The Loop Momentum Amplituhedron

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Work in progress with T. Lukowski
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Introduction and Review

Introduction

Recent advances in formulations
for observables in QFTs:
positive geometries

Geometries with some notion of "positivity"
substituting other concepts such as "locality"

Generalization of convex polytopes

this talk focus on amplitudes in N=4 sYM: amplituhedra

(picture by A. Gilmore)

Introduction

Positive geometries

* Regions with boundaries of all codimension, each bdry again a positive geometry

* Equipped with a unique differential form with logarithmic singularities along all boundaries: the canonical form

> for physically relevant positive geometries
> the canonical form is a physical quantity

Given a set of external kinematic data, there exists a geometrical object defined by imposing particular positivity constraints

From positivity:

* appropriate boundaries = locality
* appropriate factorisation into smaller pieces when approaching one of the boundaries = unitarity

Amplituhedra in N=4 sYM

1. Amplituhedron (N. Arkani-Hamed, J. Trnka)
   * momentum twistor space $(\lambda, x\lambda, \chi)$: planar sector
   * MHV part factorized out: Wilson loops -> $A_{n,k} = A_{n,2}\, W_{n,k}$
   * tree and loop amplitudes

2. Momentum Amplituhedron (D. Damgaard, LF, T. Lukowski, M. Parisi)
   * non-chiral spinor-helicity space $(\lambda, \eta \,|\, \tilde{\lambda}, \tilde{\eta})$
   * tree level amplitudes

Can we generalize this construction to include:
* loops <- this talk
* non-planar sector

Amplituhedron

Tree Amplituhedron $\mathscr{A}^{\text{tree}}_{n,k'}$: relevant for N$^{k'}$MHV Wilson loops

Image of the positive Grassmannian $G_+(k',n)$ through the map

$$\boxed{\begin{array}{c} \Phi_Z : G_+(k',n) \to G(k',k'+4) \to G(4,n) \\ C \quad \mapsto \quad Y \quad \mapsto \quad z \end{array}}$$

Momentum twistors

$$z_i = \begin{pmatrix} \lambda_i^\alpha \\ \tilde{\mu}_i^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \end{pmatrix}$$

where

$$\boxed{Y_a^I = \sum_{i=1}^n c_{ai} Z_i^I} \qquad \begin{array}{l} I = 1,2,\ldots,k'+4 \\ a = 1,2,\ldots,k' \end{array}$$

Z bosonized momentum-twistor superspace

$\{$matrix Z positive$\}$

Amplituhedron

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Momentum twistors

$$z_i = \begin{pmatrix} \lambda_i^{\alpha} \\ \tilde{\mu}_i^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \end{pmatrix}$$

The momentum twistors sit inside the bosonized space:

$$\boxed{z = Y^{\perp}Z = (CZ)^{\perp}\,Z}$$

$$Y_a^I = \sum_{i=1}^{n} c_{ai}\, Z_i^I$$

This map defines a subset in the kinematic space of z_i for which

$\{\langle i\; i+1\; j\; j+1\rangle \geq 0, i<j$

and the sequence $\{\langle 1234\rangle, \langle 1235\rangle, \ldots, \langle 123n\rangle\}$ has k' sign flips$\}$

Momentum Amplituhedron

Momentum Amplituhedron $\mathcal{M}^{\text{tree}}_{n,k}$: relevant for N^{k-2}MHV amplitudes

Image of the positive Grassmannian $G_+(k,n)$ through the map

$$\boxed{\begin{array}{l} \Phi_{(\Lambda,\tilde{\Lambda})} : G_+(k,n) \to G(k,k+2) \times G(n-k,n-k+2) \to G(2,n) \times G(2,n) \\ \qquad\qquad C \quad \mapsto \quad \tilde{Y} \qquad\qquad\qquad Y \qquad\qquad \mapsto \quad \tilde{\lambda} \qquad\qquad \lambda \end{array}}$$

where

$$\boxed{\tilde{Y}^{\dot{A}}_{\dot{\alpha}} = \sum_{i=1}^{n} c_{\dot{\alpha} i}\, \tilde{\Lambda}^{\dot{A}}_i \qquad\qquad Y^{A}_{\alpha} = \sum_{i=1}^{n} c^{\perp}_{\alpha i}\, \Lambda^{A}_i}$$

$A = (a, \alpha) = 1,\ldots, n-k+2$

$\dot{A} = (\dot{a}, \dot{\alpha}) = 1,\ldots, k+2$

$\boxed{k = k' + 2}$

$(\tilde{\Lambda}, \Lambda)$ bosonized spinor helicity variables

$\left\{\text{matrix } \tilde{\Lambda} \text{ positive} \quad \text{and} \quad \text{matrix } \Lambda^{\perp} \text{ positive}\right\}$

Momentum Amplituhedron

Momentum Amplituhedron $\mathcal{M}_{n,k}^{\text{tree}}$: relevant for N^{k-2}MHV amplitudes

Image of the positive Grassmannian G+(k,n) through the map

$$\begin{array}{ccccccc} \Phi_{(\Lambda,\tilde{\Lambda})} : G_+(k,n) & \rightarrow & G(k,k+2) & \times & G(n-k,n-k+2) & \rightarrow & G(2,n) \times G(2,n) \\ C & \mapsto & \tilde{Y} & & Y & \mapsto & \tilde{\lambda} \qquad \lambda \end{array}$$

The spinor-helicity variables sit inside the bosonized space:

$$\lambda = Y^{\perp}\Lambda = (C^{\perp}\Lambda)^{\perp}\Lambda \qquad \tilde{\lambda} = \tilde{Y}^{\perp}\tilde{\Lambda} = (C\tilde{\Lambda})^{\perp}\tilde{\Lambda}$$

$$\tilde{Y}_{\dot{\alpha}}^{\dot{A}} = \sum_{i=1}^{n} c_{\dot{\alpha}i}\,\tilde{\Lambda}_i^{\dot{A}} \qquad Y_{\alpha}^{A} = \sum_{i=1}^{n} c_{\alpha i}^{\perp}\,\Lambda_i^{A}$$

This map defines a subset in the kinematic space $(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}})$ for which

$\{\langle i\ i+1\rangle \geq 0, [i\ i+1] \geq 0, s_{i,i+1,\ldots,i+j} \geq 0\,,$

the sequence $\{\langle 12\rangle, \langle 13\rangle, \ldots, \langle 1n\rangle\}$ has $k-2$ sign flips

the sequence $\{[12], [13], \ldots, ([1n]\}$ has k sign flips $\}$

$$\langle i\ j\rangle = \lambda_{ia}\lambda_j^a, [i\ j] = \tilde{\lambda}_{i\dot{a}}\tilde{\lambda}_j^{\dot{a}},$$

$$s_{i,i+1,\ldots,i+j} = (p_i + p_{i+1} + \ldots + p_{i+j})^2$$

Boundaries and Singularities

Amplituhedra encode physical singularities of amplitudes in the structure of their boundaries

Momentum Amplituhedron

Facets: $\begin{cases} \langle i\, i+1\rangle = 0\,, & [i\, i+1] = 0 \\ s_{i,i+1\ldots,i+p} = 0\,, & p = 2,\ldots,n-4 \end{cases}$

Collinear limits

Factorizations

Full stratification known

(LF, T. Lukowski, R. Moerman)

Amplituhedron

Facets: $\langle i\; i+1\, j\, j+1\rangle = 0$

Full stratification not known

Since the MHV amplitude factored out, no boundaries corresponding to $\langle i\, i+1\rangle = 0$

Very different at tree level!

$$\langle i\; i+1\, j\, j+1\rangle = \frac{(p_i + p_{i+1} + \ldots + p_{j-1})^2}{\langle i\, i+1\rangle\langle j\, j+1\rangle}$$

Tree level momentum amplituhedron is not a simple translation of the amplituhedron construction

Loop Geometry

Loop Amplituhedron

Loop Amplituhedron $\mathscr{A}^{\text{loop}}_{n,k'}$: Image of the map

$$\begin{array}{ccccc} \Phi_Z : G_+(k',n) \times G(2,n)^\ell & \to & G(k',k'+4) \times G(2,k'+4)^\ell & \to & G(4,n) \times G(2,4)^\ell \\ C \qquad\qquad D_a & \mapsto & Y \qquad\qquad \mathscr{L} \equiv (Z_A, Z_B) & \mapsto & z \qquad\qquad (z_A, z_B) \end{array}$$

where

$$Y^I_\alpha = \sum_{i=1}^{n} c_{\alpha i} Z^I_i \qquad\qquad \mathscr{L}^I_{\gamma(a)} = \sum_{i=1}^{n} d_{\gamma i(a)} Z^I_i$$

$\gamma = 1,2$
$a = 1,\ldots,\ell$

Each loop = line (AB) = point x in dual space-time

$$\ell \leftrightarrow z_A z_B$$

Integral over the space of lines (AB) = integral over a pair of points A and B, divided by the GL(2) redundancies labeling their positions on the line

$$d^4\ell = \frac{d^4 z_A d^4 z_B}{\text{vol(GL(2))}}$$

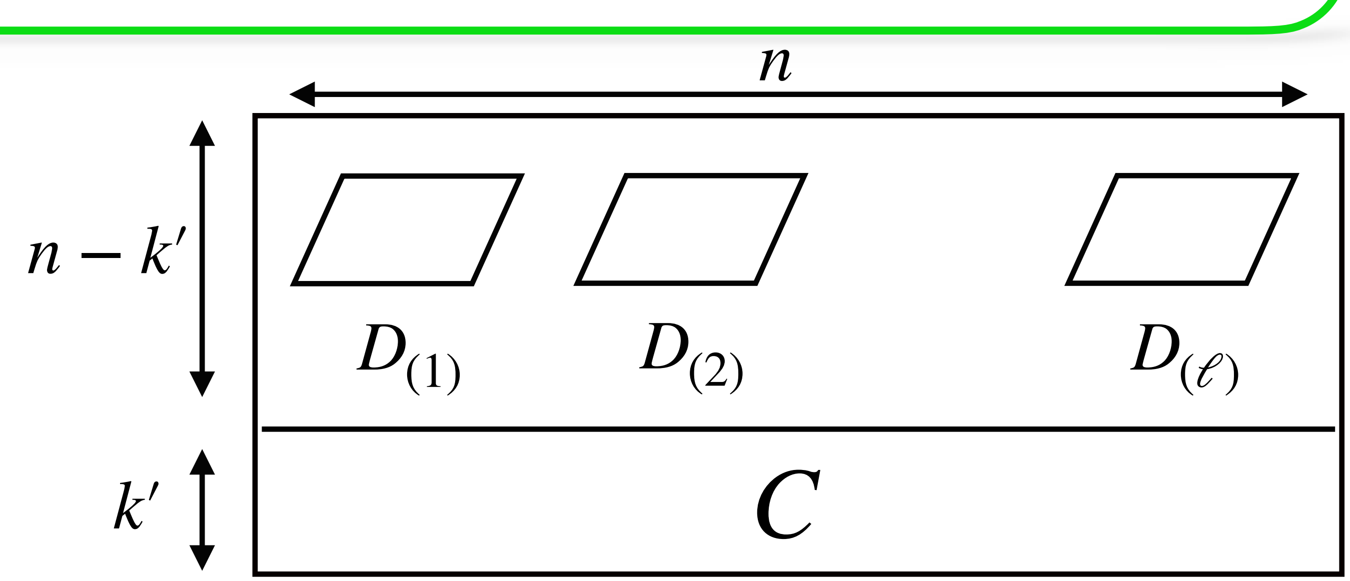
Loop Amplituhedron

Loop Amplituhedron $\mathcal{A}^{\text{loop}}_{n,k'}$: Image of the map

$$\Phi_Z : G_+(k',n) \times G(2,n)^\ell \to G(k',k'+4) \times G(2,k'+4)^\ell \to G(4,n) \times G(2,4)^\ell$$
$$\begin{array}{ccccccccc} C & & D & \mapsto & Y & & \mathcal{L} \equiv (Z_A, Z_B) & \mapsto & z & & (z_A, z_B) \end{array}$$

with

$$(C) \quad \begin{pmatrix} D_{(a_1)} \\ C \end{pmatrix} \cdots \begin{pmatrix} D_{(a_1)} \\ \vdots \\ D_{(a_\ell)} \\ C \end{pmatrix}$$ positive



"Extended positivity" from idea of hiding pairs of adjacent particles

$$\begin{array}{cccccccccccc}
A_1 & B_1 & 1 & 2 & \dots & m & A_2 & B_2 & m+1 & \dots & n \\
\end{array}$$
$$\begin{pmatrix}
1 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & * \\
0 & 1 & * & * & \dots & * & 0 & 0 & * & \dots & * \\
0 & 0 & * & * & \dots & * & 1 & 0 & * & \dots & * \\
0 & 0 & * & * & \dots & * & 0 & 1 & * & \dots & * \\
0 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & *
\end{pmatrix}$$

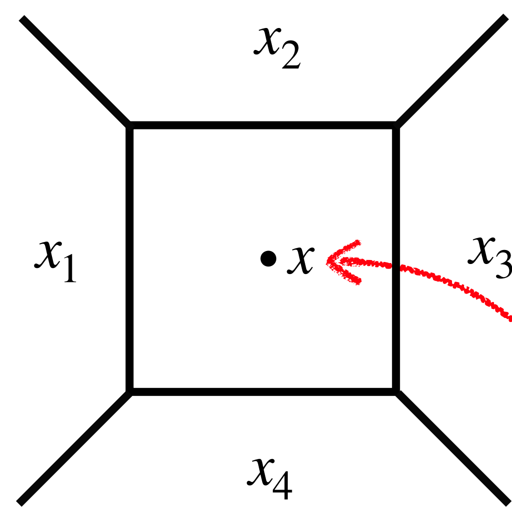
$$(C) \quad \begin{pmatrix} D_{(1)} \\ C \end{pmatrix} \begin{pmatrix} D_{(2)} \\ C \end{pmatrix} \begin{pmatrix} D_{(1)} \\ D_{(2)} \\ C \end{pmatrix}$$

Loop Amplituhedron

Main obstacle in generalizing this to ordinary momentum space:

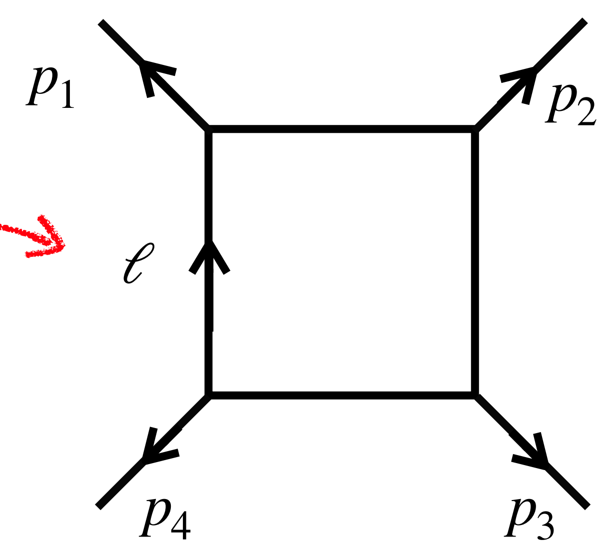
difficult to find a proper, global definition of
off-shell loop momentum

In dual space (planar theory):



uniquely defined
up to a global shift

In momentum space:



we can redefine e.g. $\ell' = \ell + p_1$
diagram by diagram:
the integrand changes

Approach 1

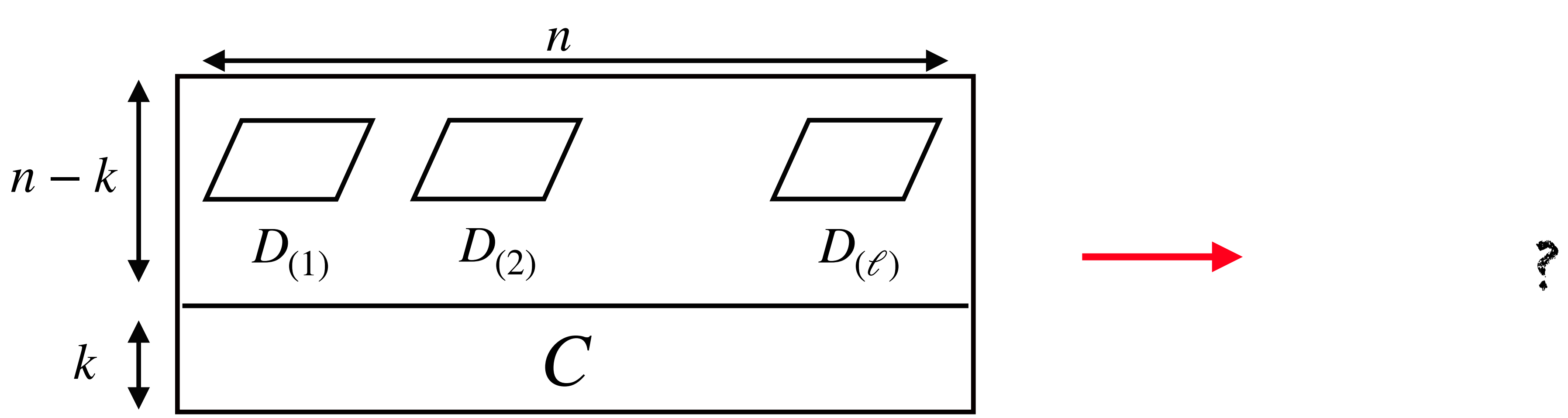
Mimic the hiding particles approach for spinor helicity space

Remember that $\tilde{Y} = C \cdot \tilde{\Lambda}$: try to extend it to include some D matrices encoding loops:

$$''\tilde{\mathscr{L}} = D \cdot \tilde{\Lambda}''$$

with some mutual positivity condition

But not enough space to take the perp: not clear what to do with λ



$\tilde{Y} = C \cdot \tilde{\Lambda}$ $\quad$ $\tilde{\mathscr{L}} = D \cdot \tilde{\Lambda}$ $\quad\quad$ $Y = C^{\perp} \cdot \Lambda$ $\quad$ $\mathscr{L} = ? \cdot \Lambda$

Moreover, the relation of $\tilde{\mathscr{L}}, \mathscr{L}$ to loop momentum not clear

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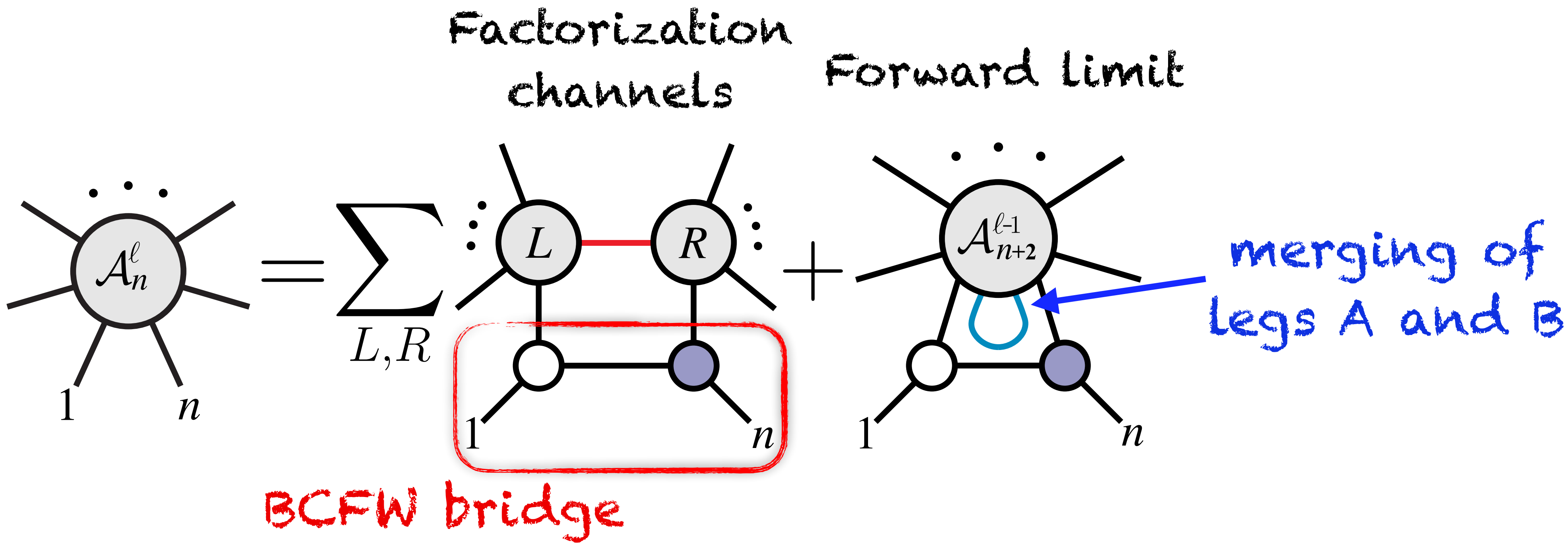
Approach 2

One parametrization of loop momentum given by BFCW construction

BFCW in spinor helicity space for loop amplitudes:

(N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, J. Trnka)



"Boundary measurement" from on-shell diagrams -> matrices C and D

Approach 2

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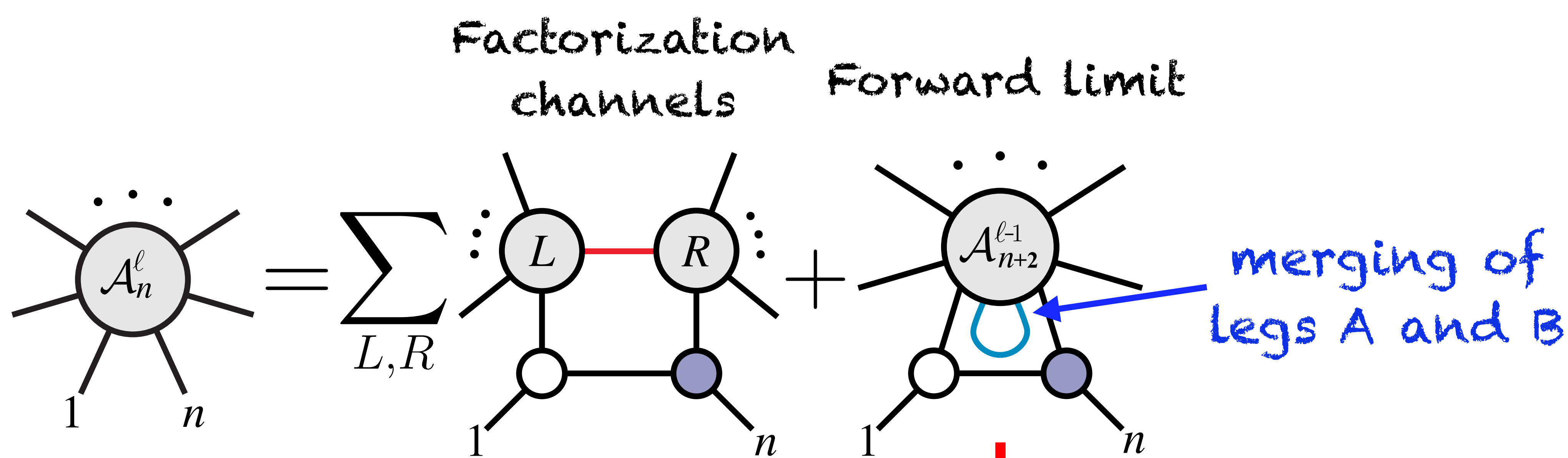

Then

$$\ell \equiv \lambda_{AB}\tilde{\lambda}_{AB} + \alpha\lambda_1\tilde{\lambda}_n \qquad d^4\ell = \frac{d^2\lambda_{AB}d^2\tilde{\lambda}_{AB}}{\mathrm{vol}(\mathrm{GL}(1))}d\alpha\langle 1\lambda_{AB}\rangle[n\tilde{\lambda}_{AB}]$$

- $\lambda_{AB}, \tilde{\lambda}_{AB}$ can be parametrised in terms of external momenta ✅
- positivity conditions not obvious ❌

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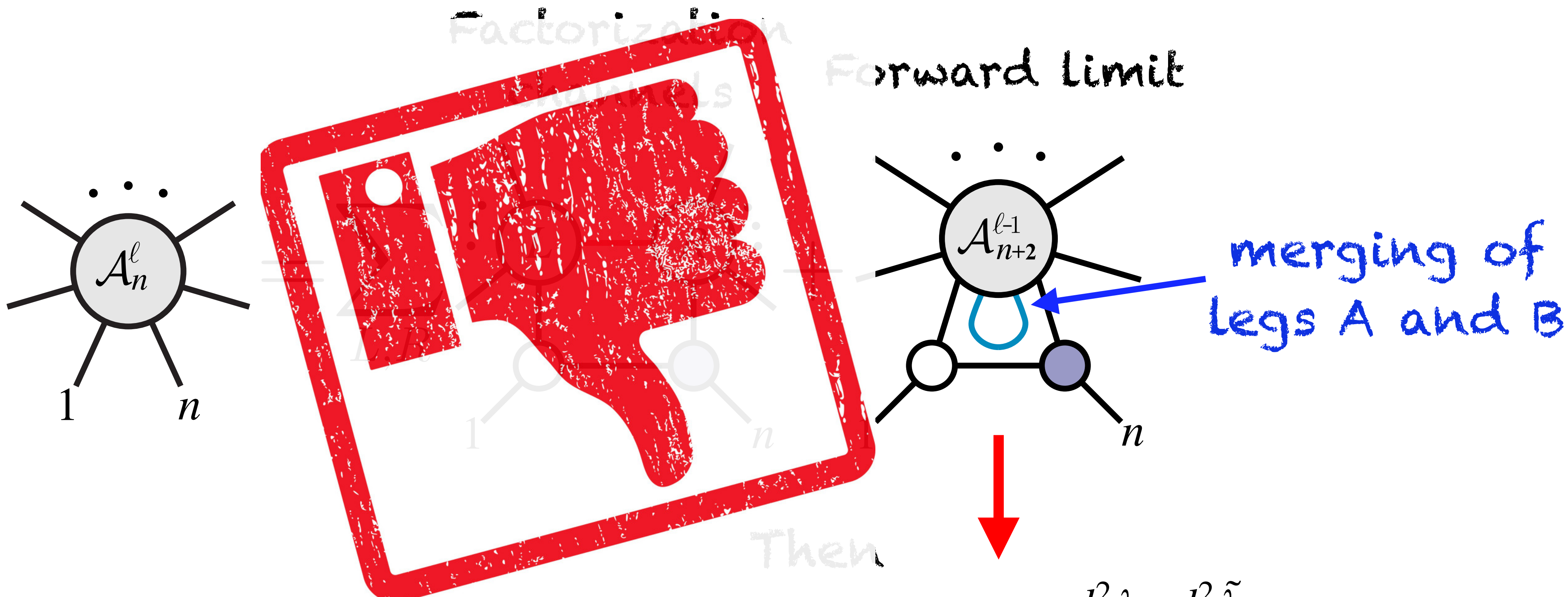

$$\ell \equiv \lambda_{AB}\tilde{\lambda}_{AB} + \alpha\lambda_1\tilde{\lambda}_n \qquad d^4\ell = \frac{d^2\lambda_{AB}d^2\tilde{\lambda}_{AB}}{\mathrm{vol(GL(1))}}d\alpha\langle 1\lambda_{AB}\rangle[n\tilde{\lambda}_{AB}]$$

- $\lambda_{AB}, \tilde{\lambda}_{AB}$ can be parametrised in terms of external momenta ✅
- positivity conditions not obvious ❌

Back to Boundaries

Amplituhedron vs Momentum Amplituhedron

singularities at tree level are different

but

singularities of loop-level integrands can be mapped between momentum twistors and momentum space

$$(\ell + \sum_j p_j)^2 = 0 \quad \leftrightarrow \quad \langle AB i\, i+1 \rangle = 0$$

$$(\ell_1 - \ell_2)^2 = 0 \quad \leftrightarrow \quad \langle (AB)_1 (AB)_2 \rangle = 0$$

Direct translation between tree amplituhedron and momentum amplituhedron not possible but maybe it can be done at loop level!

Loop Momentum Amplituhedron

Loop Momentum Amplituhedron

Direct translation: from momentum twistors to spinor-helicity variables

for external particles $z_i = \begin{pmatrix} \lambda_i \\ x_i \lambda_i \end{pmatrix}$ -- for the loop $z_{A,B} = \begin{pmatrix} \lambda_{A,B} \\ x\lambda_{A,B} \end{pmatrix}$

From amplituhedron: $z_{A,B} = \sum_i d_{(A,B)i} z_i$

Therefore:

$$\boxed{\lambda_{A,B} = \sum_i d_{(A,B)i} \lambda_i}$$ -> inherited GL(2) on λ_A, λ_B

Off-shell momentum in GL(2) invariant form:

$$\ell = \lambda_A \tilde{\lambda}^A + \lambda_B \tilde{\lambda}^B = \lambda_A \tilde{\lambda}_B - \lambda_B \tilde{\lambda}_A \qquad \lambda' \to \lambda G \,, \tilde{\lambda}' \to G^{-1}\tilde{\lambda}$$

and:

$$\boxed{x\lambda_{A,B} = \sum_i d_{(A,B)i} x_i \lambda_i \quad \text{->} \quad \tilde{\lambda}_a = \sum_{j<i} d_{ai} \frac{\langle ij \rangle}{\langle AB \rangle} \tilde{\lambda}_j}$$

- we made a choice how to relate ℓ to dual space: $\ell = x - x_1$
- this is a global choice

Loop Momentum Amplituhedron

Loop Momentum Amplituhedron $\mathscr{M}^{\text{loop}}_{n,k}$: Image of the map

$$\tilde{\Phi}_{(\Lambda,\tilde{\Lambda})} : G_+(k,n) \times G(2,n)^{\ell} \to G(2,n) \times G(2,n) \times GL(2)^{\ell}$$

$$C \qquad D_a \quad \mapsto \quad \lambda \qquad \tilde{\lambda} \qquad \ell_a$$

where

$$\lambda = Y^{\perp}\Lambda = (C^{\perp}\Lambda)^{\perp}\Lambda \qquad \tilde{\lambda} = \tilde{Y}^{\perp}\tilde{\Lambda} = (C\tilde{\Lambda})^{\perp}\tilde{\Lambda}$$ tree

$$\ell_a = \left(\sum_i d_{Ai}\lambda_i\right)\left(\sum_{j<i} d_{Bi}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right) - \left(\sum_i d_{Bi}\lambda_i\right)\left(\sum_{j<i} d_{Ai}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right)$$ loop

this map associates to every point C and D_a
the tree-level variables $(\lambda, \tilde{\lambda})$ and the loop momenta ℓ_a

Loop Momentum Amplituhedron

Loop Momentum Amplituhedron $\mathscr{M}^{\text{loop}}_{n,k}$: Image of the map

$$\tilde{\Phi}_{(\Lambda,\tilde{\Lambda})} : \begin{array}{ccccccc} G_+(k,n) & \times & G(2,n)^{\ell} & \to & G(2,n) \times G(2,n) \times GL(2)^{\ell} \\ C & & D_a & \mapsto & \lambda \qquad \tilde{\lambda} \qquad \ell_a \end{array}$$

Positivity inherited from the amplituhedron:

- for every element $C \in G_+(k,n)$ and $(\lambda,\tilde{\lambda}) \in \mathscr{M}^{\text{tree}}_{n,k}$, define

$$\boxed{\check{C} = QC}$$

with $$Q_{ij} = \frac{\langle i-1\, i\rangle \delta_{i+1,j} + \langle i\, i+1\rangle \delta_{i-1,j} + \langle i+1\, i-1\rangle \delta_{i,j}}{\langle i-1\, i\rangle\langle i\, i+1\rangle}$$

map relating spinor-helicity $G_+(k,n)$ with momentum twistor $G_+(k-2,n)$

(N. Arkani-Hamed, F. Cachazo, C. Cheung)

Importantly this implies $\boxed{\check{C} \in G_+(k-2,n)}$

Loop Momentum Amplituhedron

Loop Momentum Amplituhedron $\mathcal{M}_{n,k}^{\text{loop}}$: Image of the map

$$\boxed{\begin{array}{cccccc} \tilde{\Phi}_{(\Lambda,\tilde{\Lambda})} : G_+(k,n) \times G(2,n)^{\ell} & \to & G(2,n) \times G(2,n) \times GL(2)^{\ell} \\ C \qquad\qquad D_a & \mapsto & \lambda \qquad\qquad \tilde{\lambda} \qquad\qquad \ell_a \end{array}}$$

Positivity inherited from the amplituhedron:

- for every element $C \in G_+(k,n)$ and $(\lambda,\tilde{\lambda}) \in \mathcal{M}_{n,k}^{\text{tree}}$, define

$$\boxed{\check{C} = QC}$$

- same positivity conditions as for loop amplituhedron:

$$\left(\check{C}\right) \quad \begin{pmatrix} D_{(a_1)} \\ \check{C} \end{pmatrix} \cdots \begin{pmatrix} D_{(a_1)} \\ \vdots \\ D_{(a_\ell)} \\ \check{C} \end{pmatrix} \quad \text{positive}$$

Loop Momentum Amplituhedron

Example: MHV amplitudes

* since $k-2=0$, $\check{C}$ is an empty matrix and there is no positivity condition involving it

  -> C positive and D_a positive

* for MHV amplitudes: no need to triangulate tree level
* at one loop, amplituhedron triangulated by so-called kermits

$$K_{i,j}\ :\quad \begin{array}{cccccccc} & & & i & & & & \\ \multicolumn{8}{c}{\begin{pmatrix} 1 & 0 & \dots & \beta_1 & \beta_2 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & \beta_3 & \beta_4 \end{pmatrix}} \\ & & & & & & j & \end{array}$$

$$\boxed{\mathscr{M}_{n,2}^{1-\text{loop}} = \cup_{i<j}\, \tilde{\Phi}_{(\Lambda,\tilde{\Lambda})}(C, K_{i,j})}$$

Loop Momentum Amplituhedron

Example: MHV amplitudes

$$\mathscr{M}_{n,2}^{1-\text{loop}} = \cup_{i<j}\, \tilde{\Phi}_{(\Lambda,\tilde{\Lambda})}(C, K_{i,j})$$

The differential form

$$\Omega_{n,2}^{1-\text{loop}} = \Omega_{n,2}^{\text{tree}} \wedge \sum_{i<j} \Omega_{K_{i,j}}$$

where

$$\Omega_{K_{i,j}} = \text{dlog}\frac{(\ell - \ell^*_{1\,\text{i}})^2}{(\ell - \ell^*_{1\,\text{i+1}})^2} \wedge \text{dlog}\frac{(\ell - \sum_{\text{a=1}}^{\text{i}} \text{p}_\text{a})^2}{(\ell - \ell^*_{1\,\text{i+1}})^2} \wedge \text{dlog}\frac{(\ell - \ell^*_{1\,\text{j}})^2}{(\ell - \ell^*_{1\,\text{j+1}})^2} \wedge \text{dlog}\frac{(\ell - \sum_{\text{a=1}}^{\text{j}} \text{p}_\text{a})^2}{(\ell - \ell^*_{1\,\text{j+1}})^2}$$

spurious singularities

and $$\ell^*_{ij} = \frac{1}{\langle ij\rangle}\left(\lambda_i \sum_{l=1}^{j-1} \langle lj\rangle \tilde{\lambda}_l - \lambda_j \sum_{l=1}^{i-1} \langle li\rangle \tilde{\lambda}_l\right)$$

Conclusions
and
Open Questions

Conclusions and Open Questions

Main features of our construction:

- ☑ singularities of loop integrands can be mapped between spinor helicity and momentum twistor spaces -> allows for direct translation from loop amplituhedron
- ☑ loop momentum written in GL(2) invariant way
- ☑ global definition of loop momentum fixed by the map $\tilde{\Phi}_{(\Lambda,\tilde{\Lambda})}$

Challenges:

- ☐ definition still relies on positivity conditions that are not very natural from the point of view of spinor helicity space: improve?
- ☐ finding triangulations: is it possible to solve BCFW to find GL(2) loop momentum?
- ☐ still planar loop integrand: generalize to non-planar sector?
- ☐ ...

Thank you!