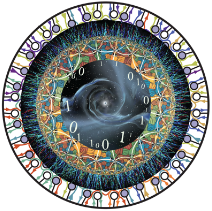
Non-planar BCFW Geometry



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Center for Quantum Mathematics and Physics

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Based on arxiv:2208.02262
SP, J Trnka, M Zheng

In this talk...

We will study tree-level $\mathcal{N} = 4$ SYM via:

- BCFW recursion relations
- Grassmannian formulation

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These methods have evinced special properties for planar amplitudes:

- Convex positive geometry for N^{k}MHV amplitudes
- Holomorphic expressions for MHV amplitudes

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- Generalize the Grassmannian geometry configuration corresponding to these non-planar diagrams
- Get holomorphic expressions for non-planar BCFW diagrams

BCFW Recursion

Introducing a complex shift,

$$\hat{\lambda}_i = \lambda_i + z\lambda_j$$
$$\hat{\tilde{\lambda}}_j = \tilde{\lambda}_j - z\tilde{\lambda}_i$$

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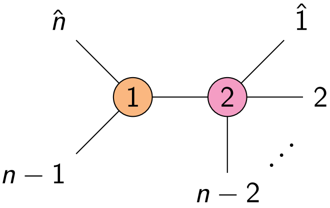
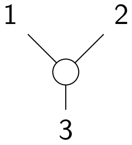
Example: $(n1)$ shift for n-point MHV amplitude has one BCFW term,



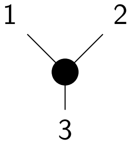
On-Shell Diagrams

$\mathcal{N} = 4$ SYM is BCFW-recursible

$\Rightarrow$ every BCFW term can be constructed by BCFW-bridging together 3-point amplitudes where external legs are on-shell,



$$= \frac{\delta^4(P)\delta^{1\times 4}(Q)}{[12][23][31]}$$



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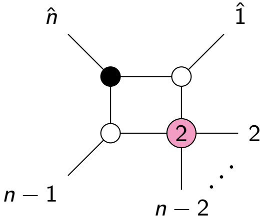
On-Shell Diagrams: MHV Example

$(n1)$ shift n-point MHV has only one BCFW term,



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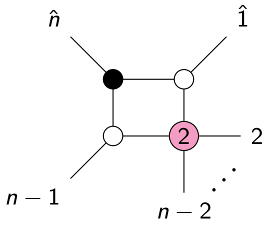
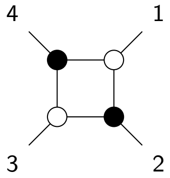
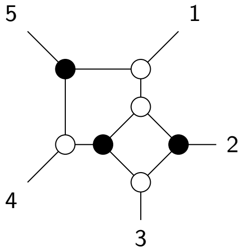


4- and 5-point for (41) and (51) shifts,





Grassmannian

Dual formulation of $\mathcal{N} = 4$ SYM BCFW terms as auxiliary $k \times n$ matrix C,

$$\oint \frac{d^{k\times n}C}{\text{vol}\,(GL(k))\prod \text{minors}} \delta^{2k}(C \cdot \widetilde{\lambda})\delta^{2(n-k)}(C^{\perp}\cdot\lambda)\delta^{0|4k}(C \cdot \widetilde{\eta})$$

where $C \in G_{+}(k, n)$.

How is this related to on-shell diagrams?

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where $C \in G_{+}(k,n)$.

How is this related to on-shell diagrams?

Each on-shell diagram corresponds to a particular cell in $G(k,n)$.
Additionally, diagrams provide a parametrization $C(\alpha_i)$.

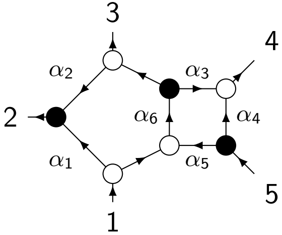
Solving the C-constraints gives $\alpha_i^*(\lambda,\tilde{\lambda})$, localizing the integral

$$\frac{\delta^4(P)}{J}\delta^8(Q)\delta^{0|4(k-2)}(C^*\cdot\widetilde{\eta})\times\prod_i\frac{1}{\alpha_i^*}$$

Edge Variables

This parametrization, can be read off from the on-shell diagram for the corresponding BCFW term.

Example: (51)-shifted 5-point MHV
Perfect orientation $\Rightarrow$ Boundary measurement $\Rightarrow$ C-matrix



$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_6 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_2\alpha_5\alpha_6 & \alpha_5\alpha_6 & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix}$$

$$\prod_i \frac{1}{\alpha_i^*} \to \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

Another way to see this is that $C = \lambda$ for MHV and $\prod_i \frac{1}{\alpha_i} = \frac{1}{\prod \text{minors}}$.

Multiple Descriptions Converge

In our example of $(n1)$-shifted n-point MHV,

$$\begin{aligned}\mathcal{A}_n^{\text{MHV}} =& \mathcal{A}_3^{\overline{\text{MHV}}} \overset{\text{BCFW}}{\times} \mathcal{A}_{n-1}^{\text{MHV}} \\ =& \text{On-shell diagram} \\ =& (2n-4)\text{-dimensional cells of } G_+(2,n) \\ =& \text{holomorphic poles}\end{aligned}$$

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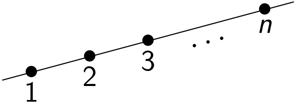
Two extra features arise in adjacent-shifted BCFW terms for MHV amplitudes:

- Positivity/convexity
- Holomorphicity

Positive Grassmannian Geometry

C-matrix is a matrix in kinematical λ-space.

For $C \in G_{+}(2, n)$, each column of $C \in \mathbb{P}^1$,



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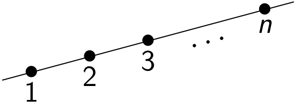


$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}$$

Positivity of minors gives $z_i - z_j > 0$ for $i > j$

$$\Downarrow$$

z_i are ordered and form a convex configuration in $\mathbb{P}^1$

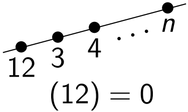
Evaluating this cell gives the Parke-Taylor amplitude,

$$PT(1, \cdots, n) = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \ldots \langle n1\rangle}$$

Holomorphicity

$PT(1, \cdots, n)$ only has poles when $\langle k k+1 \rangle \to 0$.

This can be seen directly from the geometry in $\mathbb{P}^1$,



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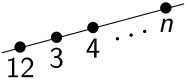
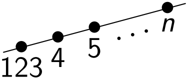


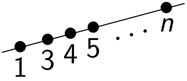
$$(12) = 0$$

Similarly for lower poles,



$$(12) = (23) = (13) = 0$$

$$P_1 \sim P_2 \sim P_3$$



$$(2*) = 0$$

$$P_2 \to 0$$

Now: Beyond Adjacent MHV

For adjacent-shifted NMHV and beyond:

- On-shell diagrams still correspond to cells in $G_{+}(k, n)$
- Is there a notion of holomorphicity?

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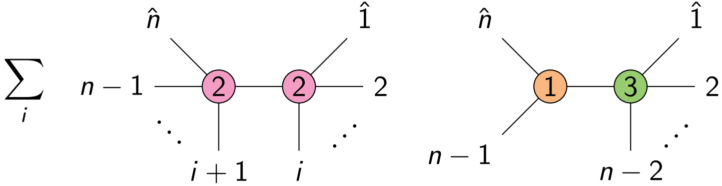
For non-adjacent BCFW shifts:

- How do we define a non-convex Grassmannian geometry?
- What about holomorphicity?

Positive Geometry for Adjacent NMHV

Two types of BCFW terms:



$= \sum$ on-shell diagrams $= \sum$ codimension $n-5$ cells of $G_+(3,n)$

$= \sum$ Grassmannian configurations in $\mathbb{P}^2$

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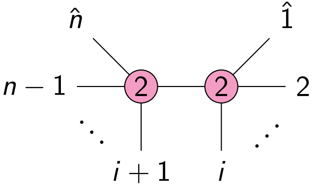
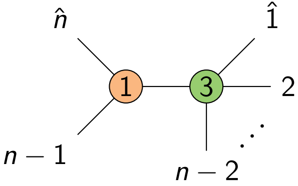
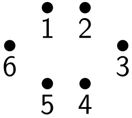
$\sum_i$ 



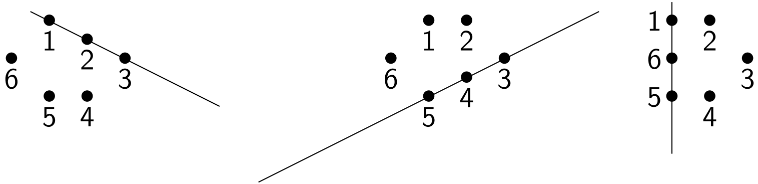
$$= \sum \text{on-shell diagrams} = \sum \text{codimension } n-5 \text{ cells of } G_{+}(3,n)$$
$$= \sum \text{Grassmannian configurations in } \mathbb{P}^2$$

For $n = 6$, there's 1 condition on a generic configuration of points



Example: (61) shifted NMHV_6

There are three BCFW terms that correspond to three codimension 1 cells in $G_{+}(3,6)$,



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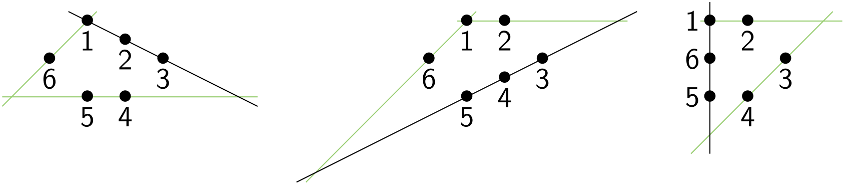

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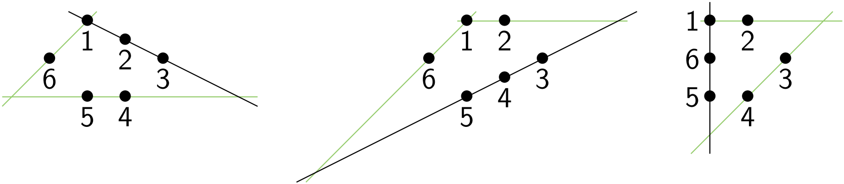


Summing over these cells gives

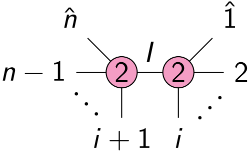
$$\mathcal{A}_6^{(3)} = \mathcal{R}_1 + \mathcal{R}_3 + \mathcal{R}_5 \,.$$

where the R-invariants are

$$\mathcal{R} = \frac{\delta^4(P)\delta^8(Q)\delta^4([56]\eta_4 + [64]\eta_5 + [45]\eta_6)}{s_{123}\langle 12\rangle\langle 23\rangle[45][56]\langle 1|23|4]\langle 3|45|6]} \,,$$

Holomorphicity for Adjacent NMHV: Diagram 1

NMHV has mixed poles. How do we recognize them as being boundaries of the Grassmannian geometry in $\mathbb{P}^2$?



$$= A^{\text{tree}}_{k=2}(i{+}1, i{+}2, \ldots, n{-}1, \hat{n}, I) \times \frac{1}{(P_2 + p_n)^2} \times A^{\text{tree}}_{k=2}(I, \hat{1}, 2, \ldots, i{-}1, i)\,.$$

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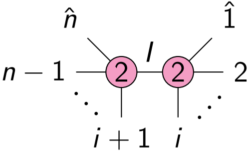

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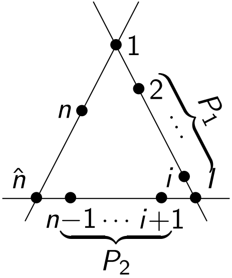
NMHV = MHV $\times$ MHV $\times$ pole which is MHV $\times$ MHV $\times$ MHV,

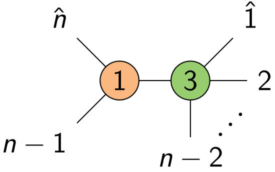


$$= \mathcal{R}_{1,i+1,n}$$
$$= PT(1, \ldots, i, I)PT(I, i{+}1, \ldots, n{-}1, \hat{n}) \times PT(\hat{n}, n, 1) \cdot \langle 1|P_1|n]^3 \times \Delta$$

where $\lambda_I = P_2|n]$

$\Rightarrow$ a holomorphic description of poles!

Holomorphicity for Adjacent NMHV: Diagram 2



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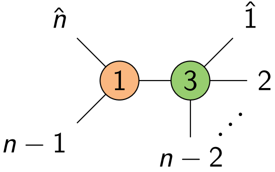
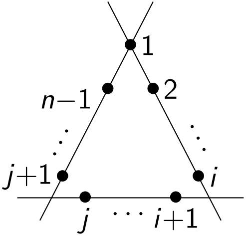
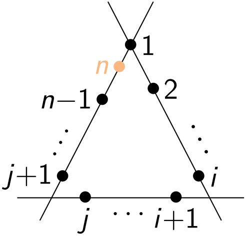


We can attach $\overline{\text{MHV}}$ to our previous expression for NMHV,

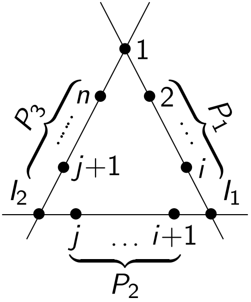
$\sum_{i,j}$ 

$+$ add n between $n-1$ and 1 $= \sum_{i,j}$ 

Holomorphicity for Adjacent NMHV: Diagram 2

Remarkably, the previous MHV $\times$ MHV $\times$ MHV structure from Diagram 1 extends to Diagram 2.

Thus $\mathcal{R}_{1,i+1,n}$ inspires the more general configurations $\mathcal{R}_{1,i+1,j+1}$ that cannot be seen as a simple rewriting of BCFW terms,



$$= \mathcal{R}_{1,i+1,j+1}$$
$$= PT(1,\ldots,i,I_1)PT(I_1,i{+}1,\ldots,j,I_2)$$
$$\times PT(I_2,j{+}1,\ldots,n,1)\langle 1P_1P_3|1\rangle^3 \times \Delta$$

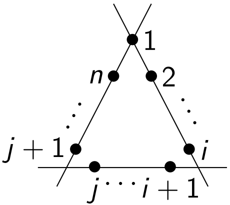
where $\lambda_{I_1} = \langle 1|P_3P_2, \lambda_{I_2} = \langle 1|P_1P_2$

Combinations of these show up in non-adjacent BCFW! Let's see how.

Aside: Reducing to Three Lines

BCFW terms are a special subset of cells of $G_+(3, n)$,

$$\mathcal{A}_n^{\text{NMHV}} = \sum_{i,j}$$



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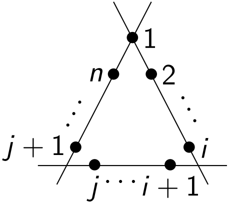
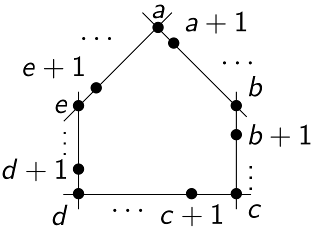
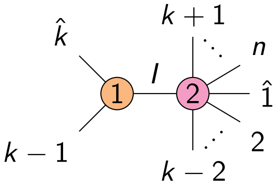


For n-point NMHV, we have $n-5$ constraints on n points in $\mathbb{P}^2$, so we are left with 5 unconstrained lines,



Non-Adjacent MHV

For a non-adjacent BCFW shift, already at MHV, there is more than one diagram:



Top cells are no longer convex with respect to ordering 1, 2, $\cdots$, n.

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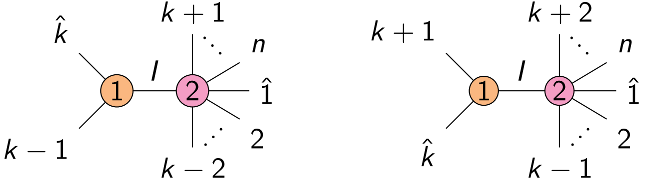

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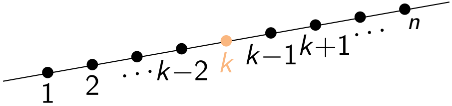
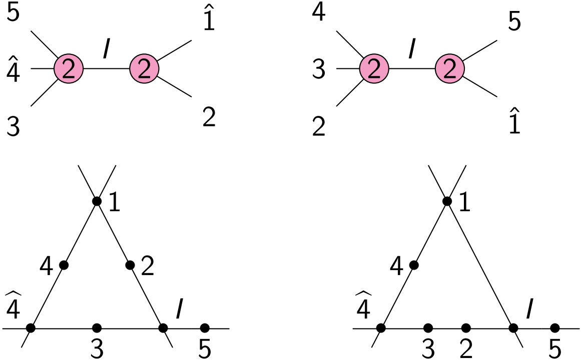
Cells are convex with respect to *some* ordering



$\Rightarrow$ Parke-Taylor factors give holomorphicity.

Non-Adjacent NMHV

Example: (14) shifted NMHV_5

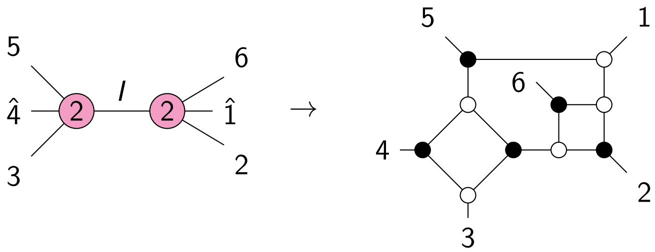


Again, convexity is lost and in this case, it cannot be restored by changing the ordering of external points.

Non-Planar On-Shell Diagrams

Non-adjacent BCFW terms generically correspond to non-planar on-shell diagrams.

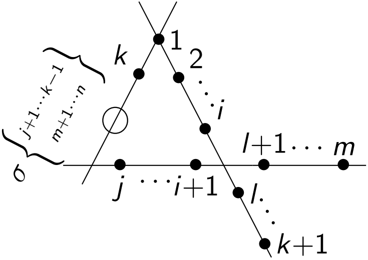
Example: (51)-shifted 6-point NMHV has a contribution from



A Lack of Positivity

For a general $(k1)$ shifted n-point amplitude,

$$A_n^{\text{NMHV}} = \sum_{\substack{i,j,l,m \\ \sigma(j+1\cdots k-1;m+1\cdots n)}}$$



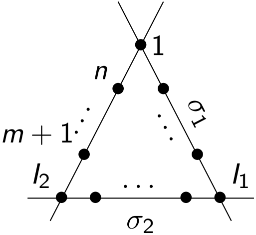
Planar Expansion of Non-Adjacent NMHV

Kleiss-Kuijf relations read

$$\mathcal{A}_n[1,\alpha,n,\beta] = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup\, \beta} \mathcal{A}_n[1,\sigma,n]$$

KK on each of the MHV lines allows us to rewrite each of the above non-planar non-convex geometries as a sum of planar convex ones!

$$\sum_{\sigma_1,\sigma_2}$$



where

$$\sigma_1 = \{2,\ldots,i\} \sqcup\!\sqcup \{i{+}1,\ldots,j\}^T, \qquad \sigma_2 = \{j{+}1,\ldots,k\} \sqcup\!\sqcup \{k{+}1,\ldots,m\}^T$$

Non-Adjacent BCFW and Holomorphicity

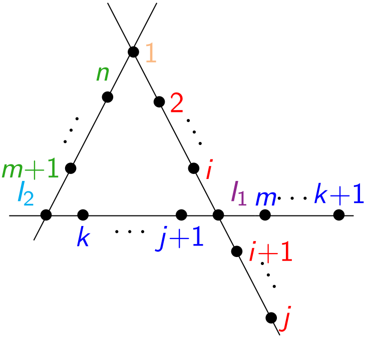
Two options:

- Non-planar geometry $= \sum$ convex configurations
  $= \sum$ products of Parke-Taylor factors for particular external orderings

Non-Adjacent BCFW and Holomorphicity

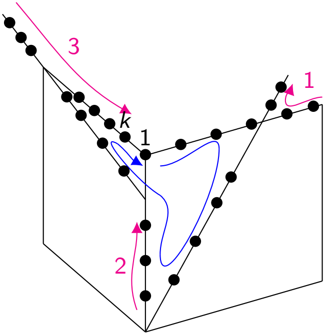
Two options:

- Non-planar geometry $= \sum$ convex configurations
  $= \sum$ products of Parke-Taylor factors for particular external orderings
- Non-planar $\mathcal{R}$-invariants that are defined directly on the non-convex geometry



$$
\begin{aligned}
&= PT(1, 2, \ldots, i, l_1, i+1, \ldots, j) \\
&\quad \times PT(l_1, j+1, \ldots, k, l_2, k+1, \ldots, m) \\
&\quad \times PT(l_2, m+1, \ldots, n, 1) \\
&\quad \times \langle 1 P_1 P_3 | 1 \rangle^3 \Delta
\end{aligned}
$$

Non-Adjacent $(1k)$-Shifted N^2MHV



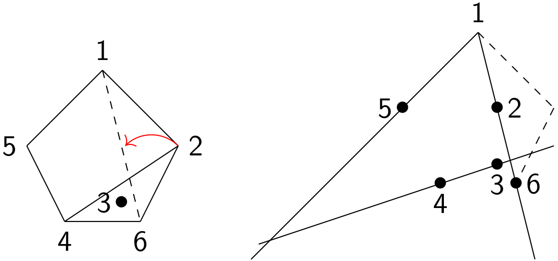
Generalized $\mathcal{R}$-invariants $=$ PT $\times$ PT $\times$ PT $\times$ PT $\times$ PT $\times J \times \Delta$.

Interesting Features

- Non-planar diagrams evaluate to

$$\sum \mathcal{R}_{1,\{P_1^a,P_1^b\},\{P_2^a,P_2^b\},P_3}$$

  where the individual orderings $P_1^a, P_1^b, P_2^a, P_2^b, P_3$ are preserved. Is there any residual dual conformal symmetry?
- Understanding BCFW geometries as boundaries of top cells:



Summary

- Before: Adjacent BCFW-shifted MHV amplitudes enjoyed an associated positive geometry and all singularities were holomorphic poles on $G_+(2, n)$

Summary

- Before: Adjacent BCFW-shifted MHV amplitudes enjoyed an associated positive geometry and all singularities were holomorphic poles on $G_+(2, n)$
- After: NMHV and beyond are also associated to only holomorphic poles on $G_+(k, n)$ cells
- Non-adjacent BCFW gives rise to a subset of non-planar on-shell diagrams for N^kMHV
- These non-positive BCFW geometries can be expressed as a sum of positive geometries with different external particle orderings
- Associated non-planar holomorphic expressions and $\mathcal{R}$-invariants can be constructed

Outlook

- Do non-planar BCFW geometries exhibit any residual dual conformal symmetry?
- What can we say about non-BCFW non-planar on-shell diagrams?
- Can we use a similar geometric construction to enumerate equivalence classes of top cells?

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Thank you for listening!