# Internal boundaries of the amplituhedron

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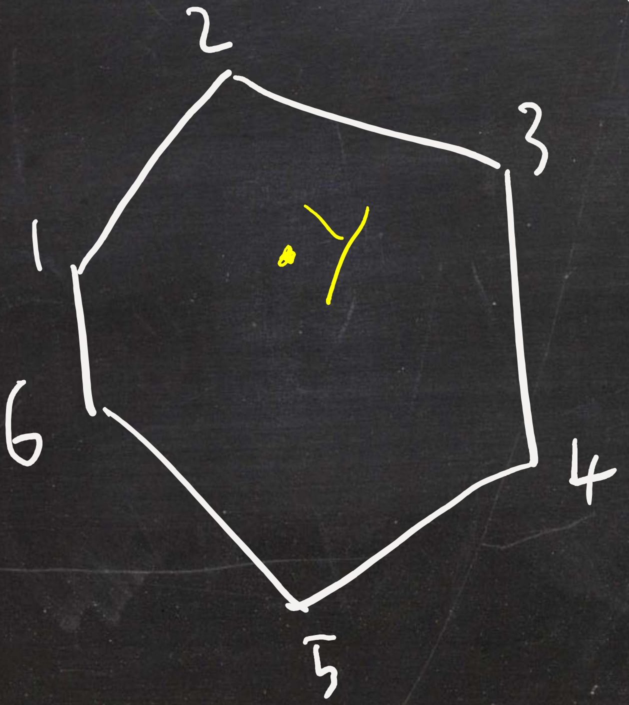
based on 2207.12464 with Gabriele Dian, Alastair Stewart

# Outline

- Intro to Amplituhedron, positive geometry, canonical form
- Loop amplituhedron needs generalisation of positive geometry!
- Reason? Internal boundaries
- Suggests weighted positive geometries (WPGs)
- Deepest cuts

# The amplituhedron

Toy model: polygons in $\mathbb{P}^2$, $Y, Z_i \in \mathbb{P}^2$



Y in polygon

$$\Downarrow$$

$$\boxed{\langle Y\, i\; i{+}1 \rangle > 0}$$

$\langle Y Z_i Z_{i+1} \rangle$

Y to the right of line $(i\; i{+}1)$

note

$$\langle Y\, 1\, i \rangle = (+,\dots,+,-,\dots,-)$$

$i=2,3,4,5$

Flips once from + to -

Flipping Number 1

# Natural Generalization [Arkani-Hamed, Trnka + Thomas]

Toy model
Polygons in $P^2$

"Tree Amplituhedron"

$A_{n,k,m}$

(physics m=4)

$Y \in P^2 = Gr(1,3) \longrightarrow Gr(k, k+m)$ k-planes in $\mathbb{R}^{k+m}$

$Z_i \in P^2 \longrightarrow P^{m+k-1}$ lines in $\mathbb{R}^{k+m}$

$\langle Y\ i\ i{+}1\rangle > 0 \longrightarrow \langle Y\ i\ i{+}1\ j\ j{+}1\rangle > 0$ (m=4)

i=1..n

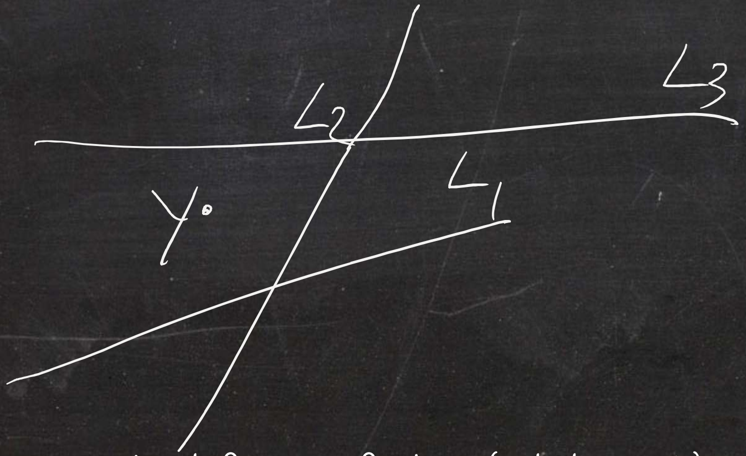
$\langle Y\ 1\ i\rangle$ has One sign flip $\longrightarrow$ $\langle Y\ 1\ 2\ 3\ i\rangle$ has k sign flips (max possible)

( Polygons in $P^2$ = $A_{n,1,2}$ amplituhedron)

# Loops too!! (Integrand)

L-Loop amplituhedron $\mathcal{A}_{n,k,l,m}$ = amplituhedron Y together with L lines $L_i$, $i=1,\dots,L$ $\in Gr(2, m+k)$

in $P^{m+k-1}$



$\langle Y L_i\, j\, j{+}1\rangle > 0$

$\langle Y L_i L_j\rangle > 0$

loop flipping number.

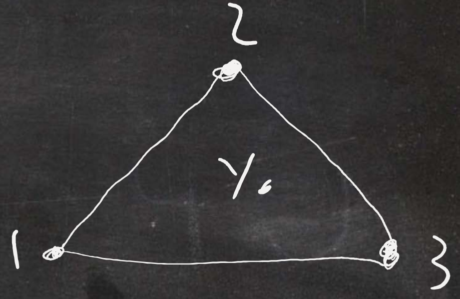
Claim: canonical form of the (n.k.l,m=4) amplituhedron = planar, l-loop, $N^k$MHV, n-point amplitude (integrand) [Arkani-Hamed,Trnka]

# Canonical Form

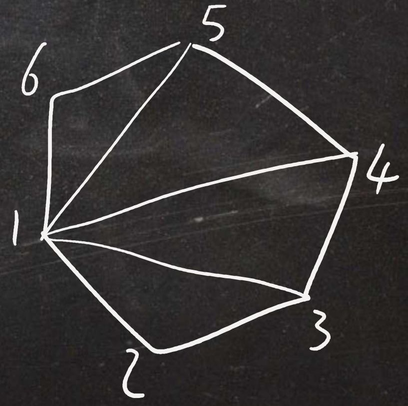
- Natural map from geometry to differential form: "canonical form"

- Eg Triangle: $A_{3,1,2}$

$$\Omega[\triangle] = [1,2,3] := \boxed{\frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}}$$



- Polygon: $A_{n,1,2}$ $\Omega[\bigcirc] =$ sum of triangles $= \sum_{i=2}^{n-1} [1, i, i+1]$



Positive geometry = region possessing a canonical form

# Amplitude = Canonical Form of Amplituhedron

- Eg 2: $A_{5,1,4}$

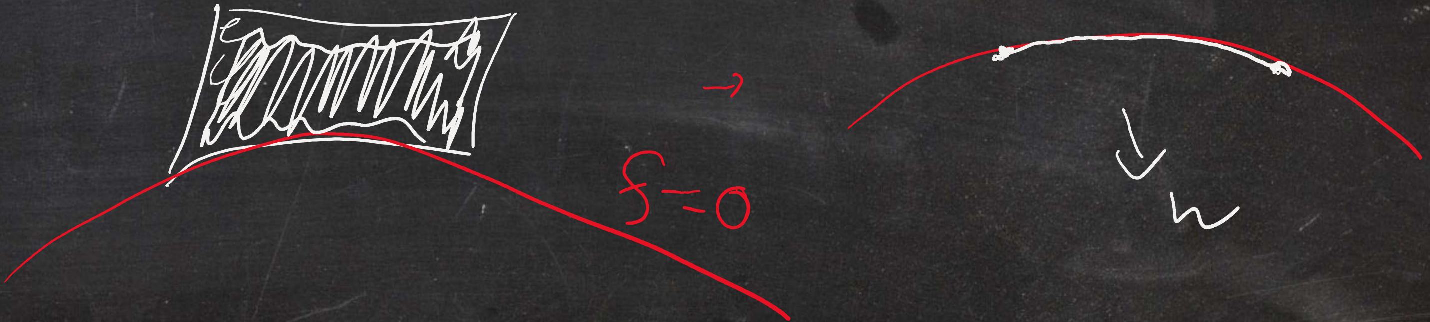
$$[12345] = \boxed{\frac{\langle Y d^4 Y\rangle \langle 12345\rangle^2}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}}$$

- $A_{n,1,4} = \sum_{i,j} [1, i, i+1, j, j+1]$

- This is precisely the n-point NMHV tree-level superamplitude in N=4 SYM !!
- Claim: The canonical form of the (n,k,4) amplituhedron is the n-point, $N^k$MHV, tree-level, planar scattering amplitude in N=4 SYM [Arkani-Hamed,Trnka]
- Similar geometry with fewer sign flips -> products of amplitudes
- Similar geometry without sign flip constraint -> square of superamplitude [Dian,PH]

# Canonical form

- Canonical Form(and hence positive geometry) defined recursively via it's residues [Arkani-Hamed, Bai, Lam]
- Canonical form = volume form; log divergences on the boundary
- residue on boundary= canonical form of boundary



boundary on f=0

⇓

Residue at f=0

$$\mathrm{Res}_{f=0}\,\Omega = \lim_{f\to 0} f\Omega = df \wedge \omega$$

canonical form of geometry on f=0

(True for all boundary components)

- Eventually reach dimension 0 boundaries
- $\Omega(\cdot) = \pm 1$ (depending on orientation)

$\Rightarrow$ Maximal residues of positive geometries = +/-1

**Note:**

Multiple residue = Residues of residue of residue ...

↓

Boundary components of boundary components of boundary components....

# (Small Aside) Multiple residue not unique
# Higher codimension boundaries not unique

- In general multiple residues depend on the order you take single residues
- Analogous statement true for boundaries
- Everything perfectly consistent!

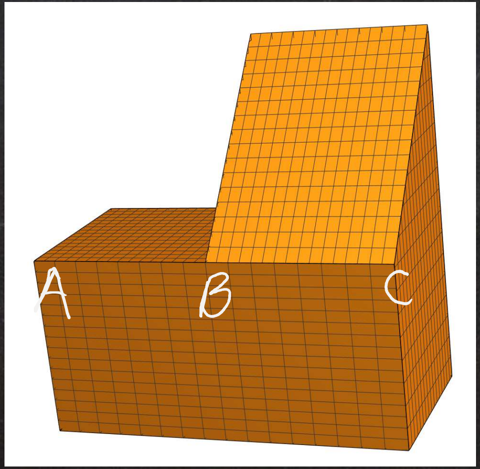
(Boundary component of)^k geometry

∦

Codimension k boundary component of geometry

(not clear what interpretation of this is for k > 1)

Simple example

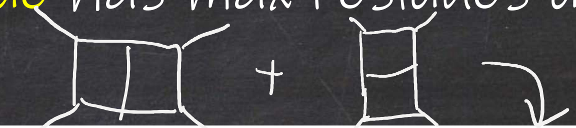


Codimension 2 boundary = [A,C]
Boundary of sloping roof boundary = [B,C]
Boundary of flat roof boundary = [A,B]

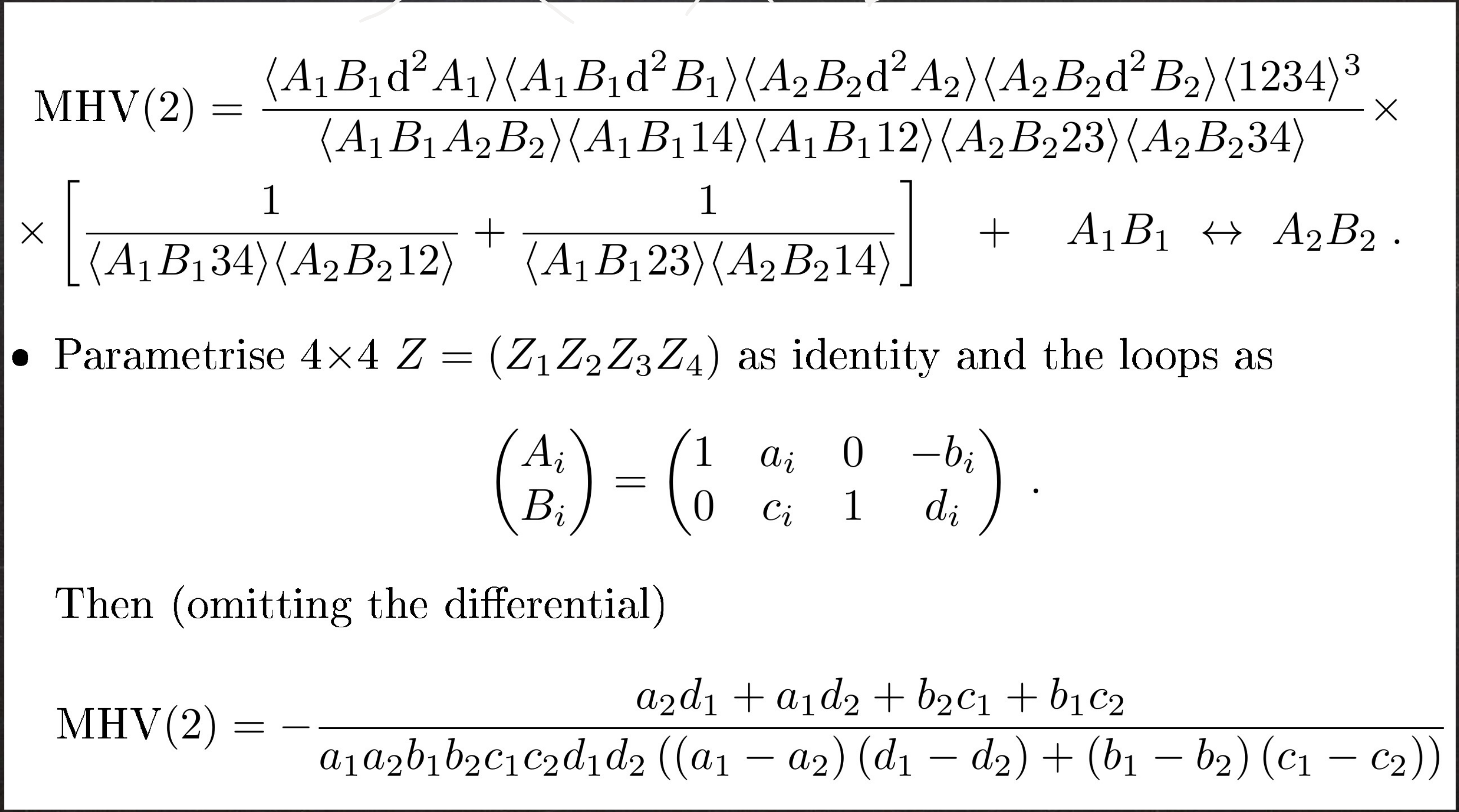
# Small Problem

Loop amplitude has max residues different from +/- 1 !!!

2 loop MHV:



eg.



$$\mathrm{MHV}(2) = \frac{\langle A_1B_1\mathrm{d}^2A_1\rangle\langle A_1B_1\mathrm{d}^2B_1\rangle\langle A_2B_2\mathrm{d}^2A_2\rangle\langle A_2B_2\mathrm{d}^2B_2\rangle\langle 1234\rangle^3}{\langle A_1B_1A_2B_2\rangle\langle A_1B_1 14\rangle\langle A_1B_1 12\rangle\langle A_2B_2 23\rangle\langle A_2B_2 34\rangle}\times$$

$$\times\left[\frac{1}{\langle A_1B_1 34\rangle\langle A_2B_2 12\rangle}+\frac{1}{\langle A_1B_1 23\rangle\langle A_2B_2 14\rangle}\right] \quad + \quad A_1B_1 \leftrightarrow A_2B_2\ .$$

- Parametrise $4\times 4$ $Z=(Z_1Z_2Z_3Z_4)$ as identity and the loops as

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 & a_i & 0 & -b_i \\ 0 & c_i & 1 & d_i \end{pmatrix}\ .$$

Then (omitting the differential)

$$\mathrm{MHV}(2) = -\frac{a_2d_1+a_1d_2+b_2c_1+b_1c_2}{a_1a_2b_1b_2c_1c_2d_1d_2\left((a_1-a_2)(d_1-d_2)+(b_1-b_2)(c_1-c_2)\right)}$$

- Now we take the residues in $b_1 = 0, c_1 = 0, b_2 = 0, c_2 = 0$
- complicated pole factorises revealing new pole

$$-\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 (a_1 - a_2)(d_1 - d_2)} .$$

- Now take the residue in $a_1$ at $a_1 = a_2$

$$-\frac{(d_1 + d_2)}{a_2 d_1 d_2 (d_1 - d_2)} .$$

- Now take residue in $d_1$ at $d_1 = d_2$,

max res=2

$$-\frac{\bigcirc\!\!\!\!\!\!2\;\;}{a_2 d_2} ,$$

# Loop amplituhedron $\neq$ positive geometry!!??

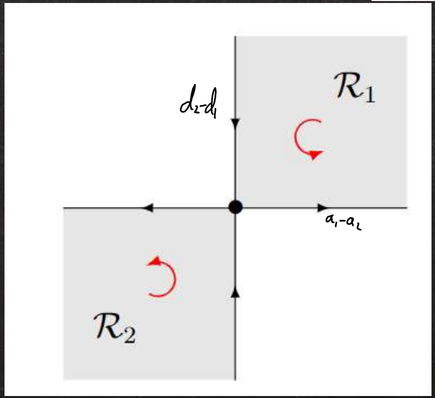
- Examine the above residues geometrically .
- Start with amplituhedron. Carefully take boundaries corresponding to each of the above residues:

$\downarrow$

$$a_i > 0, \quad d_i > 0, \quad -(a_1 - a_2)(d_1 - d_2) > 0 \rightarrow \Omega = -\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 \left(a_1 - a_2\right)\left(d_1 - d_2\right)}$$

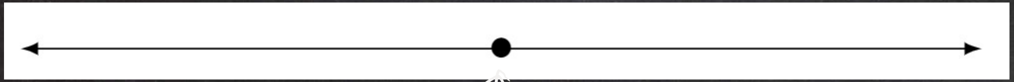
$||$

$$\mathcal{R}_1 := \{a_1, a_2, d_1, d_2 \mid a_1 > a_2 > 0 \wedge d_2 > d_1 > 0\}$$
$$\mathcal{R}_2 := \{a_1, a_2, d_1, d_2 \mid a_2 > a_1 > 0 \wedge d_1 > d_2 > 0\}$$



$R_1$, $R_2$ same orientation

$d_2-d_1=0$ boundary.



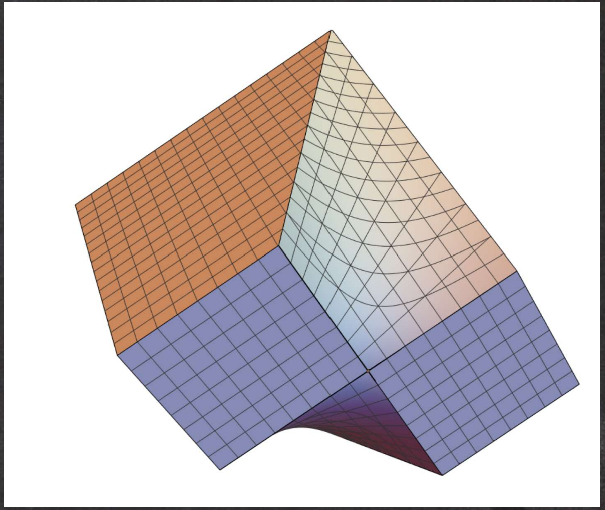
"internal boundary" separating two regions of opposite orientation (so not oriented!)

Previously unnoticed feature:

The (loop) amplituhedron contains internal boundaries!

# Simple toy example:

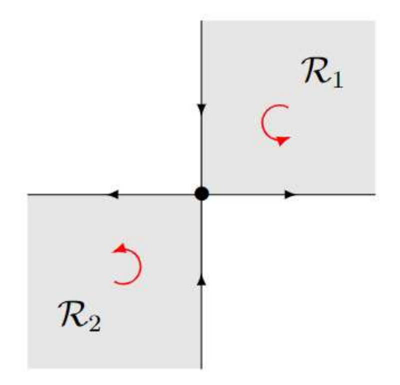
$xy + z > 0, \quad z > 0$



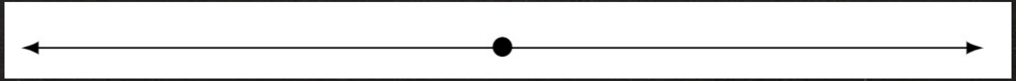
$\Omega = 2/(z(z+xy))$

Composite singularity [Arkani-Hamed, Cachazo, Cheung, and Kaplan]

Z=0 boundary:



y=0 boundary:



# Generalised positive geometry

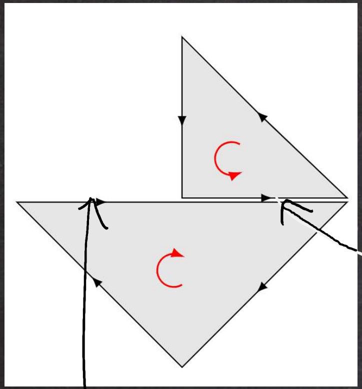
Generalized canonical form recursive def:

$$\mathrm{Res}_f \Omega = \lim_{f\to 0} f\Omega \;=\; df \wedge (\omega_{\mathrm{ext}} + 2\omega_{\mathrm{int}})$$

canonical form of
standard (external)
boundary region

canonical form of
internal boundary region

eg.



Internal boundary

External boundary

$$\Omega(R_1) = \frac{dx\,dy}{xy(x+y-1)} + \frac{2dx\,dy}{y(x+y+1)(x-y-1)}$$

(just subtract the two triangles)

$$\operatorname*{Res}_{y=0} \Omega = \lim_{y\to 0} y\,\Omega = dx\left(\frac{1}{x}-\frac{1}{x+1}\right)+2dx\left(\frac{1}{x-1}-\frac{1}{x}\right) = \Omega([-1,0]) + \boxed{2}\Omega([0,1])$$

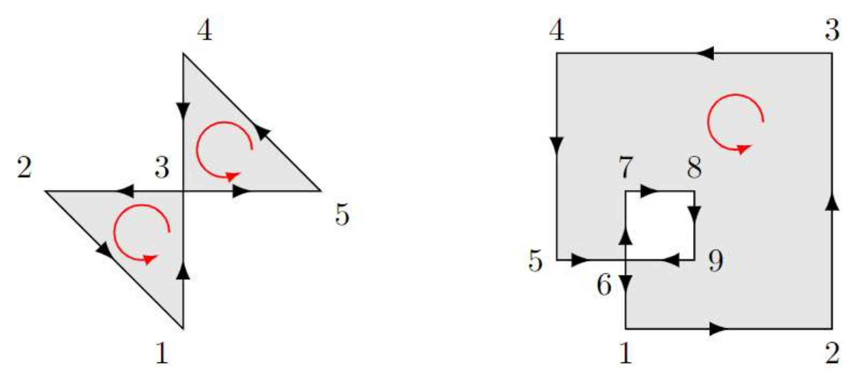
- Agrees with the formula

External boundary

Internal boundary

# GPGs more complete than PGs: Anything that triangulates in GPGs is a GPG

- not true for positive geometries eg.



# But suggests further generalisation:

## Weighted Positive Geometry (WPG)

First recall:

Orientation = volume form $O \sim \lambda O, \lambda > 0$

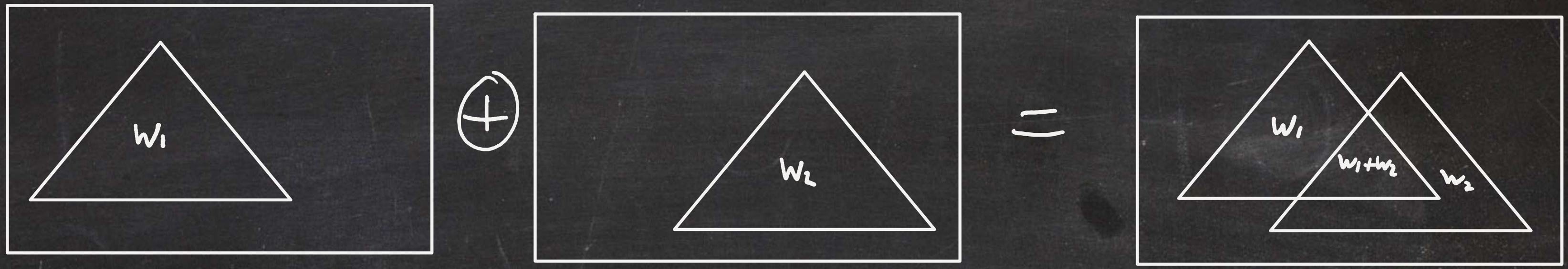
Generalise: "Weighted orientation"

$$(\underset{\text{Integer valued}}{w}, \underset{\text{Volume form}}{O}) \sim (\mathrm{sign}(\lambda) w, \lambda O') \qquad \lambda \neq 0$$

- Define geometry by a piecewise constant $\mathbb{Z}$-valued weight function w (and orientation form O)

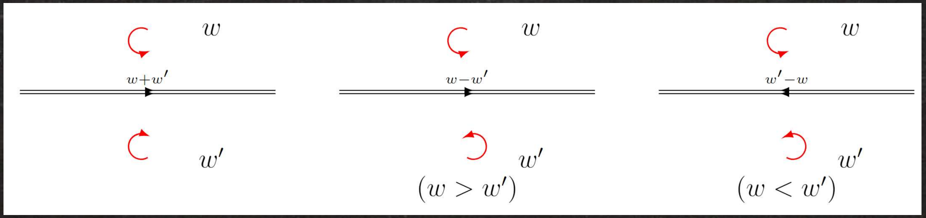
# Weighted Positive Geometry (WPG):

- Natural additive structure (geometries can overlap - makes proofs easier!)
- $(w_1, O_1) \oplus (w_2, O_2) = (w_1 + \mathrm{sign}(\lambda) w_2, O_1)$
- where $O_1 = \lambda O_2$



- Natural Projection operator onto boundaries (discontinuities):

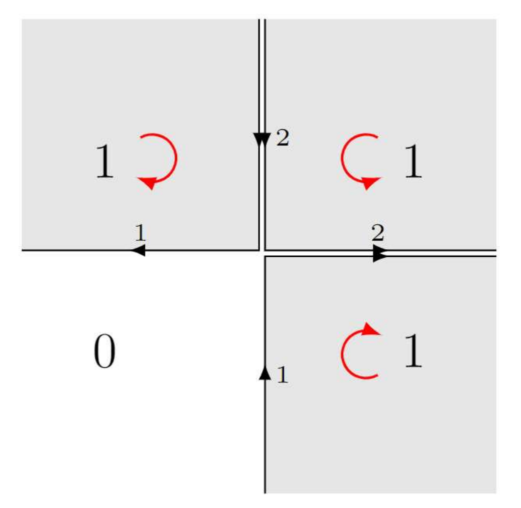
$$\Pi_{\mathcal{C}}(w, O) = (w^+|_{\mathcal{C}}, O^+|_{\mathcal{C}}) \oplus (w^-|_{\mathcal{C}}, O^-|_{\mathcal{C}})$$



# Residue of canonical form is canonical form of the projection:

$$\mathrm{Res}_{\mathcal{C}}\Omega(w,O) = \Omega(\Pi_{\mathcal{C}}(w,O))$$

eg



$$\mathrm{Res}_{y=0,x=0}\Omega = -\mathrm{Res}_{x=0,y=0}\Omega = 3$$

# WPGs -> GPGs and PGs

- GPGs are WPGs with w = 1 (in the GPG), 0 everywhere else
- Positive Geometries (PGs) are WPGs with w = ±1, 0 everywhere (so a GPG) AND induced weight on all nested boundary components is ±1, 0.
  - WPGs are a concrete realisation of the more abstract Grothendieck group of pseudo-positive geometries (formal sum) introduced in [Arkani-Hamed, Bai, Lam]

# Characterising Generalised Positive Geometries?

- any region defined by multi-linear inequalities is a GPG
- canonical form given uniquely via cylindrical decomposition: (algorithm one uses to convert integration over region into multiple integral)
- $$\mathcal{R}_i := \{x_1, \cdots, x_d\} \quad \text{st} \quad \left\{ \begin{array}{c} a_1 < x_1 < b_1 \\ a_2(x_1) < x_2 < b_2(x_1) \\ \cdots \\ a_d(x_1, \cdots, x_{d-1}) < x_d < b_d(x_1, \cdots, x_{d-1}) \end{array} \right.$$

changing variables to:

$$x'_j = -\frac{x_j - a_j}{x_j - b_j},$$

$$\mathcal{R}_i := \{x'_1, \cdots, x'_d\} \quad \text{st} \quad x'_j > 0 \quad \text{for all } j$$

$$\Omega(\mathcal{R}_i) = \prod_{j=1}^{d} \frac{dx'_j}{x'_j}$$

$$= \prod_{j=1}^{d} \left( \frac{1}{x_j - a_j} - \frac{1}{x_j - b_j} \right) dx_j$$

Push forward

Only if change of variables rational
Multilinear inequalities ensures this

# Characterising Generalised Positive Geometries?

- Multi-linear regions = very wide class
- Includes ALL amplituhedron geometries = multi-linear (inequalities defined by determinants which are multi-linear functions of the matrix components)
- But non exhaustive: there are GPGs which are not multi-linear regions
- (eg
  hyperbolic: $x y + 1 > 0$, $x > 0$ is multi-linear but...
  region involving a circle $x^2 + y^2 - 1 < 0$, $y > 0$ is not)
- Most general characterisation? Boundaries rational varieties?

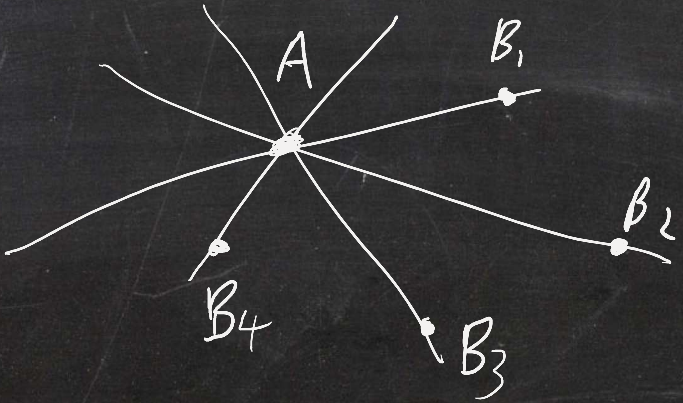
# Deepest cuts

- All loop proposal for a multiple (2L-3)-dim residue of loop amplitudes
  [Arkani-Hamed, Langer, Srikant, Trnka]

- Geometrical configuration: All loop lines intersect in a common point (or all in a plane)

$$A_i = A$$

$$L_i = (A_i B_i)$$



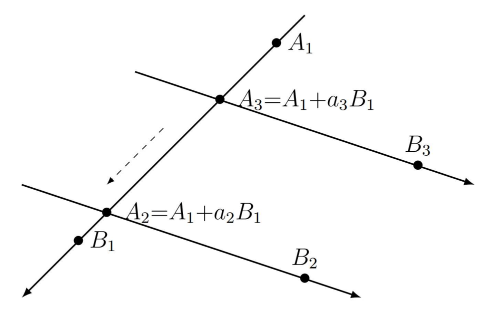
- Canonical form of this configuration can be written down at arbitrary loop order

# But how to get this result algebraically?

- We find that sequences of single residues don't give this result!
- Eg 3 loop:

$$\frac{\langle A \mathrm{d}^3 A\rangle \prod_{i=1}^{3} \langle AB_i \mathrm{d}^2 B_i\rangle}{\langle AB_1 B_2 B_3\rangle} \times$$
$$\times \left( \frac{\langle 1234\rangle^3 \langle AB_1 23\rangle}{\langle AB_1 14\rangle \langle AB_1 12\rangle \langle AB_1 34\rangle \langle AB_2 12\rangle \langle AB_2 23\rangle \langle AB_3 23\rangle \langle AB_3 34\rangle} \quad + \quad \text{symmetry} \right)$$

- "Intersect and slide"

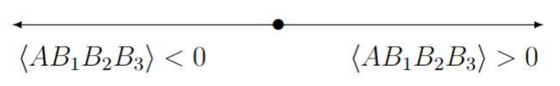


- A new pole appears at:

$$\langle AB_1 B_2 B_3\rangle = 0$$

- Geometrically: two regions with opposite orientation separated by an internal boundary

$$
\begin{aligned}
&\mathcal{R}^{\mathrm{dc}} = \mathcal{R}_1^{\mathrm{dc}} \cup \mathcal{R}_2^{\mathrm{dc}} \\
&\mathcal{R}_1^{\mathrm{dc}} = \mathcal{A}_{dc}^{(3)} \cap \{\langle AB_1B_2B_3\rangle > 0\} \qquad \text{positive orientation} \\
&\mathcal{R}_2^{\mathrm{dc}} = \mathcal{A}_{dc}^{(3)} \cap \left\{\langle AB_1B_2B_3\rangle < 0\right\} \qquad \text{negative orientation}
\end{aligned}
$$



# Higher loops?

- Above 3 loop result unique (different orderings of intersect / slide give same result up to sign)

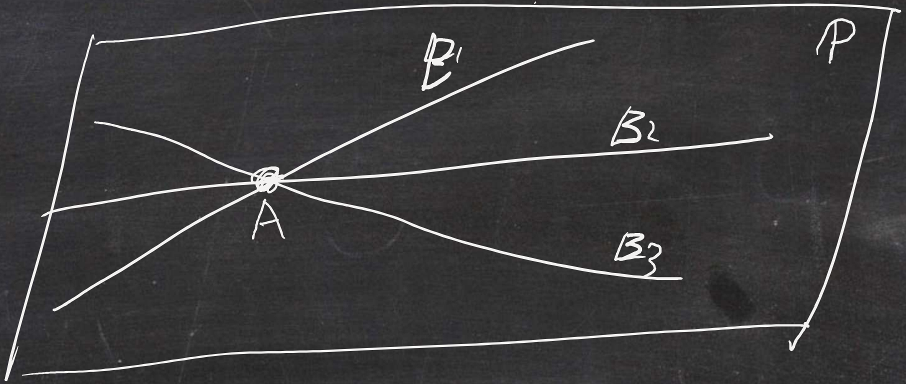
- From 4 loops: different orderings of intersect / slide give different results

- All include further <ABBB> type poles / internal boundaries

- One *can* analyse the boundary of boundary .... Corresponding to multiple residues to very high loop order
- No explicit all loop formula but very quick predictive algorithm!
- Impossible without amplituhedron
- Huge amounts of info about the amplitude
- Determine higher 4-point amplitude/correlator to even higher (that ten-) loop order?
- Construct the relevant f-graphs (rather than using a basis)

# Maximal loop-loop residue

All in one point AND all in one plane AND only three loop lines remaining



CLAIM: ANY way you reach this configuration gives the same answer (up to a numerical factor = number of internal boundaries crossed)

# Final comments:

- Internal boundaries first found in squared amplituhedron [Dian,PH]
- Amplituhedron-like (non max winding number) = amplitude x amplitude
- Sum of amplituhedron-like = squared amplituhedron = limit of correlahedron [Eden, Mason, PH]
- Squared amplitude contains non-unit max residues (but more trivially -- almost disconnected sum of positive geometries)
- This talk: So does loop amplitude, not a positive geometry!
- Loop amplituhedron = GPG or WPG

# Future:

- Look back at amplituhedron boundary structure (+ relation to symbol etc.):
  Eg [Dennen, Prlina, Spradlin, Stankowicz, Stanojevic, Volovich]

- Use cuts via amplituhedron to determine amplitude / correlator at higher loops (constructive approach?)

- Applications of weighted positive geometry? - Cosmological polytope [Arkani-Hamed, Benincasa,Postnikov], negative geometries [Arkani-Hamed, Henn, Trnka] , non-planar amplitude [Arkani-Hamed, Bourjaily, Cachazo, Postnikov, and Trnka], momentum amplituhedron [Damgaard, L. Ferro, T. Lukowski, and R. Moerman] etc.?

# Amplituhedron, Amplituhedron-like geometries

Amplituhedron: [Arkani-Hamed, Thomas, Trnka]

$$\mathscr{A}_{n,k} := \left\{ Y \in Gr(k,k+4) \;\middle|\; \begin{array}{ll} \langle Y i\, i+1\, j\, j+1\rangle > 0 & 1 \leq i < j-1 \leq n-2 \\ \langle Y i\, i{+}1\, 1\, n\rangle\, (-1)^k > 0 & 1 \leq i < n-1 \\ \{\langle Y 1 2 3 i\rangle\} & \underline{\text{has } k \text{ sign flips}} \text{ as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_>(k+4,n)$,

(tree level)

Natural generalization

Amplituhedron-like: [ Dian, PH]

$$\mathscr{H}^{(f)}_{n,k} := \left\{ Y \in Gr(k,k+4) \;\middle|\; \begin{array}{ll} \langle Y i\, i+1\, j\, j+1\rangle > 0 & 1 \leq i < j-1 \leq n-2 \\ \langle Y i\, i{+}1\, 1\, n\rangle\, (-1)^{f} > 0 & 1 \leq i < n-1 \\ \{\langle Y 1 2 3 i\rangle\} & \underline{\text{has } f \text{ sign flips}} \text{ as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_+(k+4,n)$

$0 \leq f \leq k$

- We only consider $k=n-4$

$$\mathscr{A}_{n,k} = \mathscr{H}^{(k)}_{n,k}$$

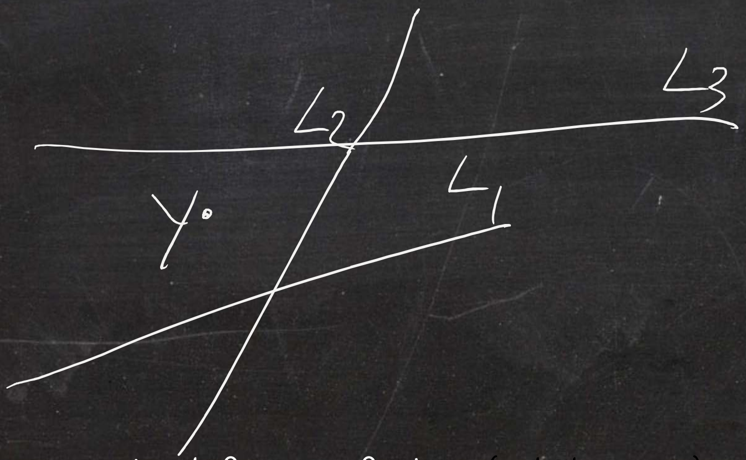
What do these give?

Loop versions also ($k=n-4$)

# Loops too!! (Integrand)

L-Loop amplituhedron $\mathcal{A}_{n,k,l,m}$ = amplituhedron Y together with L lines $L_i$, $i=1,\dots,L$ $\in Gr(2,m+k)$

in $P^{m+k-1}$



$\langle Y L_i\, j\, j{+}1\rangle > 0$

$\langle Y L_i L_j\rangle > 0$

loop flipping number.

Claim: canonical form of the $(n,k,l,m=4)$ amplituhedron = planar, $l$-loop, $N^k MHV$, $n$-point amplitude (integrand) [Arkani-Hamed,Trnka]

Amplituhedron -> amplitudes (N=4 SYM planar, perturbative integrands)

Amplituhedron-like -> products of amplitudes

Main claim:

$$\Omega\left(\mathcal{H}^{(f)}_{n,n-4}\right) = \boxed{H^{(f)}_{n,n-4} = \underbrace{A_{n,f} * A_{n,n-f-4}}_{\text{products of superamplitudes}}\,.}$$

products of superamplitudes

M=4

canonical form

- Loop version too!

# *- product

K1+k2 plane

$$\left(\prod_{a=1}^{m}\langle I_a\rangle_{k_1+m}\right) * \left(\prod_{b=1}^{m}\langle J_b\rangle_{k_2+m}\right) = \frac{(-1)^{(k_1k_2+k_2)m}}{m!}\sum_{\sigma\in S_m}\prod_{a=1}^{m}\langle Y(I_a\cap J_{\sigma(a)})\rangle_{k_1+k_2+m}$$

$$\langle Y(I\cap J)\rangle = \sum_{i\in M(I)}\langle Yi\rangle\,\langle\bar{i}J\rangle\,\mathrm{sgn}(i\bar{i})$$

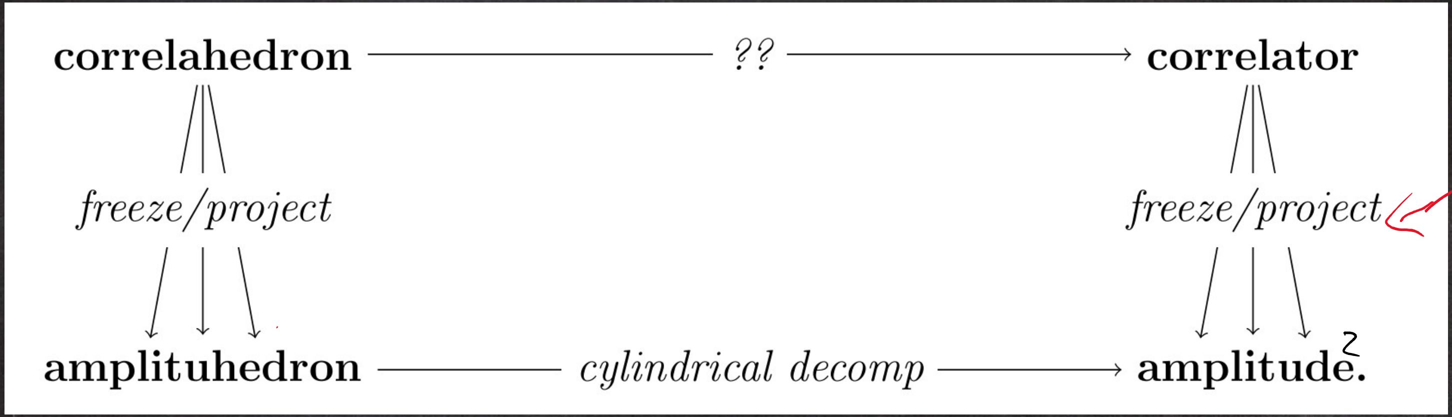
$M(I) = \binom{I}{\underline{m}}$ = set of ordered m elements in I

# Correlahedron [Eden, Mason, PH]

n+k-plane 2-planes

- Geometry $\boxed{Y \in Gr(n+k;n+k+4) \quad \langle Y X_i X_j \rangle > 0 \quad \equiv \quad G_{n;k}}$

Large dimension but simple, no winding number constraint



$$\begin{tikzcd}
\textbf{correlahedron} \arrow[rr, "??"] \arrow[d, shift right=2] \arrow[d, "\textit{freeze/project}"] \arrow[d, shift left=2] & & \textbf{correlator} \arrow[d, shift right=2] \arrow[d, "\textit{freeze/project}"] \arrow[d, shift left=2] \\
\textbf{amplituhedron} \arrow[rr, "\textit{cylindrical decomp}"] & & \textbf{amplitude}^{2}.
\end{tikzcd}$$

squared

light-like limit in amplituhedron space

- Correlahedron gives all half BPS single trace correlators [see SAGEX review!!]
- All correlators = new observation!! Consequence from [Caron-Huot, Coronado]
- previously thought just stress-tensor multiplet
- Equivalent to all IIB gravity amplitudes in AdS

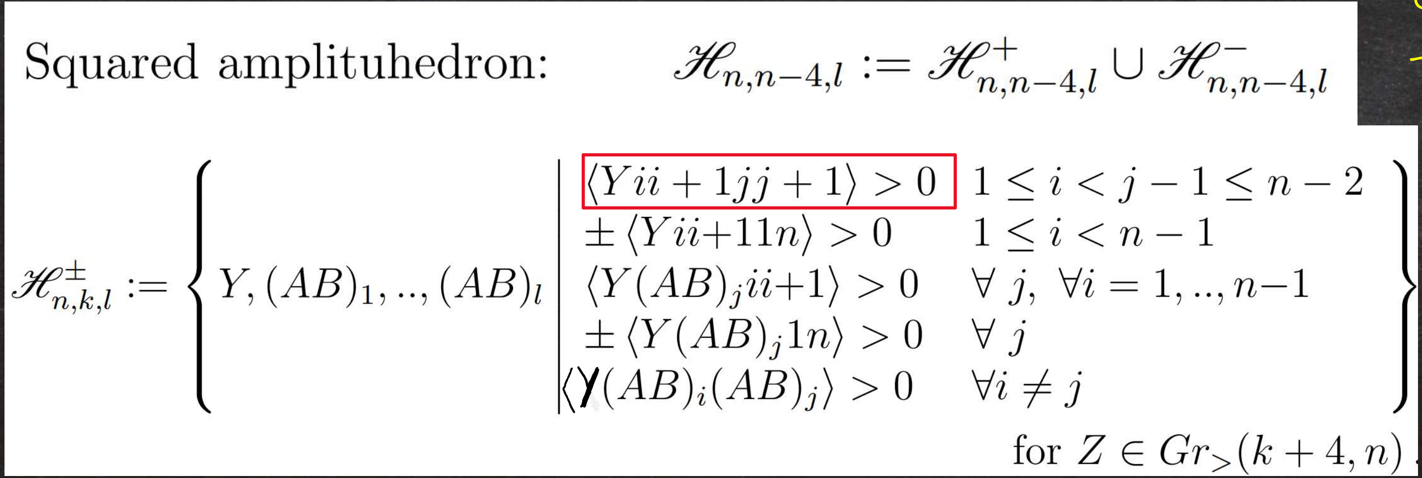
# Proofs (are hard!)

## (amplituhedron-like -> products of amplitudes)

- Alternative definition of amplituhedron-like (analogue of original amplituhedron definition via a positive C matrix Y=C.Z but now C splits into two pieces )
- Partial proof of equivalence of definitions (prove defn1 contains defn2, assuming equivalence of amplituhedron definitions)
- Prove products of onshell diagrams give a tessellation of amplituhedron-like geometries (defn2)
- Tree level only. Similar proof should work for loops. Not done in general but interesting special cases.

# "squared amplituhedron"

- Simpler object (to describe);
- physical inequalities
- no flipping constraint
- ie union over all flipping constraints



Squared amplituhedron: $\mathcal{H}_{n,n-4,l} := \mathcal{H}^{+}_{n,n-4,l} \cup \mathcal{H}^{-}_{n,n-4,l}$

$$\mathcal{H}^{\pm}_{n,k,l} := \left\{ Y, (AB)_1, .., (AB)_l \;\middle|\; \begin{array}{ll} \boxed{\langle Y i i+1 j j+1 \rangle > 0} & 1 \leq i < j-1 \leq n-2 \\ \pm \langle Y i i{+}1 1 n \rangle > 0 & 1 \leq i < n-1 \\ \langle Y (AB)_j i i{+}1 \rangle > 0 & \forall\ j,\ \forall i = 1, .., n{-}1 \\ \pm \langle Y (AB)_j 1 n \rangle > 0 & \forall\ j \\ \langle Y (AB)_i (AB)_j \rangle > 0 & \forall i \neq j \end{array} \right\}$$

for $Z \in Gr_>(k+4, n)$.

Cyclic + twisted cyclic

physical inequalities only

# squared amplituhedron = union of amplituhedron-like

squared amplituhedron :

$$\mathscr{H}_{n,n-4,l} = \bigcup_{f,l'} \mathscr{H}^{(f,l')}_{n,n-4,l}$$

Orientations match precisely!

$$\overset{?}{\Longrightarrow} \quad H_{n,n-4,l} = \sum_{f,l'} H^{(f,l')}_{n,n-4,l} = \sum_{f,l'} A_{n,f,l'} * A_{n,n-4-f,l-l'}$$

$$= \left(A_n^{*2}\right)_{n-4,l}$$

(agreeing with correlahedron)

Squared amplituhedron -> Square of amplitude!

# Problem:

- Canonical form (amplitude from amplituhedron) means max residues $\boxed{= 0,\ +/-1}$
- the maximal residues of the squared amplituhedron are not only +/- 1

e.g. $\boxed{(A^2)_{6,2} = 2A_{6,2} + A_{6,1} * A_{6,1}}$

max residues = $\boxed{0,\ +/-2,\ +/-4}$