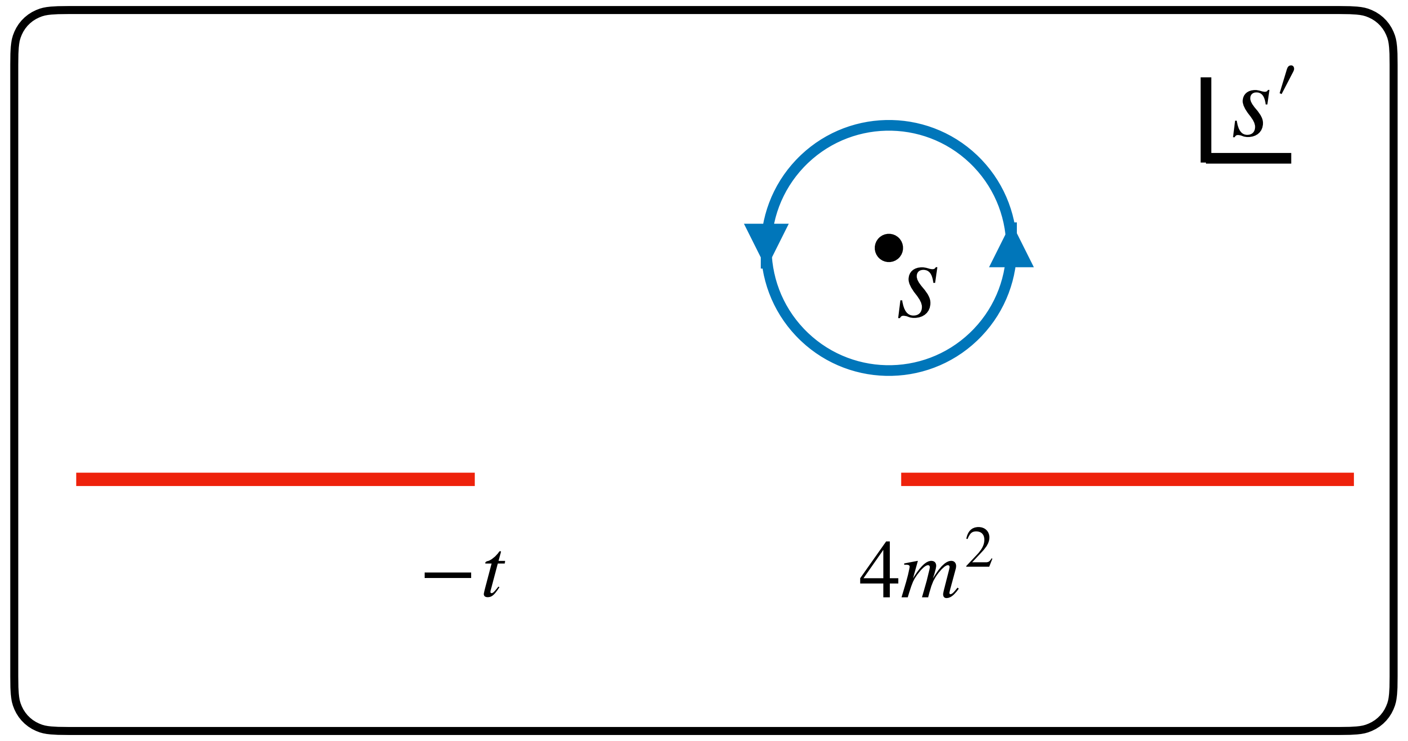
$A(s,t) =$ 

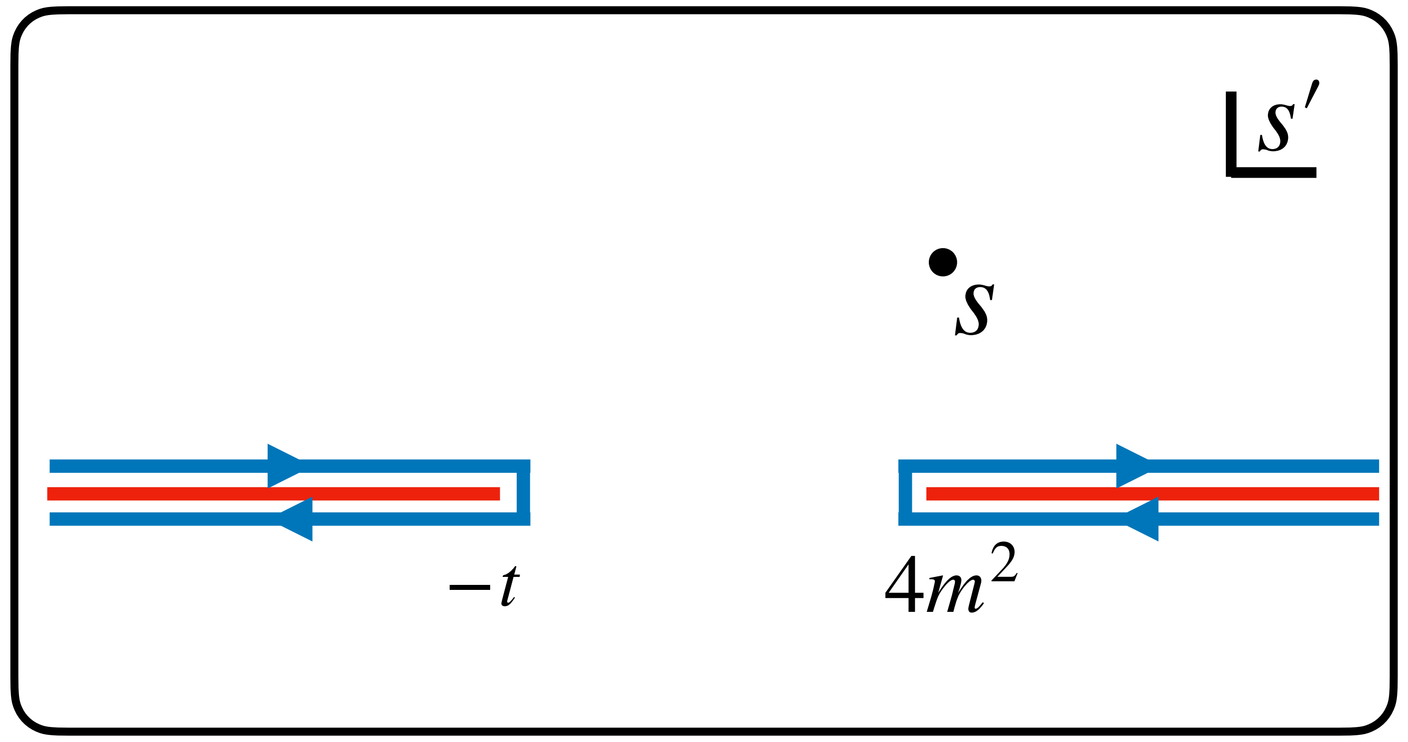
$$\mathrm{Im}[A(s,t)] \quad \longrightarrow \quad A(s,t)$$

Why do we like that?

- Unitarity $\Rightarrow$ $\mathrm{Im}[A]$ satisfies positivity properties.
- $\mathrm{Im}[A]$ is often simpler than the full amplitude.

Dispersion Relations for the S-Matrix





$$A(s,t) = \oint \frac{ds'}{2\pi i}\frac{1}{s'-s}A(s',t) \longrightarrow A(s,t) = \int \frac{ds'}{2\pi}\frac{1}{s'-s}\mathrm{Im}[A(s',t)]$$

Ingredients:

1. Analyticity = causality
2. Regge bound $\frac{|A(s,t)|}{s^n} \to 0$ as $s \to \infty$

Flat space	**AdS space**
S-matrix	correlation functions in a CFT
S-matrix dispersion relations	CFT dispersion relations

Motivation

1. Implement the conformal bootstrap for holographic CFTs.

2. Fruitful interface between S-matrix and conformal bootstrap methods.

Outline

I. Derivations

1. Position space
2. Mellin space
3. Lightray integrals

II. Applications

1. Bounds on AdS effective field theories
2. Planar $\mathcal{N} = 4$ SYM

Derivation 1: Position Space

The Four-Point Function

$$A(s,t) \longrightarrow \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

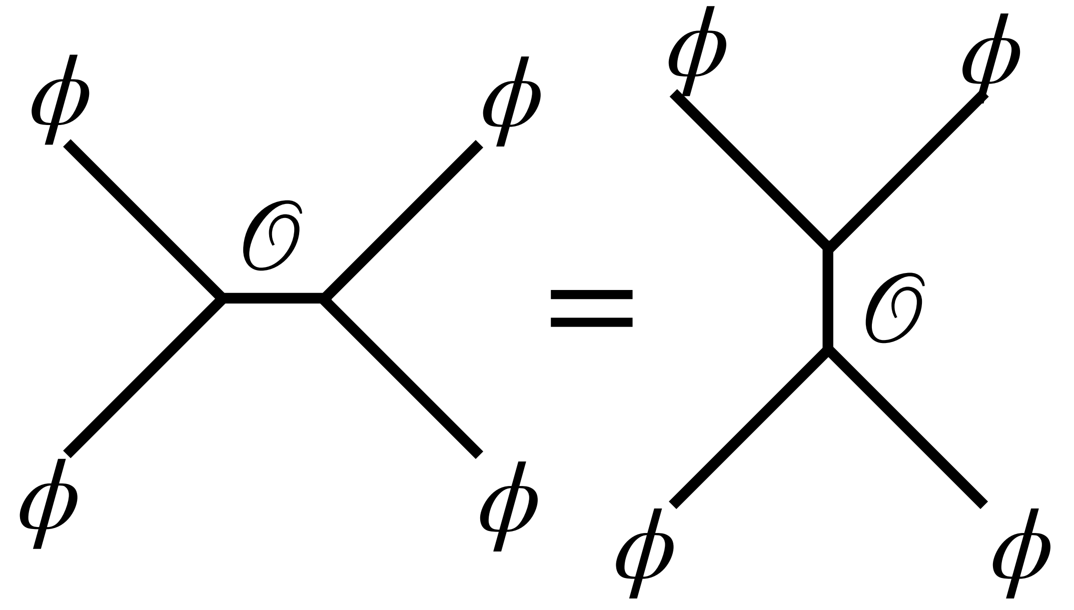
cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_3)\rangle\langle \phi(x_2)\phi(x_4)\rangle \, \mathcal{G}(u,v)$

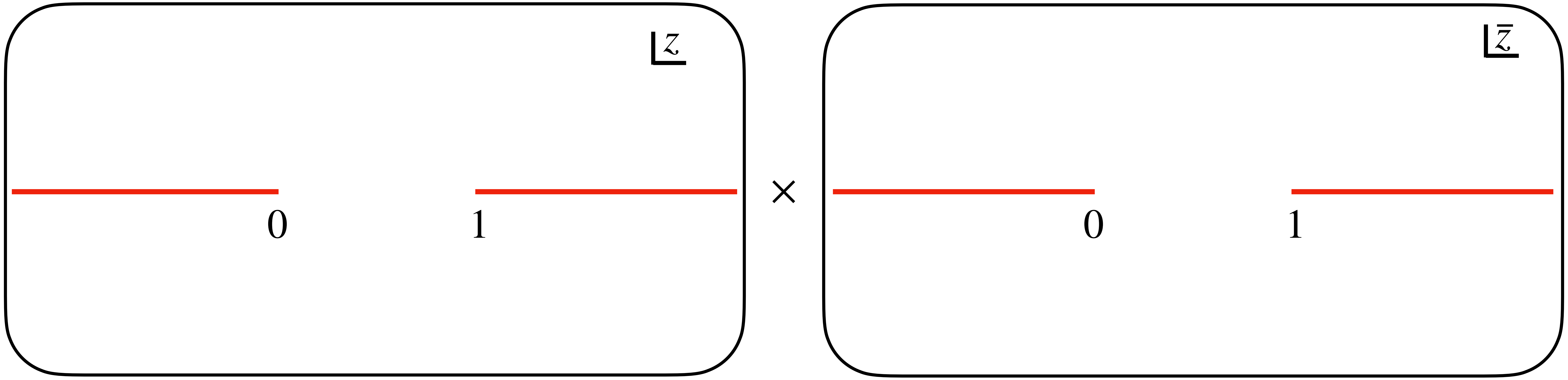
conformal blocks

- conformal block expansion: $\mathcal{G}(u,v) = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 G_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v) = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 G_{\Delta_\mathcal{O},J_\mathcal{O}}(v,u)$



Analyticity

- Change variables $\mathscr{G}(u, v) \to \mathscr{G}(z, \bar{z})$, where $u = z\,\bar{z}$, $v = (1-z)\,(1-\bar{z})$.

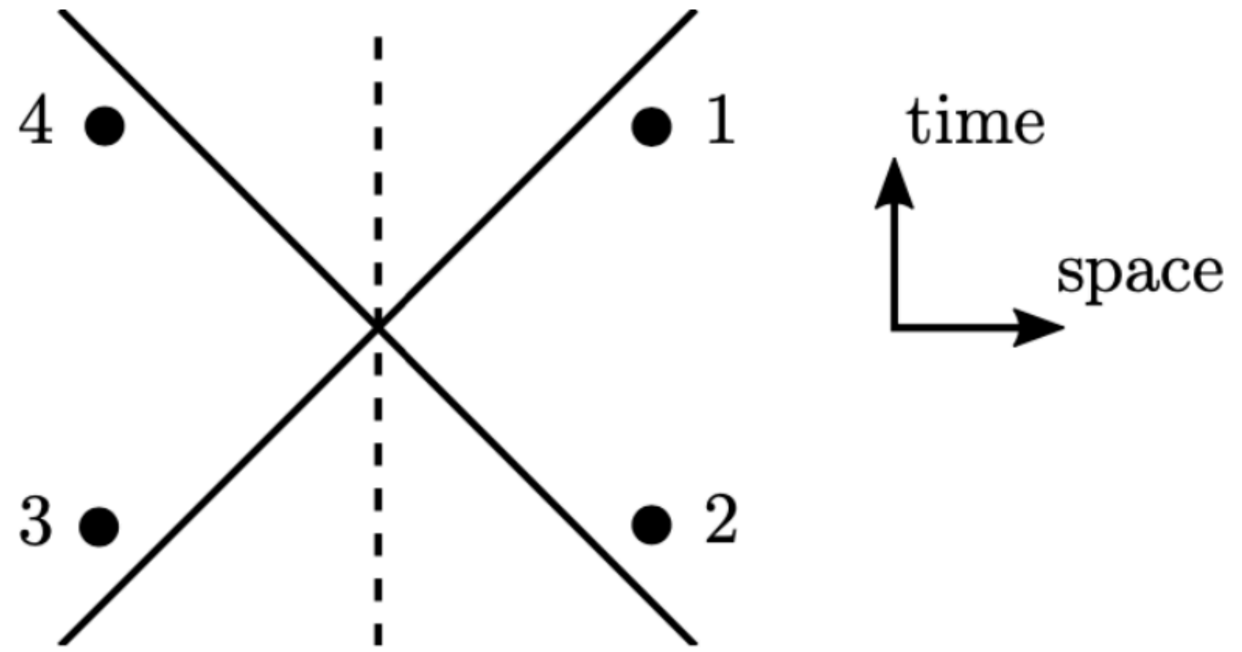


- OPE limits: s-channel $z, \bar{z} \to 0$; t-channel $z, \bar{z} \to 1$; u-channel $z \to +i\infty$, $\bar{z} \to -i\infty$.
- Regge limit (u-channel): $z, \bar{z} \to +i\infty$ with fixed $z/\bar{z}$.
- Regge bound: $|\mathscr{G}(z, \bar{z})| \sim |z|^{J_*-1}$, $J_* =$ Regge spin, nonperturbatively $J_* \leq 1$.

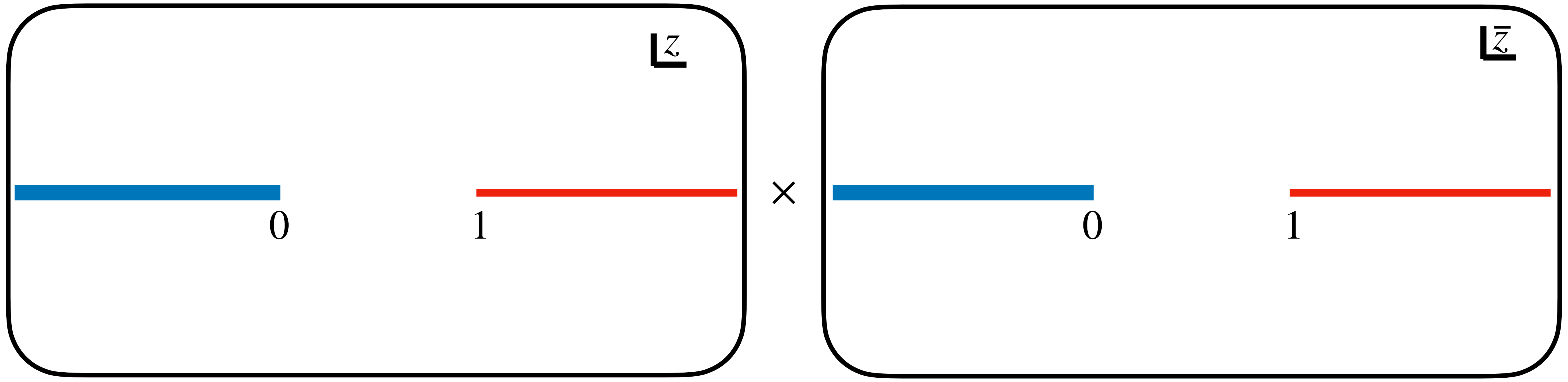
The Double Commutator

[Hartman, Jain, Kundu '15]
[Caron-Huot '17]

$$\mathrm{Im}[A(s,t)] \longrightarrow \mathrm{dDisc}[\mathcal{G}(z,\bar{z})] \sim \langle \Omega | [\phi(x_1),\phi(x_2)][\phi(x_3),\phi(x_4)] | \Omega \rangle$$

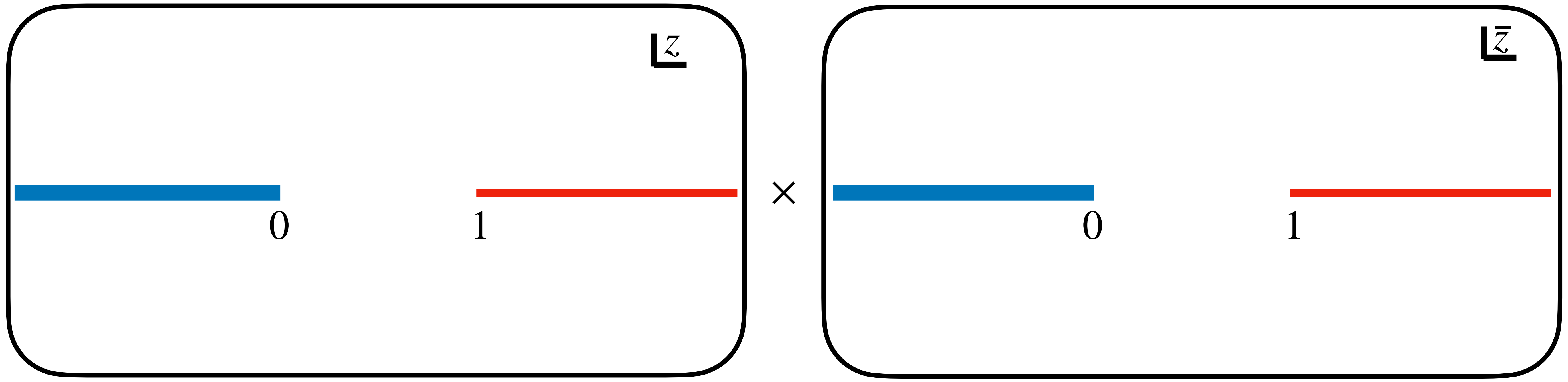


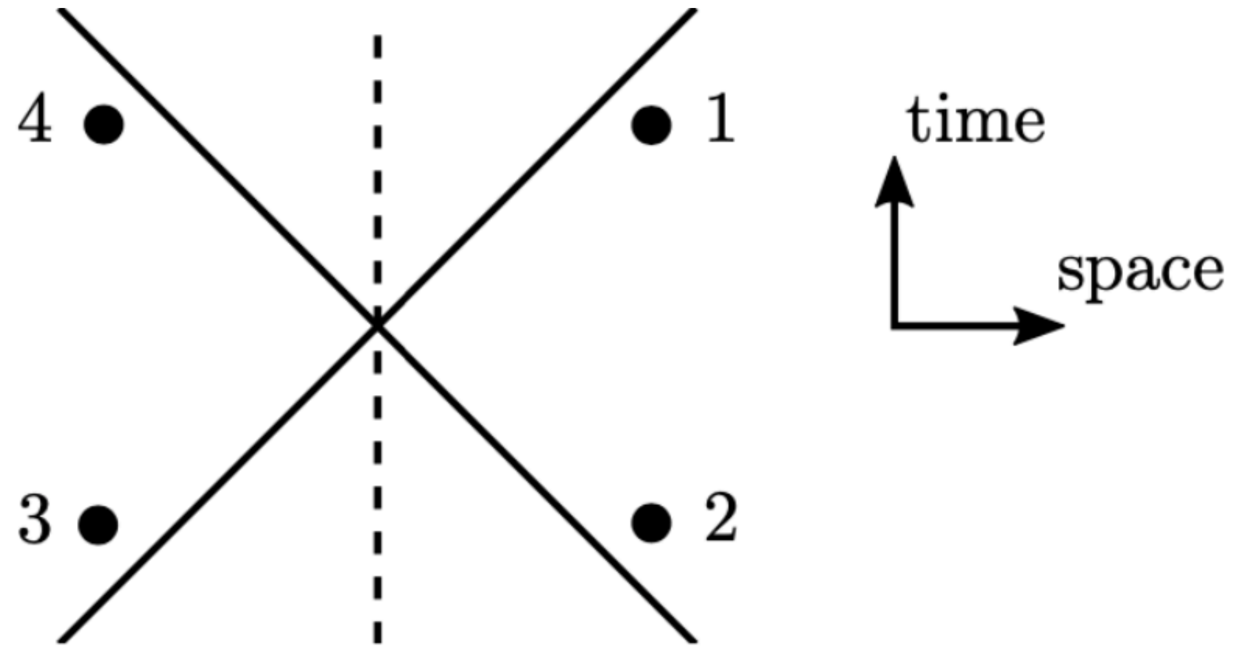
- $z, \bar{z} < 0.$
- $$2\mathrm{dDisc}[\mathcal{G}(z,\bar{z})] = \mathcal{G}(z+i\epsilon,\bar{z}-i\epsilon) + \mathcal{G}(z-i\epsilon,\bar{z}+i\epsilon) - \\ -\mathcal{G}(z+i\epsilon,\bar{z}+i\epsilon) - \mathcal{G}(z-i\epsilon,\bar{z}-i\epsilon)\,.$$



- $\mathrm{dDisc}[G_{\Delta,J}] = 2\sin^2\left[\frac{\pi}{2}(\Delta - J - 2\Delta_\phi)\right] G_{\Delta,J} \Rightarrow \mathrm{dDisc}[G_{2\Delta_\phi+2n+J,J}] = 0.$
- $\mathrm{dDisc}[\mathcal{G}] \geq 0.$

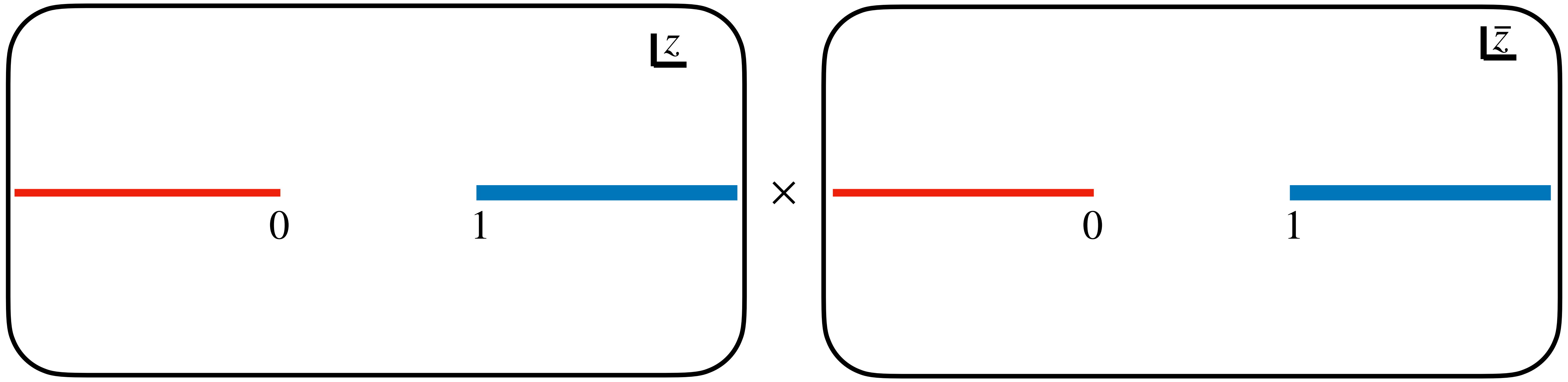
- $\mathrm{dDisc}_s[\mathcal{G}(z,\bar{z})] \sim \langle \Omega \,|\, [\phi(x_1), \phi(x_2)][\phi(x_3), \phi(x_4)] \,|\, \Omega \rangle$

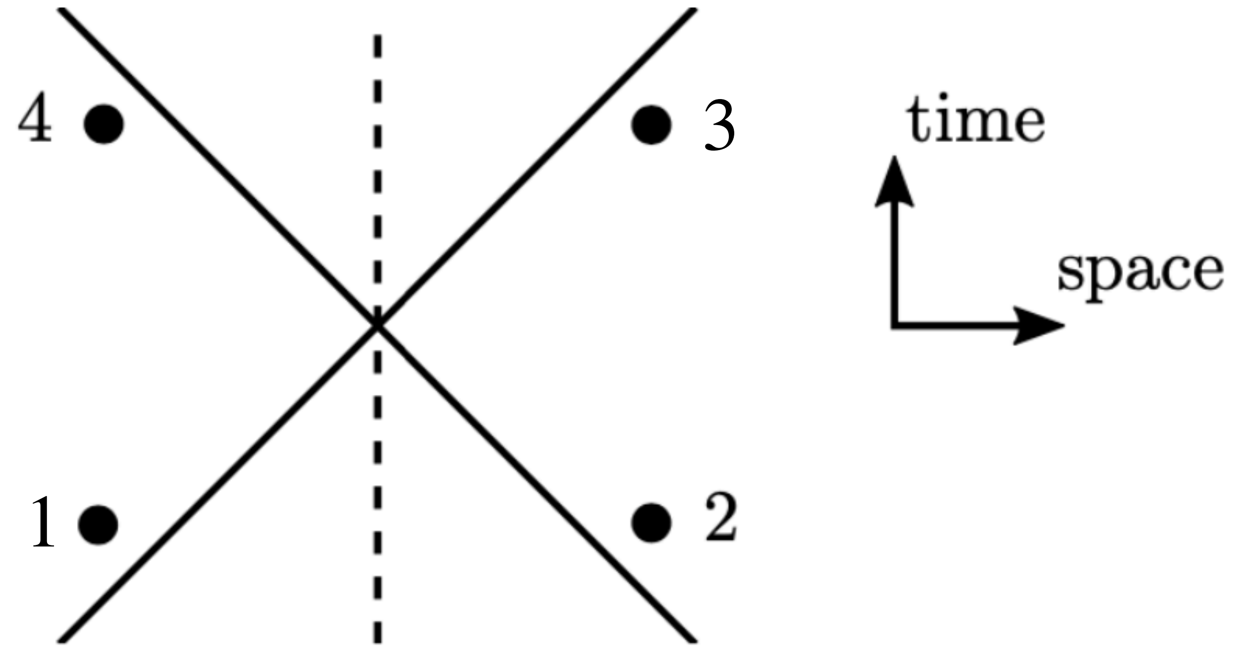




$z, \bar{z} < 0$

- $\mathrm{dDisc}_t[\mathcal{G}(z,\bar{z})] \sim \langle \Omega \,|\, [\phi(x_1), \phi(x_4)][\phi(x_2), \phi(x_3)] \,|\, \Omega \rangle$





$z, \bar{z} > 1$

Conformal Dispersion Relation

[Carmi, Caron-Huot '19]
[DM, Rastelli, Zhou '19]

Theorem (Carmi, Caron-Huot '19): Suppose the Regge intercept satisfies $J_* < 0$. Then

$$\mathcal{G}(z, \bar{z}) = \iint_{-\infty}^{0} dw d\overline{w}\, K_s(z, \bar{z}; w, \overline{w})\, \mathrm{dDisc}_s[\mathcal{G}(w, \overline{w})] + \iint_{1}^{\infty} dw d\overline{w}\, K_t(z, \bar{z}; w, \overline{w})\, \mathrm{dDisc}_t[\mathcal{G}(w, \overline{w})].$$

- K_s, K_t are known functions involving elliptic integrals, found by an combination of computation, self-consistency and guesswork.

Questions:

- How can we derive conformal dispersion relations systematically?
- Physical theories only satisfy the weaker condition $J_* \leq 1 \Rightarrow$ need to introduce subtractions. How do we do that?

Derivation 2: Mellin Space

Mellin Space

[Penedones '10]
[Penedones, Silva, Zhiboedov '19]

- The Mellin representation of a conformal four-point function

$$\mathcal{G}(u,v) = \iint_{-i\infty}^{+i\infty} \frac{ds dt}{(4\pi i)^2} u^{\frac{s}{2}-\Delta_\phi} v^{\frac{t}{2}-\Delta_\phi} \Gamma\left(\Delta_\phi - \frac{s}{2}\right)^2 \Gamma\left(\Delta_\phi - \frac{t}{2}\right)^2 \Gamma\left(-\Delta_\phi + \frac{s+t}{2}\right)^2 M(s,t).$$

Mellin amplitude

- It is the inverse of the formal Mellin transform

$$\Gamma\left(\Delta_\phi - \frac{s}{2}\right)^2 \Gamma\left(\Delta_\phi - \frac{t}{2}\right)^2 \Gamma\left(-\Delta_\phi + \frac{s+t}{2}\right)^2 M(s,t) = \iint_0^\infty \frac{du dv}{uv} u^{\Delta_\phi - \frac{s}{2}} v^{\Delta_\phi - \frac{t}{2}} \mathcal{G}(u,v).$$

Idea: $M(s,t)$ is a closer analogue of a scattering amplitude than $\mathcal{G}(u,v)$.

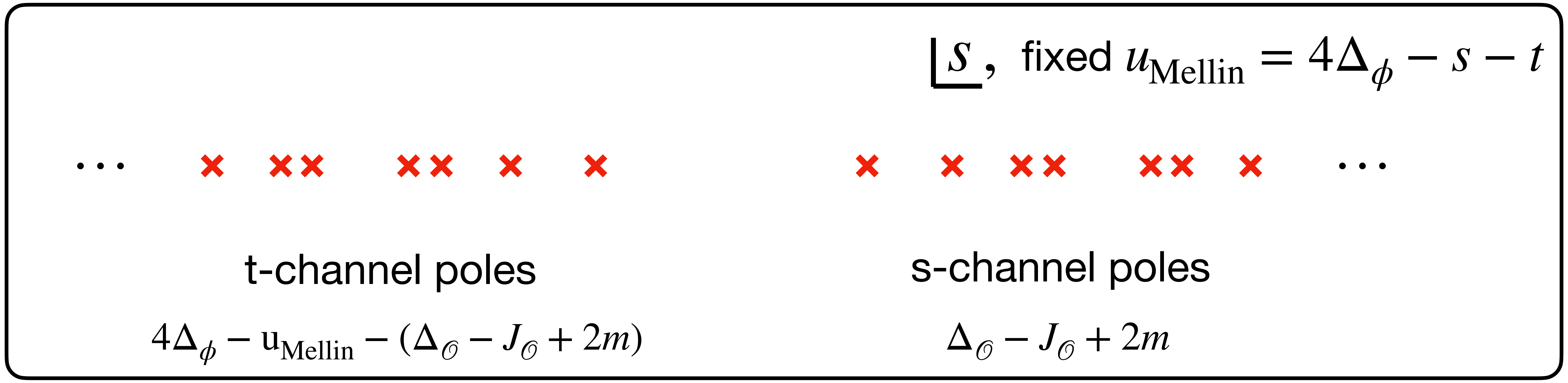
Properties of the Mellin Amplitude

- Crossing symmetry: $M(s,t) = M(t,s) = M(4\Delta_\phi - s - t, t)$

- An operator $\mathcal{O}$ in the $\phi \times \phi$ OPE becomes a sequence of poles of $M(s,t)$

$$\mathcal{G}(u,v) = \ldots + (f_{\phi\phi\mathcal{O}})^2 G_{\Delta_\mathcal{O}, J_\mathcal{O}}(u,v) + \ldots \quad \Rightarrow \quad M(s,t) \sim (f_{\phi\phi\mathcal{O}})^2 \frac{\mathcal{Q}(t)}{s - (\Delta_\mathcal{O} - J_\mathcal{O} + 2m)}$$

$$m = 0,1,2,\ldots$$

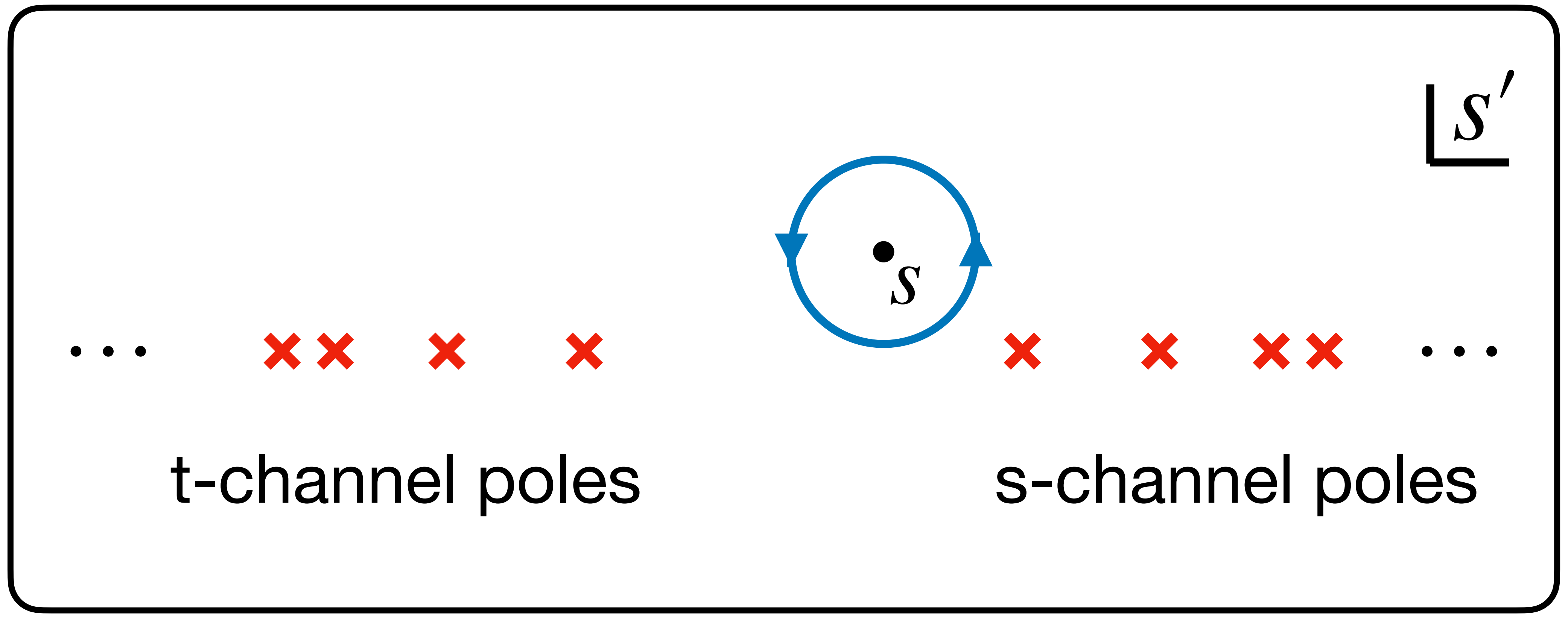


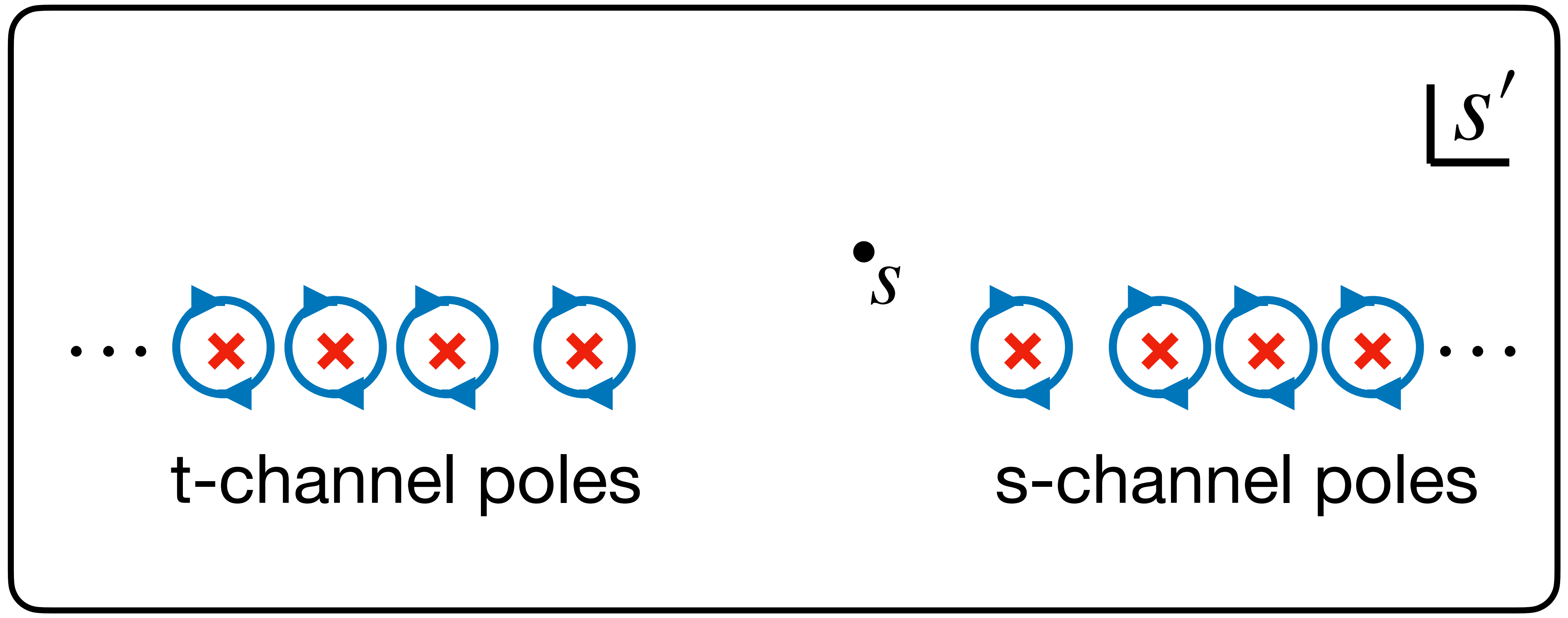
- Regge bound: $\mathcal{G}(z,\bar{z}) \sim |z|^{J_* - 1} \Leftrightarrow M(s,t) \sim |s|^{J_*}$ as $s \to \infty$ with fixed $s + t$.

Dispersion Relation in Mellin Space

[Penedones, Silva, Zhiboedov '19]

fixed-u dispersion relation





$$M(s,t) = \oint \frac{ds'}{2\pi i} \frac{M(s', t+s-s')}{s'-s} \longrightarrow M(s,t) = \sum_i \frac{\mathrm{Res}_i[M(s',t')]}{s-s_i} + \sum_j \frac{\mathrm{Res}_j[M(s',t')]}{s-s_j}$$

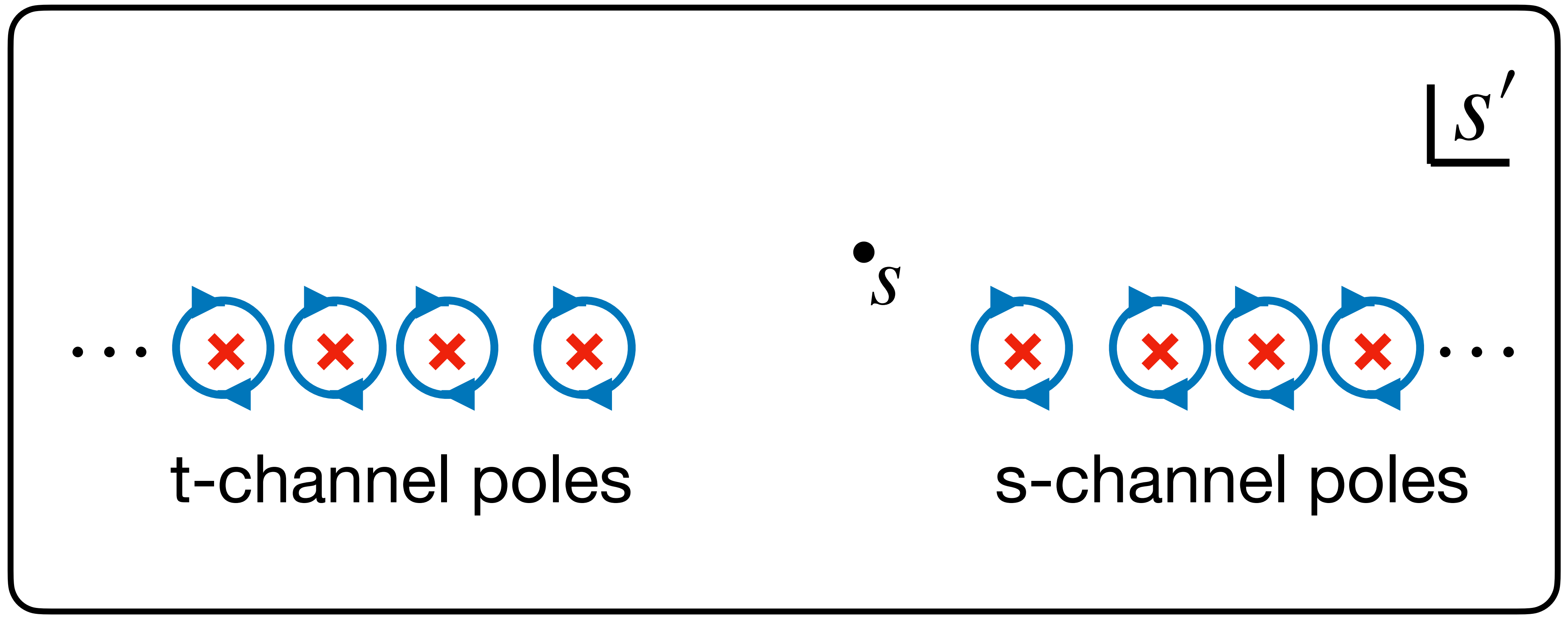
assume $J_* < 0$

s-channel poles

t-channel poles

Position Space ⇔ Mellin Space

$$M(s,t) = \underbrace{\sum_i}_{\text{s-channel poles}} \frac{\mathrm{Res}_i[M(s',t')]}{s - s_i} + \underbrace{\sum_j}_{\text{t-channel poles}} \frac{\mathrm{Res}_j[M(s',t')]}{s - s_j}$$



Idea: Perform the inverse Mellin transform back to position space.

⇒ recover the original position-space dispersion relation!

$$\mathcal{G}(u,v) = \iint\limits_{\substack{1>2\\4>3}} du'dv'\, K_s(u,v;u',v')\, \mathrm{dDisc}_s[\mathcal{G}(u',v')] + \iint\limits_{\substack{3>2\\4>1}} du'dv'\, K_t(u,v;u',v')\, \mathrm{dDisc}_t[\mathcal{G}(u',v')]$$

[Caron-Huot, DM, Rastelli, Simmons-Duffin ’20]

Bonus: An explicit formula for the dispersion kernel:

$$K_s(u,v;u',v') = \frac{1}{2\pi^2 u'v'} \iiint_{-i\infty}^{+i\infty} \frac{dxdydz}{(2\pi i)^3} \frac{\Gamma(-x)^2\Gamma(-y)^2\Gamma(z+1)^2}{(z-x)\Gamma(z-x-y)^2} \left(\frac{u}{v'}\right)^x \left(\frac{v}{v'}\right)^y \left(\frac{v'}{u'}\right)^z$$

Subtractions

[Penedones, Silva, Zhiboedov '19]
[Caron-Huot, DM, Rastelli, Simmons-Duffin '20]

Problem: The amplitude does not vanish in the Regge limit $\Rightarrow$ need subtractions.

- Regge bound: $J_* \leq 1 \ \Rightarrow \ M(s,t) = O(|s|)$ as $s \to \infty \ \Rightarrow$ single subtraction is enough.
- Natural choice: $M(s,t) = \oint \frac{ds'}{2\pi i} \frac{(s-2\Delta_\phi)(t-2\Delta_\phi)}{(s'-2\Delta_\phi)(t+s-s'-2\Delta_\phi)} \frac{M(s',t+s-s')}{s'-s}$.
- Again, can transform to position space

$$\mathcal{G}(u,v) = \iint\limits_{\substack{1>2\\4>3}} du'dv'\, K_{2,s}(u,v;u',v')\, \mathrm{dDisc}_s[\mathcal{G}(u',v')] + \iint\limits_{\substack{3>2\\4>1}} du'dv'\, K_{2,t}(u,v;u',v')\, \mathrm{dDisc}_t[\mathcal{G}(u',v')]$$

- In this form, the position-space dispersion relation is a theorem*.

Using the OPE in the Dispersion Relation

$$\mathcal{G}(u,v) = \iint\limits_{\substack{1>2\\4>3}} du'dv'\, K_{2,s}(u,v;u',v')\,\mathrm{dDisc}_s[\mathcal{G}(u',v')] + \iint\limits_{\substack{3>2\\4>1}} du'dv'\, K_{2,t}(u,v;u',v')\,\mathrm{dDisc}_t[\mathcal{G}(u',v')]$$

Idea: Expand the four-point function on both sides using conformal blocks.

- $$\sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 G_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(u,v) = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 [P_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(u,v) + P_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(v,u)]$$ **c.f. [Gopakumar, Kaviraj, Sen, Sinha]**
- Polyakov-Regge blocks: $$P_{\Delta,J}(u,v) = \iint\limits_{\substack{1>2\\4>3}} du'dv'\, K_{2,s}(u,v;u',v')\,\mathrm{dDisc}_s[G_{\Delta,J}(u',v')]$$ **[DM, Rastelli, Zhou '19]**
- Constraints on the CFT data $\Delta_{\mathcal{O}}, J_{\mathcal{O}}, (f_{\phi\phi\mathcal{O}})^2$ = reorganized conformal bootstrap equations.

Dispersive CFT Sum Rules

$$\sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 G_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v) = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 [P_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v) + P_{\Delta_\mathcal{O},J_\mathcal{O}}(v,u)]$$

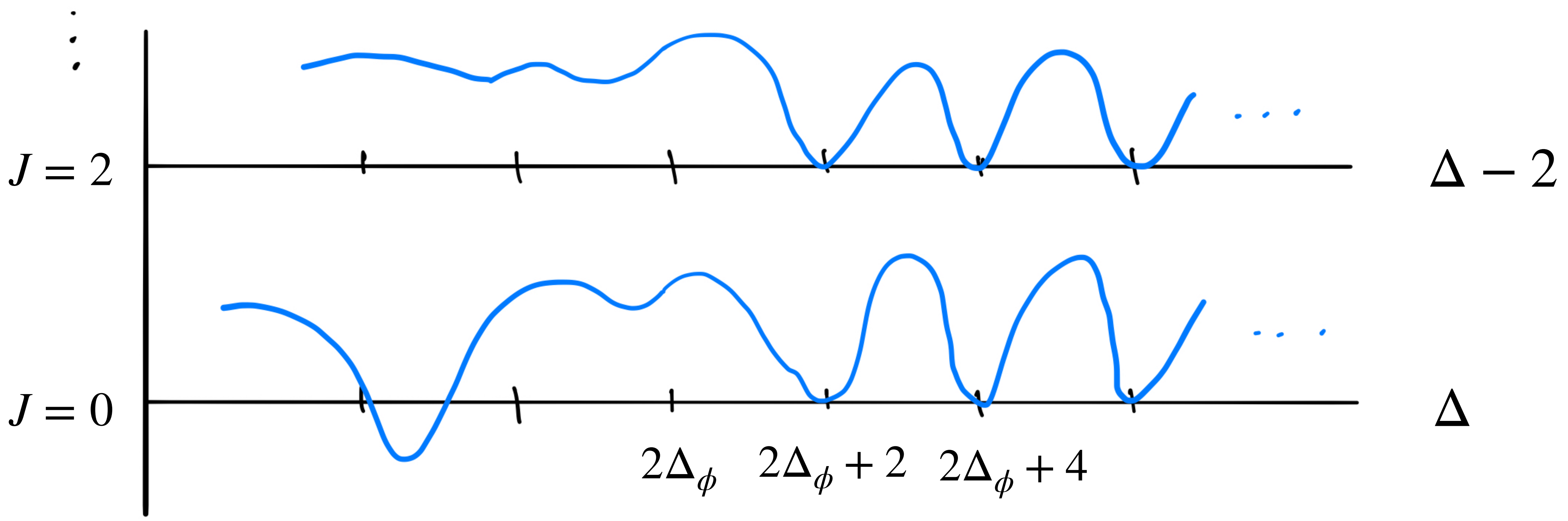
- Take the $u \to 0$ limit of this sum rule. Get a simpler sum rule

$$\sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 B_{2,v}[G_{\Delta_\mathcal{O},J_\mathcal{O}}] = 0 \qquad B_{2,v}[G_{\Delta,J}] = 2\sin^2\left[\tfrac{\pi}{2}(\Delta - J - 2\Delta_\phi)\right] \iint du'dv' \frac{(v'-u')\, G_{\Delta,J}(u',v')}{u'v'\sqrt{v^2+u'^2+v'^2-2(vu'+vv'+u'v')}}$$

[Caron-Huot, DM, Rastelli, Simmons-Duffin '20]

- $B_{2,v}[G_{\Delta,J}]$ has double zeros on all double-trace dimensions $\Delta = 2\Delta_\phi + 2n + J$ for $n > 0$.

$B_{2,v}[G_{\Delta,J}]$



very useful in many applications!

precursor in 1D CFTs, see Lucía's talk

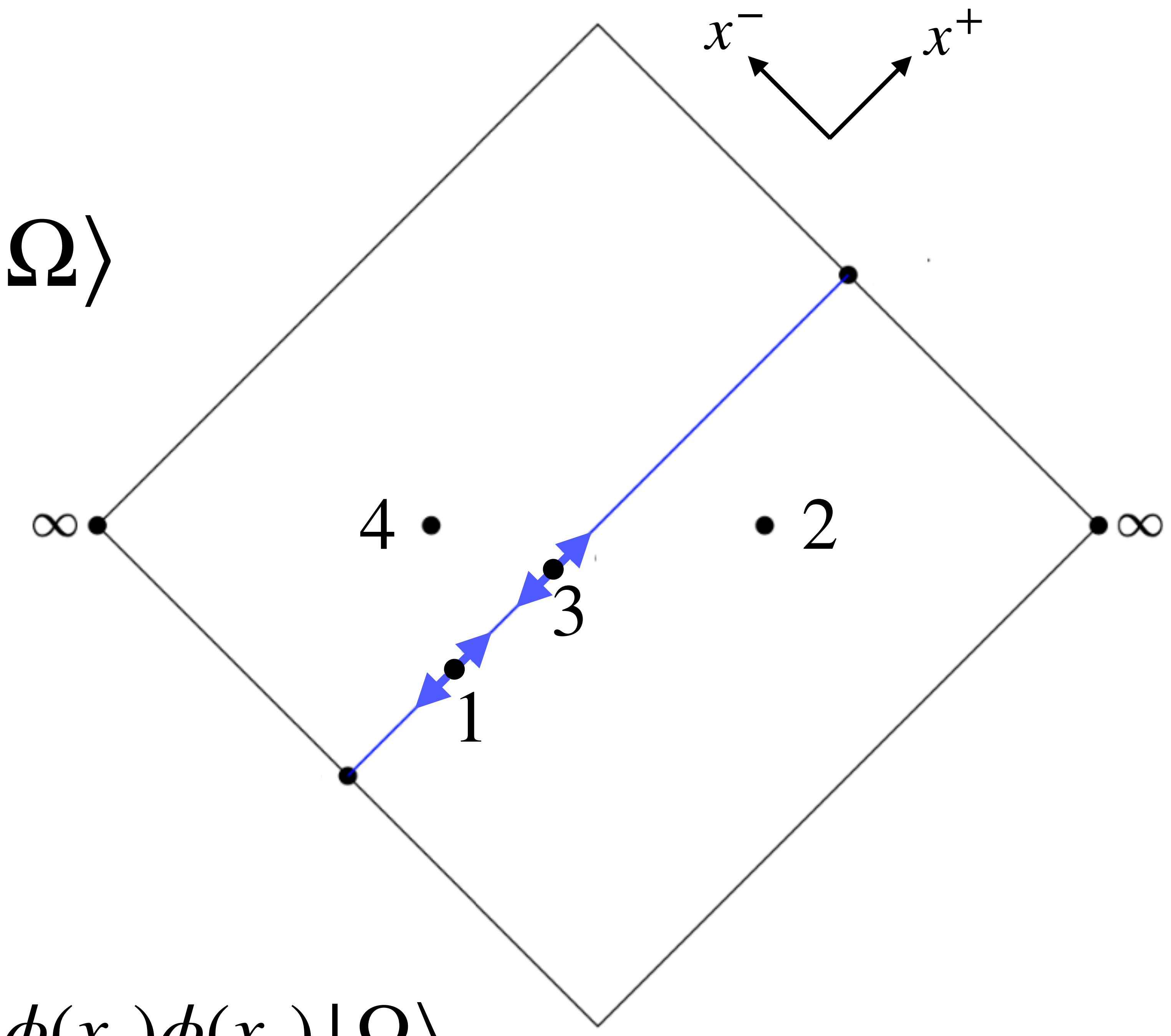
[DM '16]
[DM, Paulos '18]

Derivation 3: Causality and Lightray Integrals

Dispersive Sum Rules from Causality

- Causality: $[\phi(x_i), \phi(x_j)] = 0$ for $x_i - x_j$ spacelike.
- $\langle \Omega | \phi(x_4)[\phi(x_3), \phi(x_1)]\phi(x_2) | \Omega \rangle = 0$ for $x_1 - x_3$ spacelike.
- $\phi(x_i)\phi(x_j)$ OPE converges if $\phi(x_i)\phi(x_j)$ acts on the vacuum.

$$\langle \Omega | \underbrace{\phi(x_4)\phi(x_3)\phi(x_1)\phi(x_2)}_{\text{s-channel}} | \Omega \rangle = \langle \Omega | \underbrace{\phi(x_4)\phi(x_1)\phi(x_3)\phi(x_2)}_{\text{t-channel}} | \Omega \rangle$$



- To get a dispersive sum rule, integrate x_1, x_3 along spacelike-separated null rays against a kernel $h(x_1, x_3)$:

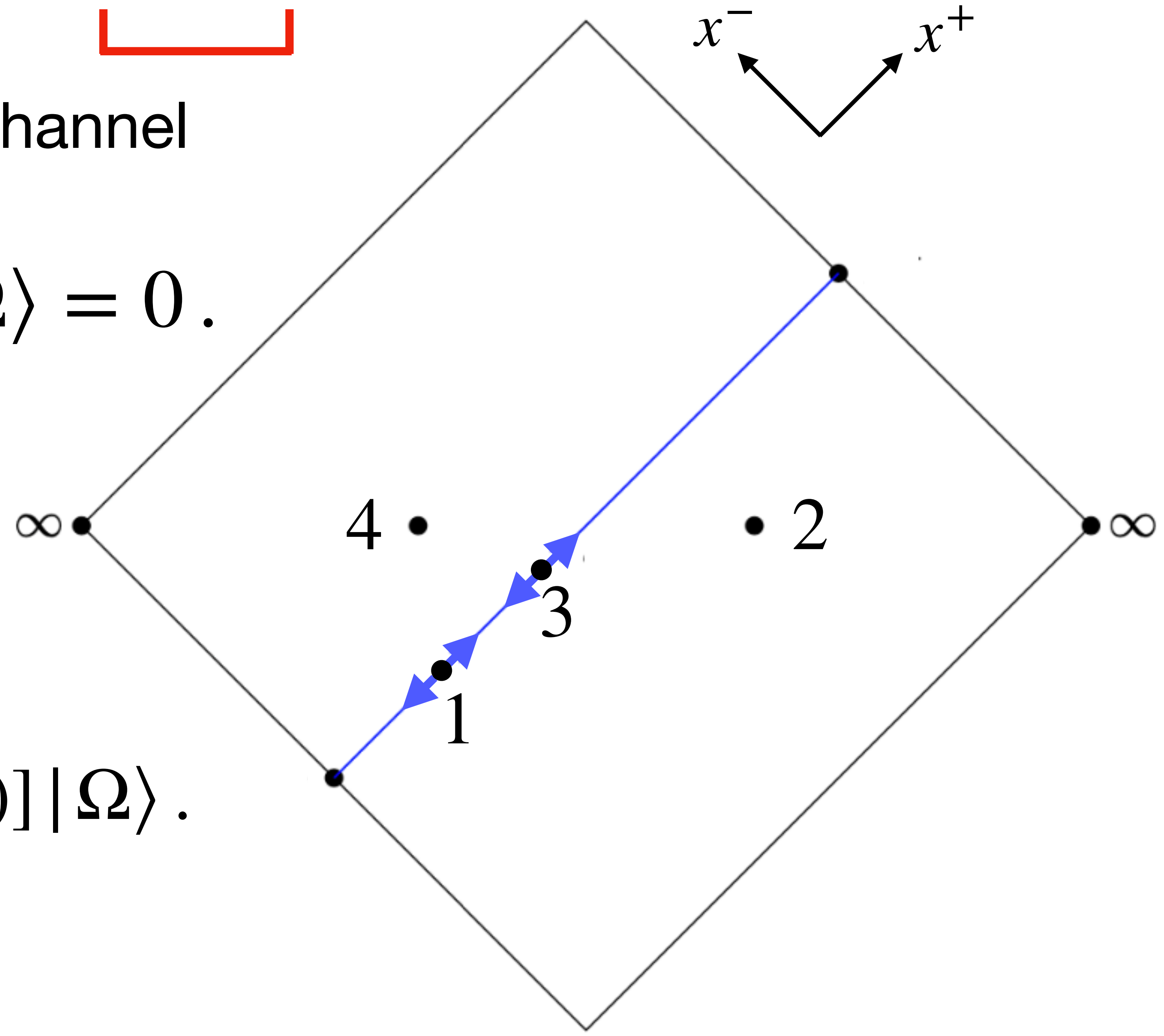
$$\iint_{-\infty}^{\infty} dx_1^+ dx_3^+ h(x_1, x_3) \langle \Omega | \underbrace{\phi(x_4)\phi(x_3)\phi(x_1)\phi(x_2)}_{\text{s-channel}} | \Omega \rangle = \iint_{-\infty}^{\infty} dx_1^+ dx_3^+ h(x_1, x_3) \langle \Omega | \underbrace{\phi(x_4)\phi(x_1)\phi(x_3)\phi(x_2)}_{\text{t-channel}} | \Omega \rangle$$

Dispersive Sum Rules from Causality

[Caron-Huot, DM, Rastelli, Simmons-Duffin '20]
c.f. [Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov '19]

$$\iint_{-\infty}^{\infty} dx_1^+ dx_3^+ h(x_1,x_3)\langle\Omega|\underbrace{\phi(x_4)\phi(x_3)}\underbrace{\phi(x_1)\phi(x_2)}|\Omega\rangle = \iint_{-\infty}^{\infty} dx_1^+ dx_3^+ h(x_1,x_3)\langle\Omega|\underbrace{\phi(x_4)\phi(x_1)}\underbrace{\phi(x_3)\phi(x_2)}|\Omega\rangle$$

s-channel t-channel



- Null-integrated local operators annihilate the vacuum: $\int_{-\infty}^{\infty} dx^+ \phi(x)|\Omega\rangle = 0\,.$
- In the absence of $h(x_1,x_3)$, we recover the double commutator

$$\iint_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle\Omega|\phi(x_4)\phi(x_3)\phi(x_1)\phi(x_2)|\Omega\rangle \to \iint_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle\Omega|[\phi(x_4),\phi(x_3)][\phi(x_1),\phi(x_2)]|\Omega\rangle\,.$$

- The integrated crossing equation therefore becomes the dispersive sum rule (superconvergence)

$$\iint_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle\Omega|[\underbrace{\phi(x_4),\phi(x_3)}][\underbrace{\phi(x_1),\phi(x_2)}]|\Omega\rangle = \iint_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle\Omega|[\underbrace{\phi(x_4),\phi(x_1)}][\underbrace{\phi(x_3),\phi(x_2)}]|\Omega\rangle\,.$$

s-channel t-channel

Dispersive Sum Rules from Causality

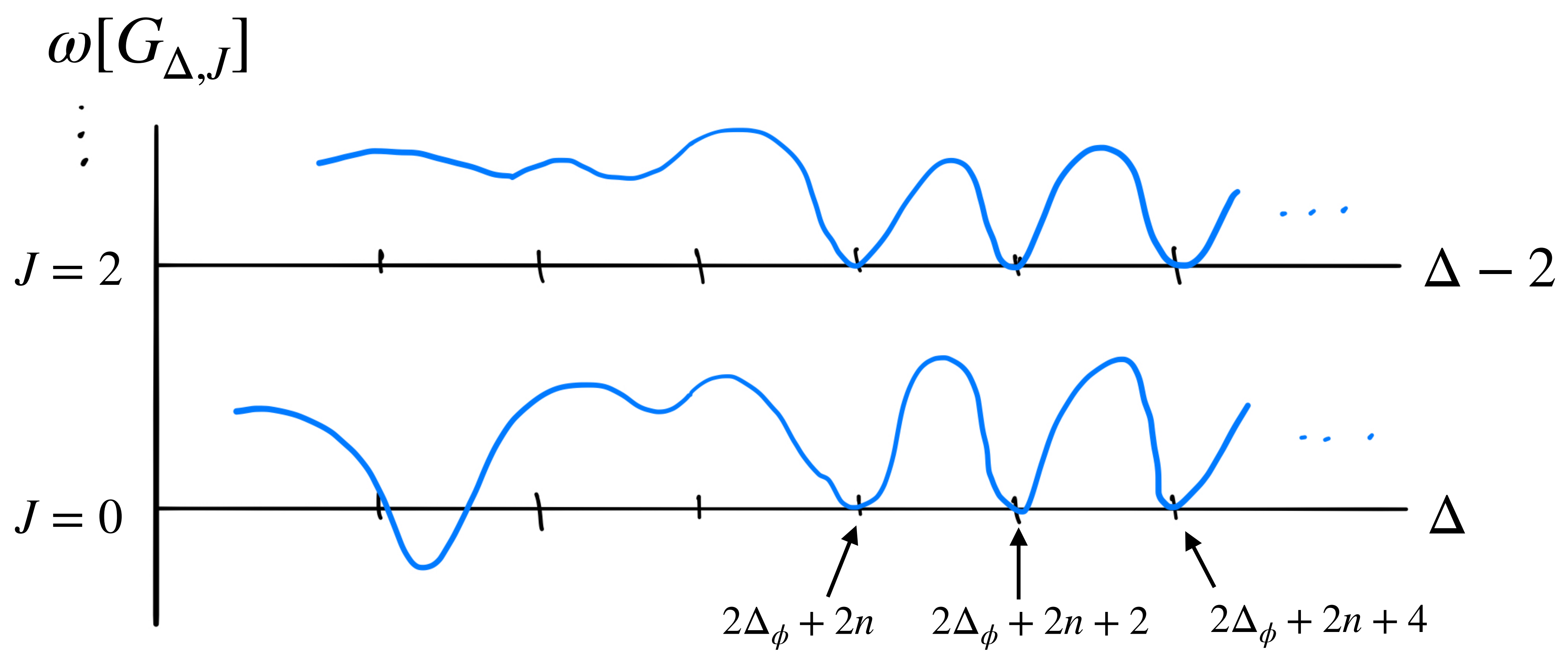
$$\iint\limits_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle \Omega | [\phi(x_4), \phi(x_3)][\phi(x_1), \phi(x_2)] | \Omega \rangle = \iint\limits_{-\infty}^{\infty} dx_1^+ dx_3^+ \langle \Omega | [\phi(x_4), \phi(x_1)][\phi(x_3), \phi(x_2)] | \Omega \rangle \, .$$

$h(x_1, x_3)$ s-channel $h(x_1, x_3)$ t-channel

- The kernel $h(x_1, x_3)$ is necessary to ensure convergence of the integrals = subtraction.
- It causes the sum rule to not vanish on light double traces.

Final result: A sum rule $\sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 \, \omega[G_{\Delta_\mathcal{O}, J_\mathcal{O}}] = 0$

- $\omega[G_{\Delta,J}]$ has double zeros on all double-traces $\Delta = 2\Delta_\phi + 2n + J$ with high enough n.

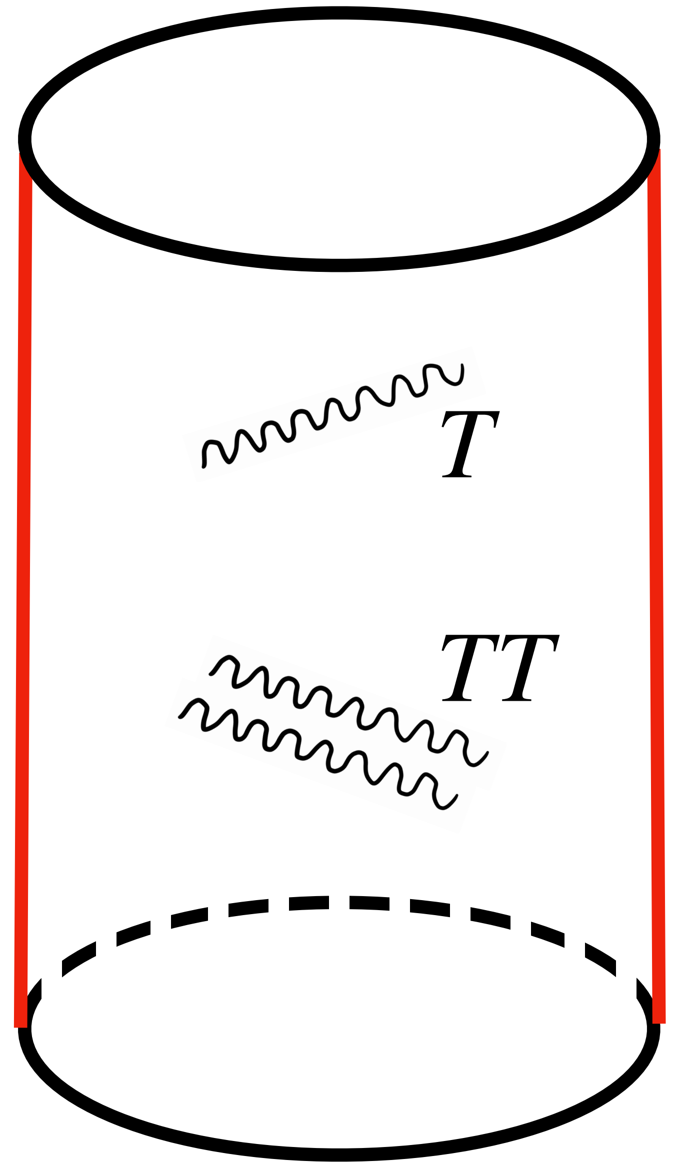


- Any ω with this property is called a dispersive sum rule / dispersive functional.

Application 1: Bounds on EFTs in AdS

Holographic CFTs

- Holography: Every CFT_d is a quantum gravity in AdS_{d+1}.
  - Semiclassical gravity: spacetime is large in Planck units $\ell_{\text{AdS}} M_{\text{Planck}} \gg 1$.
  - For $E \ll M_{\text{Planck}}$: collection of weakly-interacting particles (graviton, ...)
- Dual CFT: graviton ~ $T_{\mu\nu}$ $\quad c \sim \langle T_{\mu\nu} T_{\mu\nu} \rangle \sim (\ell_{\text{AdS}} M_{\text{Planck}})^{D-2} \gg 1$
  - Single-particle states ~ single-trace operators $\phi, T_{\mu\nu}, \ \ldots$
  - Multi-particle states ~ multi-trace operators $\phi \Box^n \partial^J \phi, TT, \phi T, \phi\phi\phi, \ldots$
- $c = \infty$: **mean field theory**
  - Anomalous dimensions suppressed by $1/c$: $\Delta_{n,J} = 2\Delta_\phi + 2n + J + O(1/c)$.
- Dispersive sum rules naturally incorporate this structure by suppressing the double traces.



Low-Energy EFT with Gravity in AdS

General relativity: $$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,(-2\Lambda + R)$$

Higher-derivative corrections: $$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,\left(-2\Lambda + R + \alpha_2 R^2 + \alpha_3 R^3 + \dots\right)$$

Dimensional analysis: $\alpha_2 = \frac{\#}{M_{\text{gap}}^2}$, $\alpha_3 = \frac{\#}{M_{\text{gap}}^4}$, where M_{gap} is the scale of new physics.

Question: Why not $\alpha_2 = \frac{10^{10}}{M_{\text{gap}}^2}$, $\alpha_3 = \frac{10^{100}}{M_{\text{gap}}^4}$?

$\Delta_{\text{gap}} = \ell_{\text{AdS}} M_{\text{gap}}$

scaling dimension of the lightest single-trace operator not included in the EFT.

Effective Field Theory from Causality

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,\left(-2\Lambda + R + \alpha_2 R^2 + \alpha_3 R^3 + \ldots\right)$$

Question: Why not $\alpha_2 = \frac{10^{10}}{M_{\text{gap}}^2}$, $\alpha_3 = \frac{10^{100}}{M_{\text{gap}}^4}$?

Partial answer: $|\alpha_2| \gg \frac{1}{M_{\text{gap}}^2}$ or $|\alpha_3| \gg \frac{1}{M_{\text{gap}}^4}$ leads to a causality violation.

[Camanho, Edelstein, Maldacena, Zhiboedov '14]

New goal: Derive **exact** bounds on the $O(1)$ coefficients, e.g. $-\frac{a_-}{M_{\text{gap}}^4} \leq \alpha_3 \leq \frac{a_+}{M_{\text{gap}}^4}$.

$\Rightarrow$ In particular, a proof that large c and large Δ_{gap} in a CFT imply bulk locality.

[Heemskerk, Penedones, Polchinski, Sully '09]

Gravity + Scalar in AdS$_D$

$$S = \int d^Dx\sqrt{-g}\left[-\frac{1}{2}(\nabla\Phi)^2 - \lambda\,\Phi^4 + g_2\,\nabla^4\Phi^4 + g_3\,\nabla^6\Phi^4 + \ldots\right] + S_{\text{grav.}}$$

- The only light single-traces: $\phi(x)$, $T_{\mu\nu}(x)$.
- The EFT couplings are encoded in the double-trace anomalous dimensions $\gamma_{n,J}$.

$$\Delta_{n,J} = 2\Delta_\phi + 2n + J + \gamma_{n,J}$$

- We will derive constraints on couplings by applying dispersive functionals to

$$\langle 0\,|\,\phi(x_4)[\phi(x_3),\phi(x_1)]\phi(x_2)\,|\,0\rangle = 0\,.$$

Why this works: The dispersive functionals extract the EFT couplings and relate them to the OPE contributions of the heavy states, about which we are agnostic, except that they come with definite signs due to unitarity.

Bounds on EFT Couplings from Functionals

[Caron-Huot, DM, Rastelli, Simmons-Duffin '21]

$$\langle 0|\phi\phi\phi\phi|0\rangle = \underbrace{G_1 + \sum G_{[\phi\phi]_{n,J}} + G_{T_{\mu\nu}} + G_{[\text{composites}]}}_{\text{IR: } \Delta<\Delta_{\text{gap}}} + \underbrace{\sum G_{\text{heavy}}}_{\text{UV: } \Delta>\Delta_{\text{gap}}}$$

1. Split IR and UV contributions and apply a dispersive functional ω.

$$\omega_{\text{IR}} = \sum_{\Delta<\Delta_{\text{gap}}} |f_{\Delta,J}|^2\, \omega[G_{\Delta,J}] \qquad \omega_{\text{UV}} = \sum_{\Delta\geq\Delta_{\text{gap}}} |f_{\Delta,J}|^2\, \omega[G_{\Delta,J}]$$

$$\omega_{\text{IR}} + \omega_{\text{UV}} = 0$$

2. ω_{IR} is computable in terms of the EFT couplings g_2, g_3, and Newton's constant G.

3. Choose ω so that $\omega_{\text{UV}} \geq 0$. We get the inequality $-\omega_{\text{IR}} \geq 0$.

   $\Rightarrow$ Bounds on G, g_2, g_3.

4. Optimize over ω to find the best possible bounds.

The Flat-Space Limit

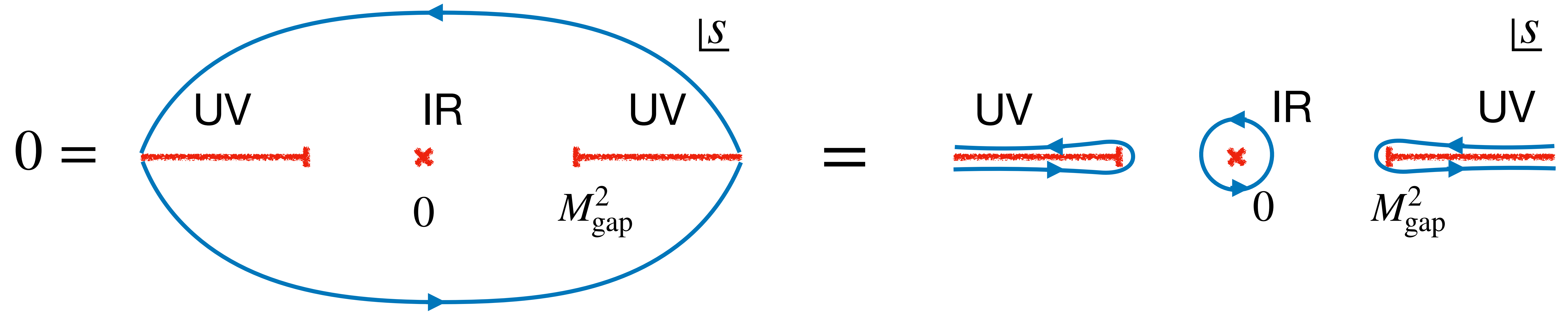
[Caron-Huot, DM, Rastelli, Simmons-Duffin '21]

- Flat-space limit: $\ell_{\text{AdS}} \to \infty$. The four-point correlator becomes the S-matrix.

$$\langle 0 | \phi\phi\phi\phi | 0 \rangle \to A(s,t)$$

- Expansion in the IR: $A(s,t) = 8\pi G \left(\frac{tu}{s} + \frac{us}{t} + \frac{st}{u} \right) + g_2(s^2 + t^2 + u^2) + g_3(stu) + \dots$

- The causality constraint $\omega_{\text{IR}} + \omega_{\text{UV}} = 0$ becomes an S-matrix dispersion relation:



[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06]
[Bellazzini, Miro, Rattazzi, Riembau, Riva '20]
[Tolley, Wang, Zhou '20]
[Caron-Huot, Van Duong '20]
[Arkani-Hamed, Huang, Huang '20]

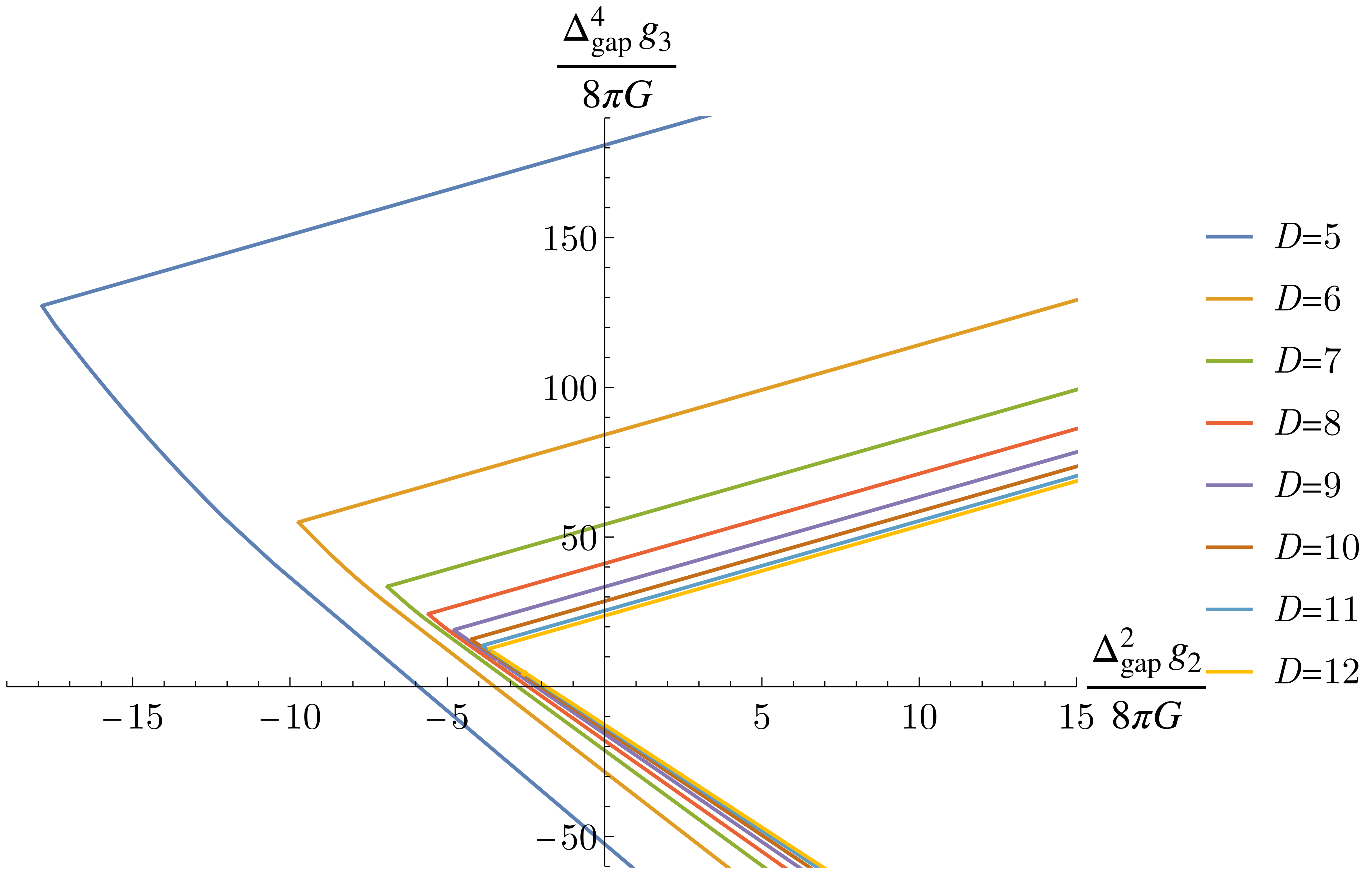
Strategy: 1. find the optimal flat-space bound using semidefinite programming [Simmons-Duffin '15] [Simmons-Duffin, Landry '19]

2. uplift the optimal functional to AdS

Results

[Caron-Huot, DM, Rastelli, Simmons-Duffin '21]

Bound at the leading order at large Δ_{gap}:



Open problem: Systematics of $1/\Delta_{\text{gap}}$ corrections.

Application 2: Planar $\mathcal{N} = 4$ SYM

The AdS Virasoro-Shapiro Amplitude

- Consider the four-point function of the stress tensor supermultiplet in $\mathcal{N} = 4$ SYM with central charge $c = \frac{N^2-1}{4}$ and 't Hooft coupling λ.

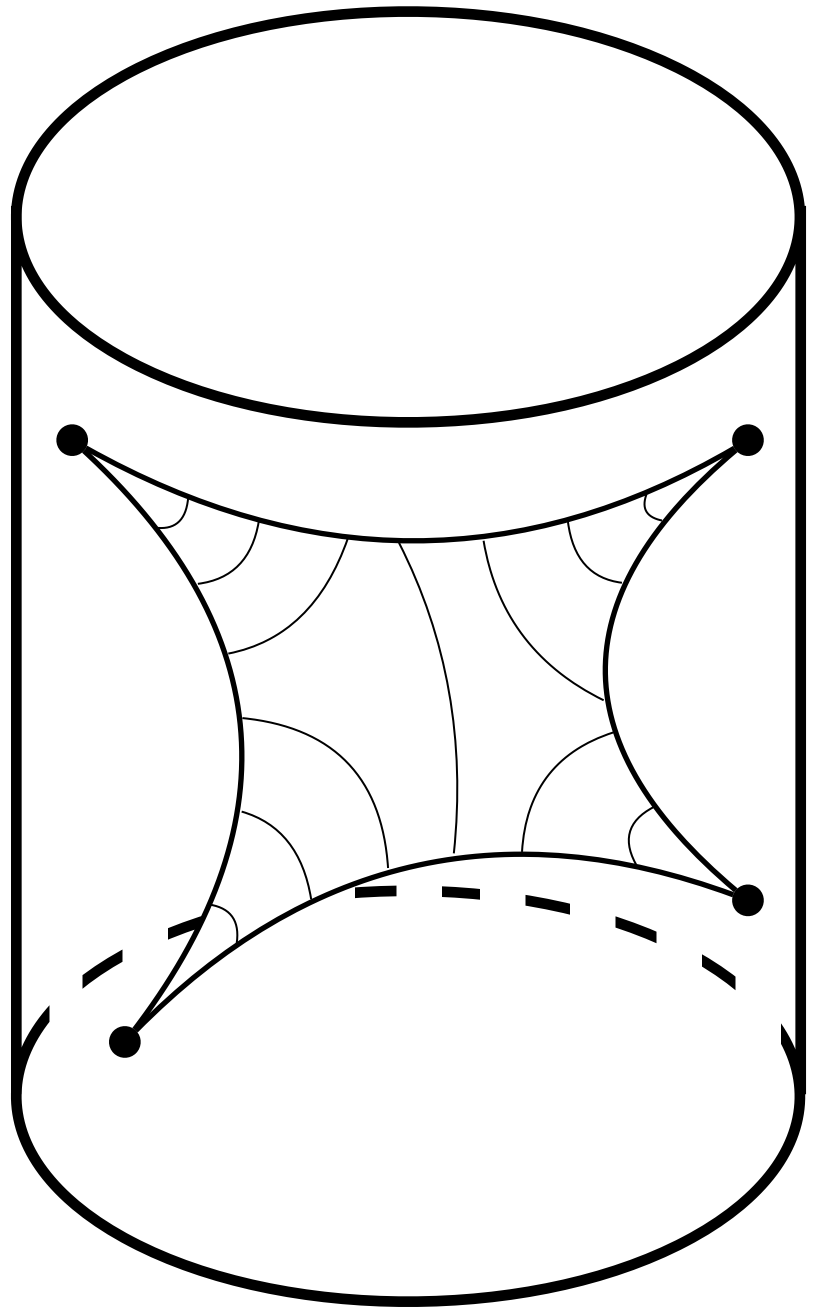
$$\mathcal{G}(u,v) = \mathcal{G}^{(\text{disc.})}(u,v) + \frac{1}{c}\mathcal{G}^{(0)}_{\lambda}(u,v) + O(c^{-2})$$

- $\mathcal{G}^{(0)}_{\lambda}(u,v)$ is the AdS version of the Virasoro-Shapiro amplitude

$$A_{\text{flat}}(s,t) = \frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 t)\Gamma(-\ell_s^2 u)}{\Gamma(\ell_s^2 s+1)\Gamma(\ell_s^2 t+1)\Gamma(\ell_s^2 u+1)}.$$

- Dimensionless parameter $\lambda = \left(\frac{\ell_{\text{AdS}}}{\ell_s}\right)^4 \begin{cases} \lambda \ll 1\text{: perturbative Yang-Mills} \\ \lambda \gg 1\text{: supergravity+stringy corrections} \end{cases}$

- Currently, no closed formula for $\mathcal{G}^{(0)}_{\lambda}(u,v)$ for general λ.



The OPE

- Conformal block expansion: $\mathscr{G}^{(0)}_{\lambda} = \mathscr{G}_{\text{protected}} + \sum_{\substack{\text{single}\\\text{traces}}} f(\lambda)^2 \, G_{\Delta(\lambda),J} + \sum_{\substack{\text{double}\\\text{traces}}} \delta(f^2 \, G)$.

- Single-traces = massive string modes: $\Delta(\lambda) = O(\lambda^{1/4})$ for $\lambda \to \infty$.

- Dimensions $\Delta(\lambda)$ in principle available from integrability (quantum spectral curve).

- OPE coefficients $f(\lambda)^2$ currently not known at finite λ (c.f. the hexagon approach).

Stringy Corrections from Dispersive Sum Rules

[Alday, Hansen, Silva '22]

- Consider the Mellin amplitude $M_\lambda(s,t)$ of the planar four-point function $\mathcal{G}_\lambda^{(0)}(u,v)$.
- Poles at single-trace dimensions: $M_\lambda(s,t) \sim f(\lambda)^2 \dfrac{\mathcal{Q}(t)}{s-\Delta(\lambda)+J-2m}$.
- $\lambda \gg 1$: $\quad M_\lambda(s,t) = \dfrac{8}{(s-2)(t-2)(u-2)} + \sum_{a,b=0}^{\infty} \dfrac{(s^2+t^2+u^2)^a (stu)^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \left[\dfrac{\alpha_{a,b}}{\lambda^0} + \dfrac{\beta_{a,b}}{\lambda^{\frac{1}{2}}} + O(\lambda^{-1})\right]$
- The flat-space limit $s,t,\lambda \to \infty$ fixes $\alpha_{a,b}$. For example $\alpha_{a,0} = \dfrac{8^a}{\Gamma(2a+6)}\zeta(2a+3)$.
- Dispersion relations express $\alpha_{a,b}$, $\beta_{a,b}$ as sums over the stringy modes.
- Dispersion relations + integrability + flat-space limit + localization:

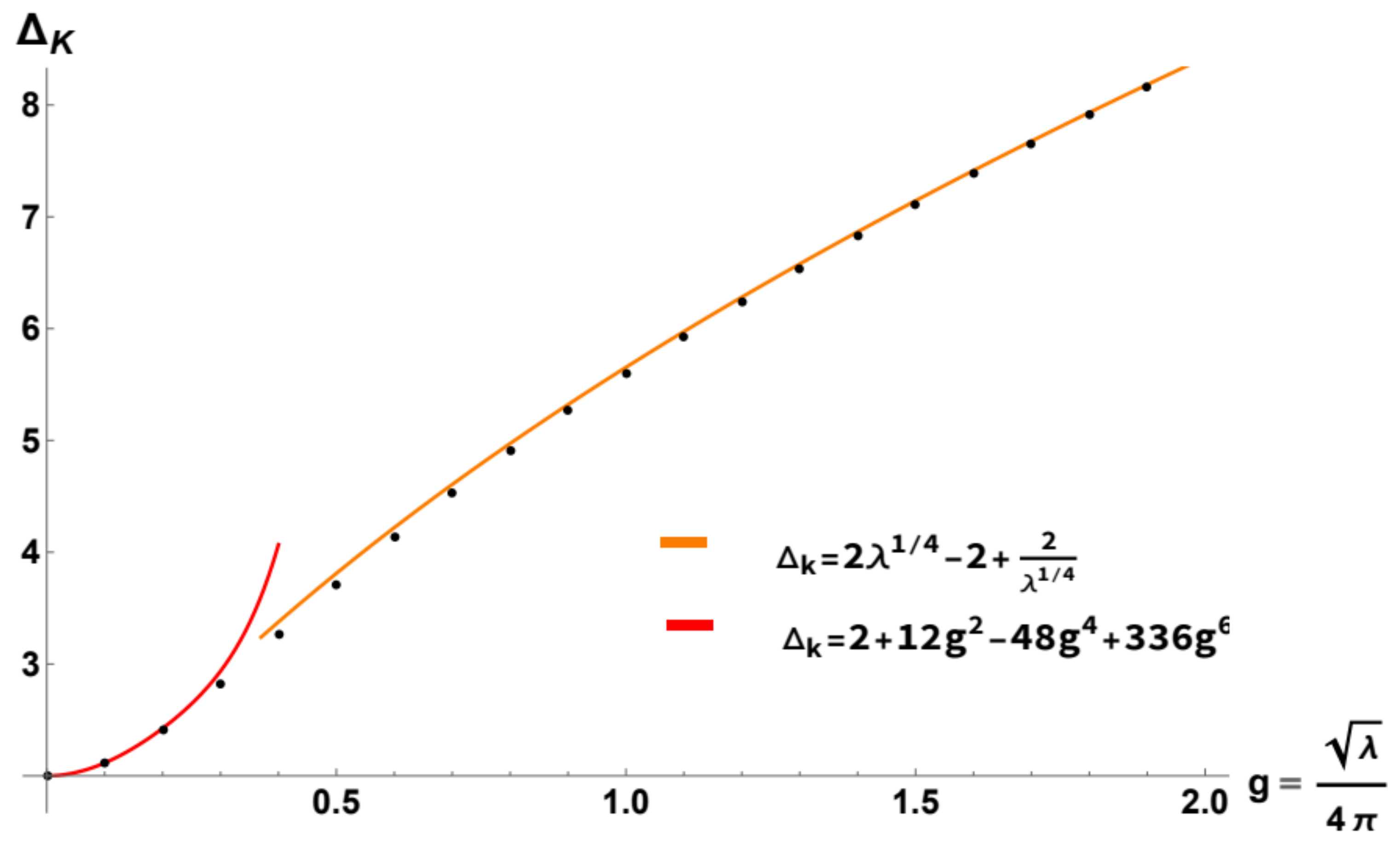
$$\beta_{a,0} = \frac{8^a}{\Gamma(2a+6)}\left[2(a^2+5a-4)\zeta(2a+3,1) - \frac{1}{3}(4a^3+12a^2-a-6)\zeta(2a+4)\right]$$

Bounds on the Konishi OPE Coefficient

[Caron-Huot, Coronado, Trinh, Zahraee ’22]

$$\mathcal{G}^{(0)}_{\lambda} = \mathcal{G}_{\text{protected}} + \sum_{\substack{\text{single}\\\text{traces}}} f(\lambda)^2\, G_{\Delta(\lambda),J} + \sum_{\substack{\text{double}\\\text{traces}}} \delta(f^2\, G)$$

- Konishi = scalar single-trace operator on the leading Regge trajectory.



[Gromov, Kazakov, Vieira ’09]

Idea: Prove an upper bound on $(f_{\text{Konishi}}(\lambda))^2$ at finite λ from unitarity and crossing of $\mathcal{G}^{(0)}_{\lambda}(u, v)$.

c.f. [Cavaglia, Gromov, Julius, Preti ’21]

Problem: The double-trace contributions in the OPE violate positivity.

Solution: Use dispersive sum rules which completely remove all double traces!

- Possible thanks to better-than-usual Regge behaviour due to $\mathcal{N} = 4$ supersymmetry.

Results

[Caron-Huot, Coronado, Trinh, Zahraee ’22]

- Infinite family of dispersive sum rules ω_i, $i = 1,2,\ldots$

$$\omega_i^{\text{protected}} + \sum_{\substack{\text{single}\\\text{traces}}} f(\lambda)^2\, \omega_i[G_{\Delta(\lambda),J}] = 0$$

- Numerical bootstrap together with spectral input from integrability gives:

- Weak coupling $\frac{\sqrt{\lambda}}{4\pi} = 0.1$: $(f_{\text{Konishi}}(\lambda))^2 \leq 0.3015$. Perturbation theory: $(f_{\text{Konishi}}(\lambda))^2 \approx 0.30067$

- Strong-ish coupling $\frac{\sqrt{\lambda}}{4\pi} = 0.3$: $(f_{\text{Konishi}}(\lambda))^2 \leq 0.299$. Other methods: $(f_{\text{Konishi}}(\lambda))^2 \in [0.24, 0.33]$.

Question: Is the bound exactly saturated by planar $\mathcal{N} = 4$ SYM for any λ?

Conclusions

- Four-point functions in conformal field theories admit a dispersive representation in terms of the double commutator.
- When we use the OPE inside the dispersion relation, we get a large class of constraints on the CFT data = dispersive sum rules.
- In holographic theories, dispersive sum rules suppress double-trace operators and express higher-derivative couplings as positive sums over single-trace operators.
- This can be used to systematically implement positivity bounds on AdS EFTs.
- It teaches us new facts about the AdS Virasoro-Shapiro amplitude.

Additional Topics

- Using dispersion relations to implement peturbation theory [Bissi, Dey, Hansen ’19]
  c.f. Agnese’s talk [Carmi, Penedones, Silva, Zhiboedov ’20]
- s-t-u-symmetric dispersion relations [Gopakumar, Sinha, Zahed ’21]
- Dispersion relations in momentum space [Meltzer ’21]
- Multiple correlators [Trinh ’21]
- boundaries, defects, 1D CFTs [Barrat, Gimenez-Grau, Liendo ’22] [Bianchi, Bonomi ’22] [Cordova, He, Paulos ’22] [Knop, DM ’22]

Future Directions

- Dispersion relations for higher-point functions ⇒ study multi-trace operators.

- Spinning external operators ⇒ study graviton scattering in AdS.

- Study CFT dispersion relations in the flat-space limit ⇒ analyticity properties of S-matrices
  see Lucía's talk

- Generalize dispersive sum rules to exhibit double zeros at interacting double traces ⇒ bootstrap bounds saturated by interacting theories.

Thank you!

Application 3: Saturation of Bootstrap Bounds

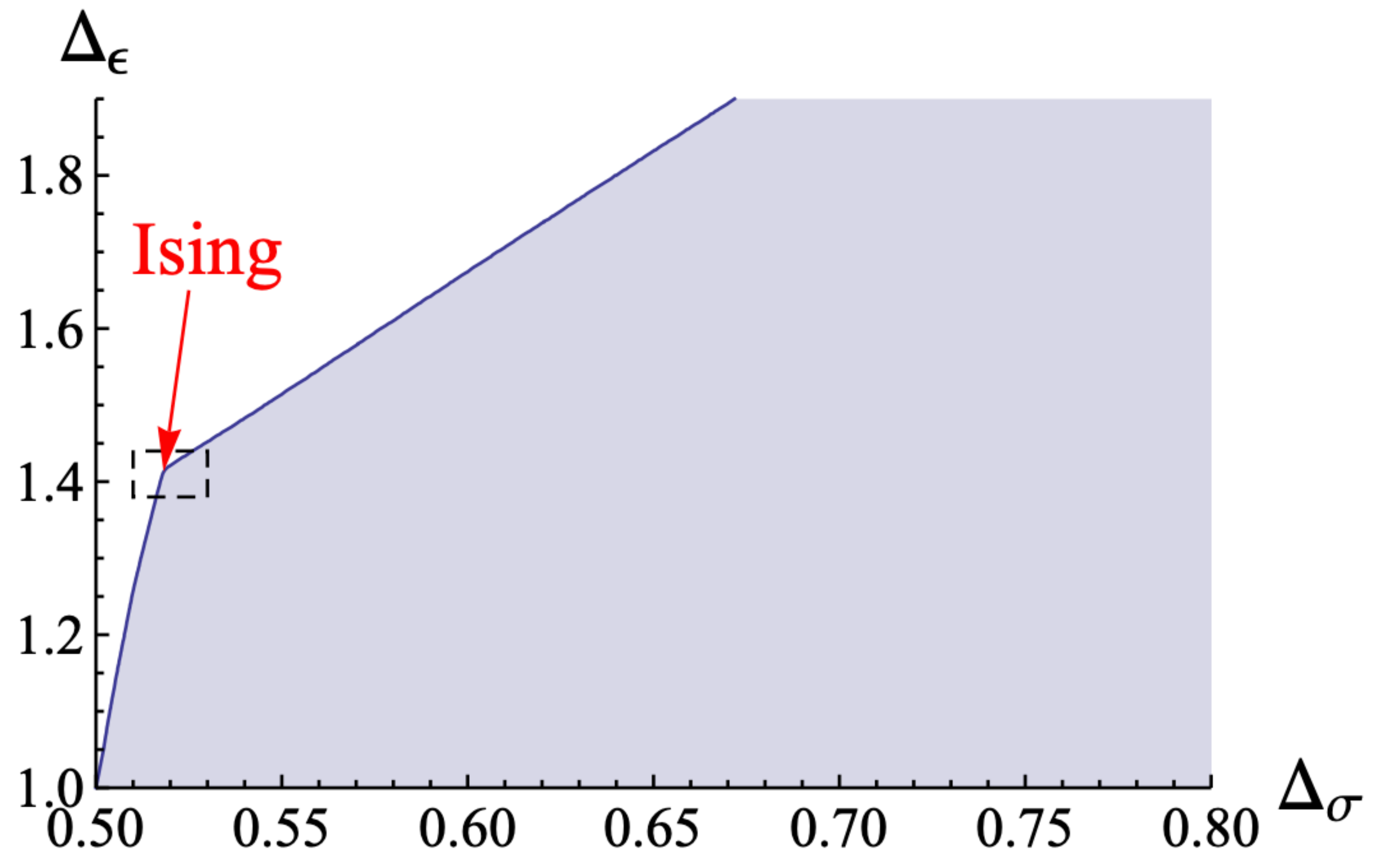
Conformal Bootstrap Bounds

- Suppose we are given a spectrum (Δ_1, J_1), (Δ_2, J_2), ... of primary operators.
- Suppose we are given a functional ω, giving rise to the sum rule $\sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2\, \omega[G_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}] = 0$.
- If $\omega[G_{\Delta_i,J_i}] \geq 0$ for all (Δ_i, J_i), then such spectrum can not occur in a unitary CFT.

Example: To bound the scalar gap, we seek functionals such that

1. $\omega[G_{\text{id}}] = 1$
2. $\omega[G_{\Delta,0}] \geq 0 \quad \forall \Delta \geq \Delta_*$
3. $\omega[G_{\Delta,J}] \geq 0 \quad \forall J > 0,\ \Delta \geq J + d - 2$

Important task: Prove the optimal saturation *analytically.*



[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi '12]

Bounding the Lightest Spinning Operator

Question: What is the optimal upper bound on the smallest Δ of a fixed spin $J = 2, 4, \ldots,$ appearing in the $\phi \times \phi$ OPE?

Conjecture: The bound is saturated by mean field theory, i.e. $\Delta \leq 2\Delta_\phi + J$ (for any spacetime dimension $d \geq 2$, Δ_ϕ and $J > 0$).

- Spectrum of mean field theory: double-trace operators $\phi \Box^n \partial^J \phi, \quad \Delta = 2\Delta_\phi + 2n + J$.
- To prove the conjecture, we need a functional ω such that $\omega[G_{\Delta',J'}] \geq 0$ unless $J' = J$ and $\Delta' \leq 2\Delta_\phi + J$.
- If it exists, such functional must have double zeros on all double-trace dimensions $\Delta' = 2\Delta_\phi + 2n + J'$ except for $J' = J$ and $n = 0$.

  $\Rightarrow$ ω is a dispersive functional!

The Extremal Functional

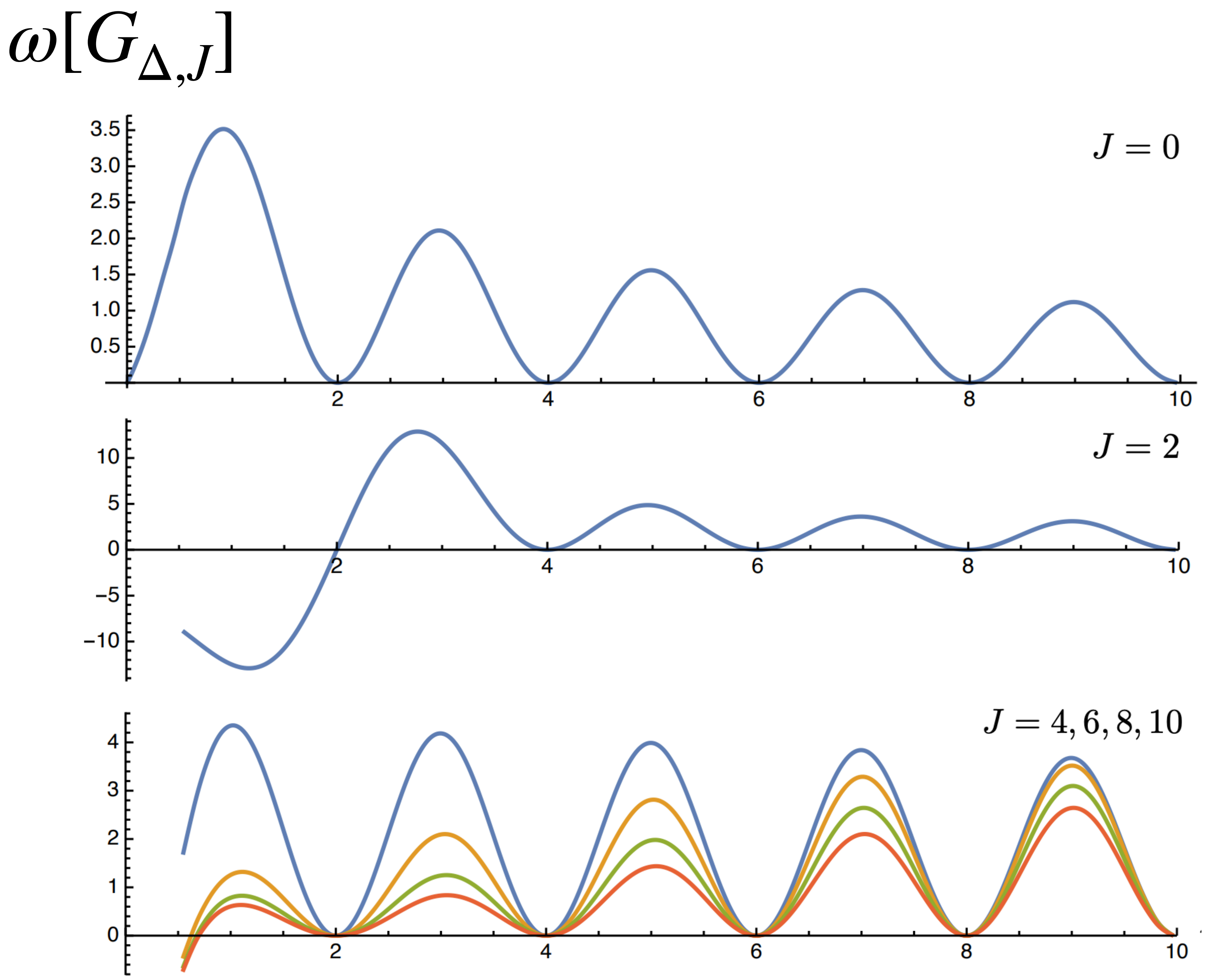
[Caron-Huot, DM, Rastelli, Simmons-Duffin '20]

- The dispersive functional $B_{2,v}$ almost does the job. Only need to ensure the correct structure of zeros on the leading double-traces.

- $\omega = \int_0^\infty dv\, h(v)\, B_{2,v}$, where $h(v)$ is uniquely fixed.

    - Positivity not guaranteed, but holds for sufficiently small Δ_ϕ. It does not hold in general!

gap maximization at spin=2



Open problem: Construct positive dispersive functionals which prove saturation by mean field theory for general d, Δ_ϕ, J.