Three things we've learned about infrared divergences this year

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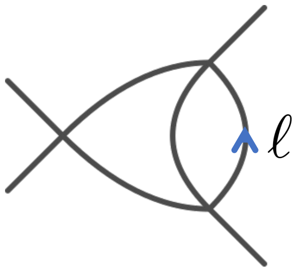
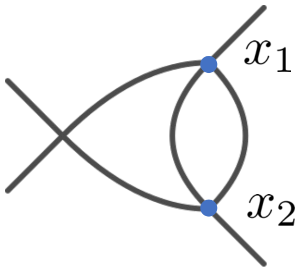
together with Nima Arkani-Hamed (IAS)
and Aaron Hillman (Princeton)

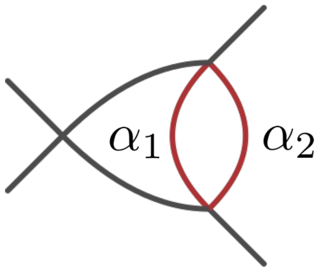
We have multiple ways of thinking about UV divergences



loop momentum
blowing up



space-time points
colliding



worldline segments
shrinking

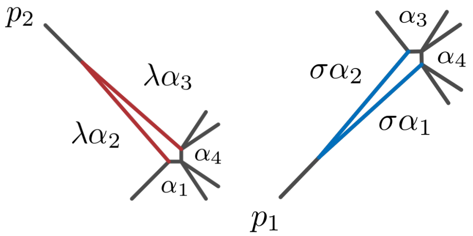
Fascinating connections with algebraic geometry, topology, and combinatorics, but also new computational tools

On the other hand, most of the intuition for IR divergences comes from the loop momentum space

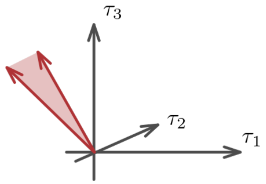
Question:

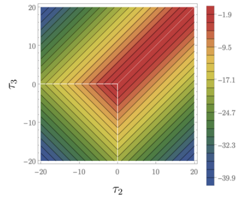
What is the meaning of IR divergences in Schwinger parameters?

1. Worldline interpretation

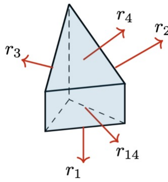


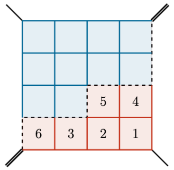
2. Divergences as redundancies

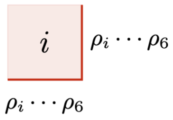




3. Computational scheme







Feynman integrals in Schwinger parametrization

(concrete examples later on)

Schwinger parameters

$$\int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}} \mathcal{N} e^{i\mathcal{F}/\mathcal{U}}$$

Space-time dimension $\mathrm{D} = 4 - 2\varepsilon$

(Translating back to the loop momenta: $\ell_a^\mu = \frac{1}{\mathcal{U}} \sum_{\substack{\text{spanning} \\ \text{trees } T}} p_{T,a}^\mu \prod_{e \notin T} \alpha_e$)

Momentum flowing through edge a along T

The integrand $\frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}}\mathcal{N}e^{i\mathcal{F}/\mathcal{U}}$ features Symanzik polynomials

(concrete examples later on)

$$\mathcal{U} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} \alpha_e$$

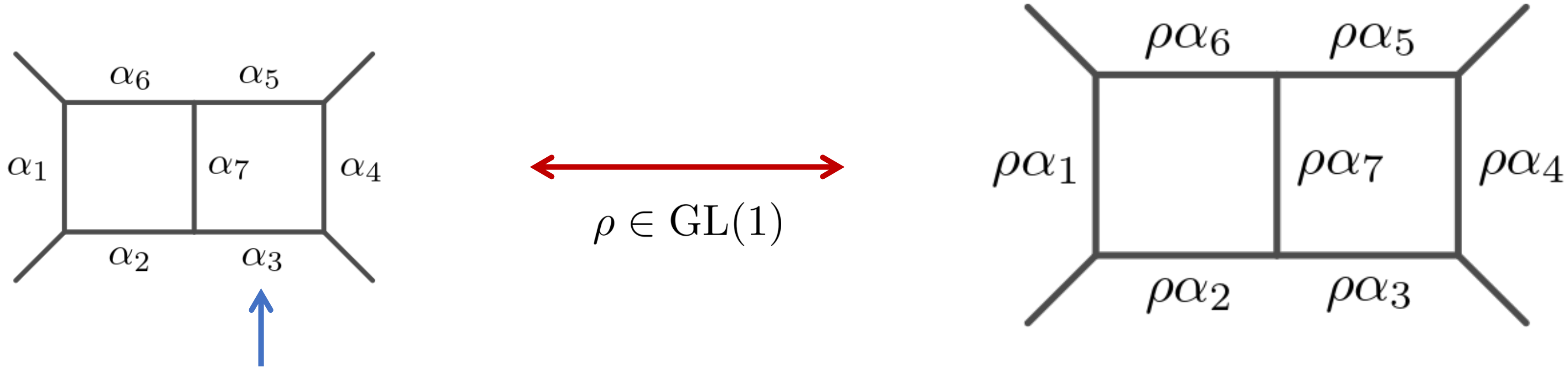
$$\mathcal{F} = \sum_{\substack{\text{spanning} \\ \text{2-trees } T_L \sqcup T_R}} p_{T_L}^2 \prod_{e \notin T_L, T_R} \alpha_e - \mathcal{U} \sum_{e=1}^{\mathrm{E}} m_e^2 \alpha_e$$

$$\mathcal{N} = \cdots = 1$$

In this talk,
but can be easily included

Overall GL(1) redundancy

(reparametrization invariance)



Schwinger parameters
(~lengths of the edges)

Modding out by the GL(1) redundancy

UV divergence factored out, $d = \mathrm{E} - \mathrm{LD}/2$

$$\int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}} e^{i\mathcal{F}/\mathcal{U}} = \#\Gamma(d) \int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathrm{GL}(1)} \frac{1}{\mathcal{U}^{\mathrm{D}/2-d}\mathcal{F}^{\mathrm{d}}}$$

Integrate out ρ

Fixing the GL(1) gauge, e.g.,

$\sum_{e=1}^{\mathrm{E}} \alpha_e = 1$ "Feynman parameters"

$\alpha_{\mathrm{E}} = 1$ This talk

Big picture:

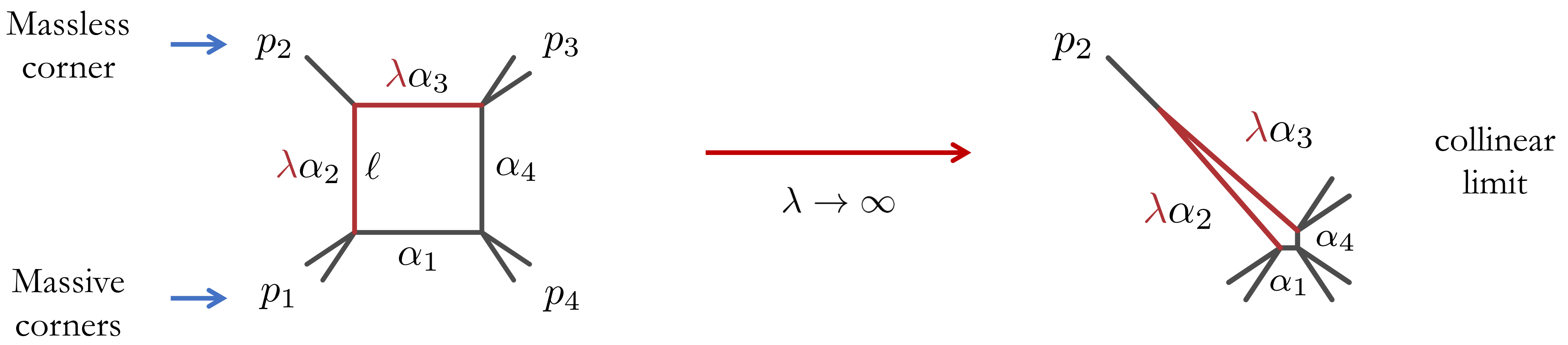
Singularities as saddle points of the worldline action $\mathcal{V} = \mathcal{F}/\mathcal{U}$

$$\alpha_e \, \partial_{\alpha_e} \mathcal{V} = 0 \qquad \text{for} \qquad e = 1, 2, \ldots, \mathrm{E}$$

How many constraints on the external kinematics?

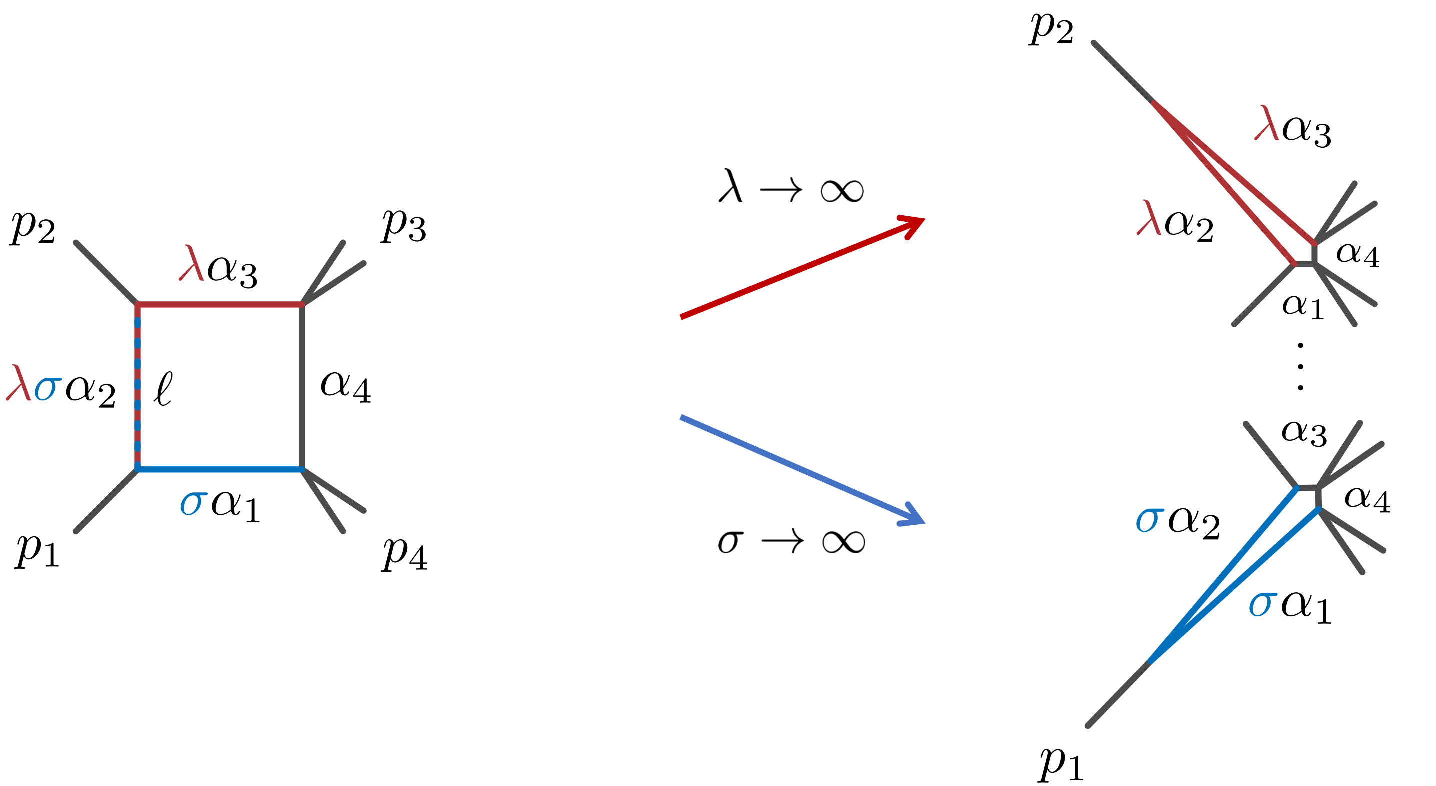
Codimension-0	**Codimension-1**	**Codimension-2 and higher**
(any value of external kinematics)	(e.g. $s = 4m^2$)	(e.g. $s = t = 0$)
UV/IR divergences	Normal/anomalous thresholds	Absence puts interesting constraints
[this talk]	[related talks/posters by Cordova, Correia, Hannesdottir, He, Henn, McLeod, Pokraka, Zhiboedov, ...]	[SM, Telen ‘21] [Hannesdottir, SM ‘22]

Simplest example of an IR divergence



Cross-check: $\ell^\mu = -p_2^\mu \frac{\alpha_3}{\alpha_2 + \alpha_3} + \mathcal{O}(\lambda^{-1})$

Next-to-simplest example of an IR divergence



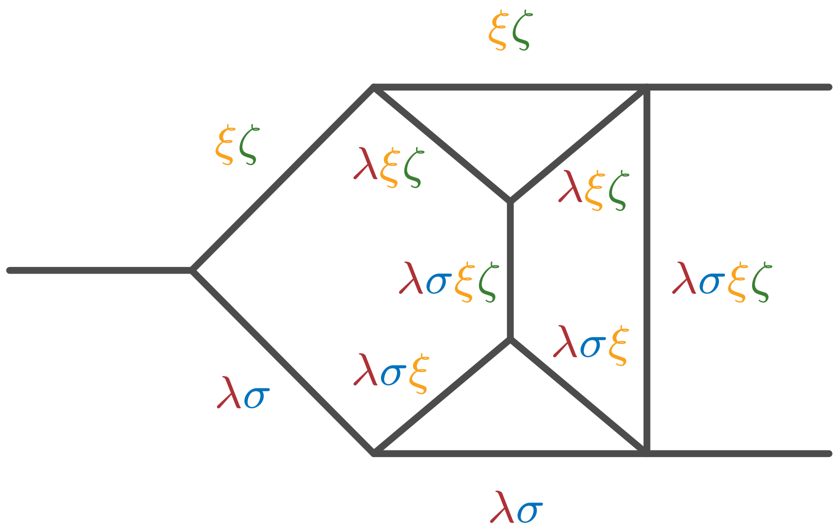
soft-collinear limit

Cross-check: $$\ell^\mu = \frac{p_1^\mu \sigma\alpha_1 - p_2^\mu \lambda\alpha_3}{\sigma\alpha_1 + \lambda\sigma\alpha_2 + \lambda\alpha_3} + \mathcal{O}(\lambda^{-1}, \sigma^{-1})$$

Related to [Yelleshpur '19]

Worldline scalings can become quite involved

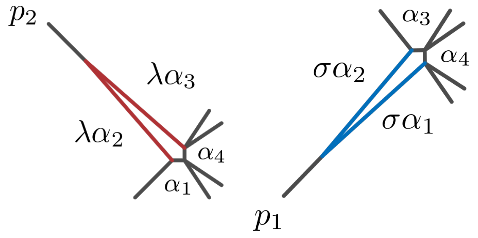
(e.g., four-loop contribution to the QCD cusp anomalous dimension)



$\lambda, \sigma, \xi, \zeta \to \infty$

We'll see that they can be entirely classified for any diagram

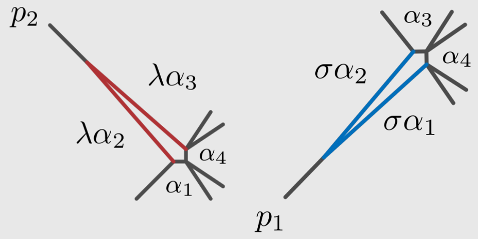
1. Worldline interpretation



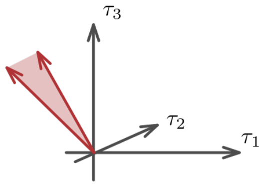
IR divergences come from worldline segments expanding at different rates

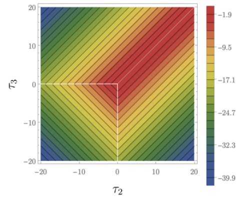
(cf. UV divergences)

1. Worldline interpretation

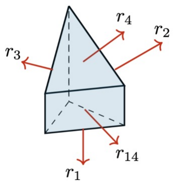


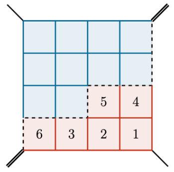
2. Divergences as redundancies

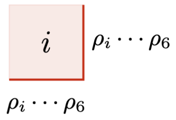




3. Computational scheme







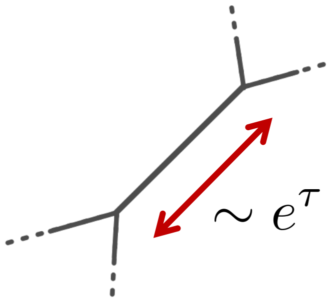
First change the variables

$$\alpha_e = \exp(\tau_e) \quad \text{for every edge} \quad e$$

Integration measure:

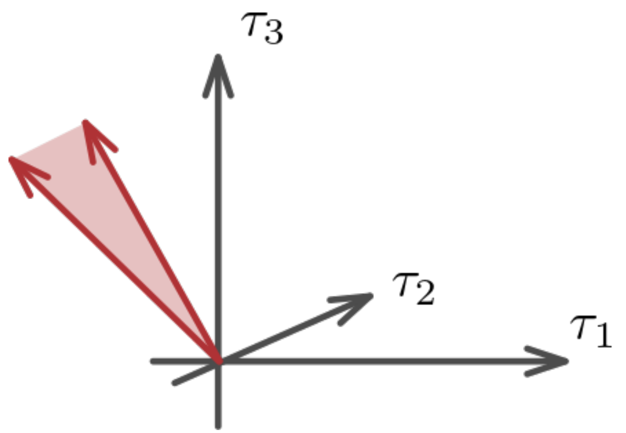
$$\int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha / \prod_{e=1}^{\mathrm{E}} \alpha_e}{\mathrm{GL}(1)} (\cdots) = \int_{-\infty}^\infty \frac{\mathrm{d}^{\mathrm{E}}\tau}{\mathrm{GL}(1)} (\cdots)$$

All divergences at infinity: $\tau_e \to \begin{cases} +\infty & \text{IR} \\ -\infty & \text{UV} \end{cases}$



UV/IR divergences comes from approaching infinity from different directions

(avoid kinematic singularities with $\tau_e \to \tau_e + i\varepsilon\partial_{\alpha_e}\mathcal{V}$) [SM ‘21]



We can linearize this problem using tropical geometry!

(Other applications of tropical geometry: [Tourkine, Cachazo, Early, Guevara, Sepulveda, Borges, Umbert, Panzer, Borinsky, Arkani-Hamed, He, Lam, Spradlin, Salvatori, Lukowski, Parisi, Williams, Drummond, Foster, Gurdogan, Kalousios, Papathanasiou, Henke, Eberhardt, SM, ...])

Tropical approximation of the integrand

$$\frac{\mathrm{d}^{\mathrm{E}}\tau}{\mathrm{GL}(1)}\left(\frac{e^{\sum_{e=1}^{\mathrm{E}}\tau_e}}{\mathcal{U}^{\mathrm{D}/2-d}\mathcal{F}^{\mathrm{d}}}\right) \approx \frac{\mathrm{d}^{\mathrm{E}}\tau}{\mathrm{GL}(1)}e^{\mathrm{Trop}}$$

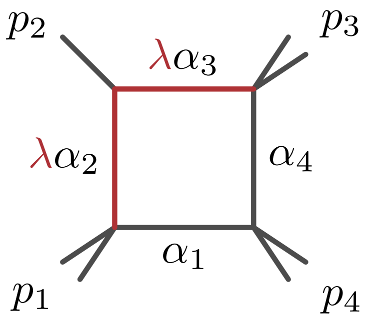
Exponentially accurate at infinity

where

$$\mathrm{Trop} = \sum_{e=1}^{\mathrm{E}}\tau_e - (\mathrm{D}/2 - d)\max(\mathcal{U}) - d\max(\mathcal{F})$$

Keep only the leading monomials

Example: collinear divergence in D=4

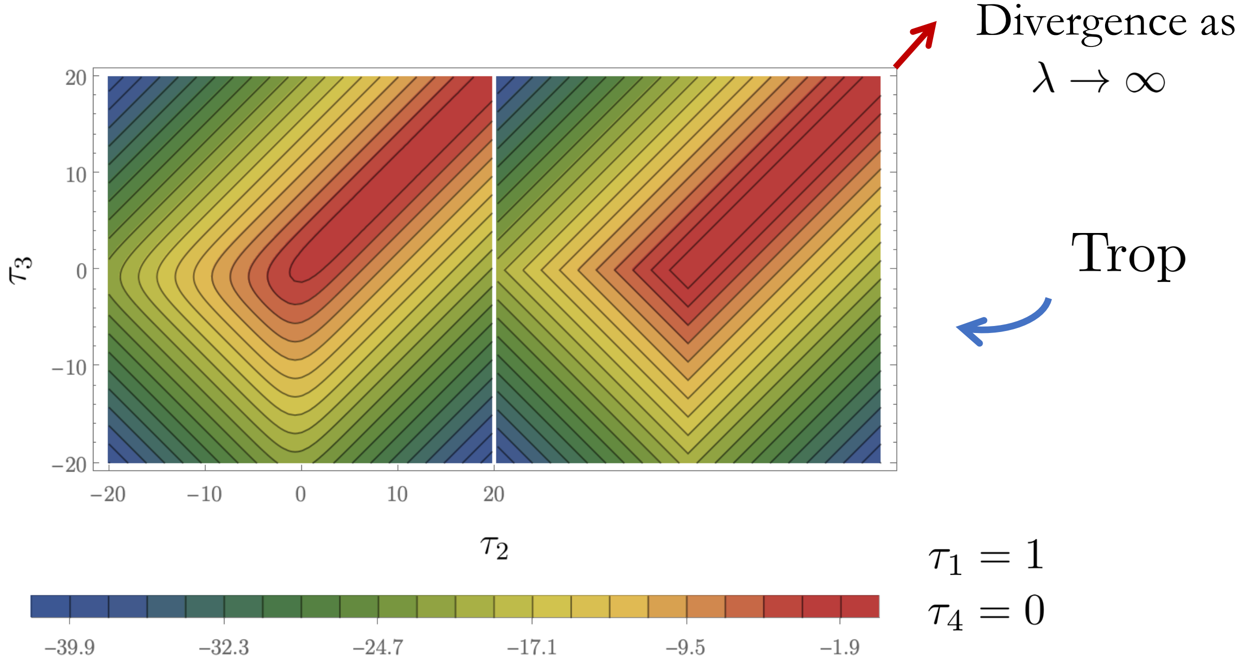


$$\mathcal{U} = e^{\tau_1} + e^{\tau_2} + e^{\tau_3} + e^{\tau_4}$$

$$\mathcal{F} = s\, e^{\tau_1+\tau_3} + t\, e^{\tau_2+\tau_4} + p_1^2\, e^{\tau_1+\tau_2} + p_3^2\, e^{\tau_3+\tau_4} + p_4^2\, e^{\tau_4+\tau_1}$$

$$\text{Trop} = \tau_1+\tau_2+\tau_3+\tau_4 - 2\max(\tau_1+\tau_3,\ \tau_2+\tau_4,\ \tau_1+\tau_2,\ \tau_3+\tau_4,\ \tau_4+\tau_1)$$

$$\log\left(\frac{e^{\tau_1+\tau_2+\tau_3+\tau_4}}{\mathcal{U}^0\mathcal{F}^2}\right)$$



Enhanced symmetry from GL(1) to GL(1)^2 $\implies$ divergence

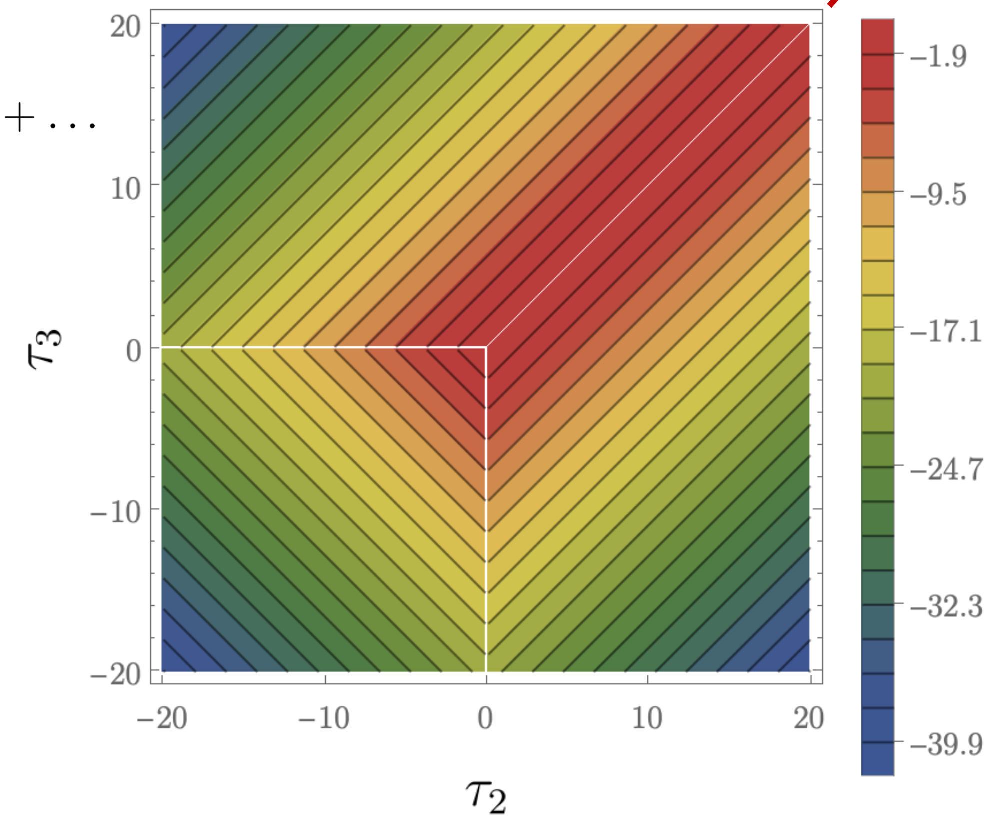
$$\frac{d^4\alpha}{(s\alpha_1\alpha_3 + t\alpha_2\alpha_4 + p_1^2\alpha_1\alpha_2 + p_3^2\alpha_3\alpha_4)^2} + \dots$$

Invariant under

GL(1) : $\alpha_e \to \rho\alpha_e$

GL(1) : $(\alpha_2, \alpha_3) \to \lambda(\alpha_2, \alpha_3)$

Trop = 0
(divergent ray)

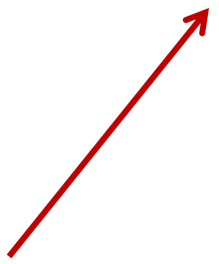


What kind of a divergence?

$$\int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha / \prod_{e=1}^{\mathrm{E}} \alpha_e}{\mathrm{GL}(1)} e^{\mathrm{Trop}}$$

$$\mathrm{Trop} \rightarrow \begin{cases} > 0 & \text{power-law} \\ = 0 & \text{logarithmic} \\ < 0 & \text{finite} \end{cases}$$

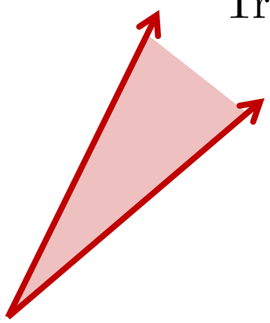
How bad of a divergence?



Trop $= 0$
ray

$\log$ or $\dfrac{1}{\epsilon}$

e.g., collinear

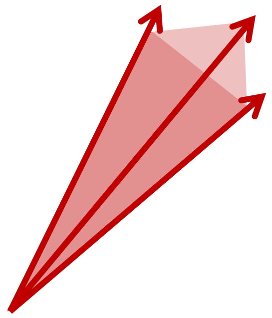


Trop $= 0$
cone

$\log^2$ or $\dfrac{1}{\epsilon^2}$

e.g., soft-collinear

$\cdots$



Trop $= 0$
cone

$\log^k$ or $\dfrac{1}{\epsilon^k}$

generalizations
$k \leqslant 2\text{L}$

Modding out by the enhanced GL(1)$^{k+1}$ redundancy

(specialize to logarithmic from now on)

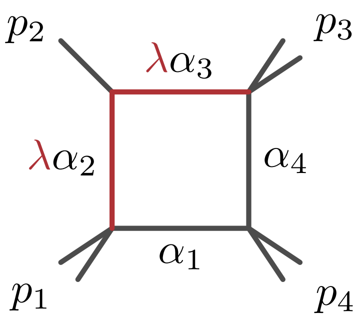
Volume of the cone $\propto \frac{1}{\epsilon^k}$

$$\Gamma(d) \int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathrm{GL}(1)} \frac{1}{\mathcal{U}^{\mathrm{D}/2-d}\mathcal{F}^d} = \Gamma(d) \sum_{\substack{\text{divergent}\\ \text{cones } C}} \mathrm{vol}(C) \int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathrm{GL}(1)^{k+1}} \frac{1}{\mathcal{U}_C^{\mathrm{D}/2-d}\mathcal{F}_C^d} + \mathcal{O}(\epsilon^{-k+1})$$

Fix the GL(1)$^{k+1}$ redundancy

Simplified Symanzik polynomials

Example: collinear divergence in D=4-2ϵ



Fix $\alpha_3 = \alpha_4 = 1$

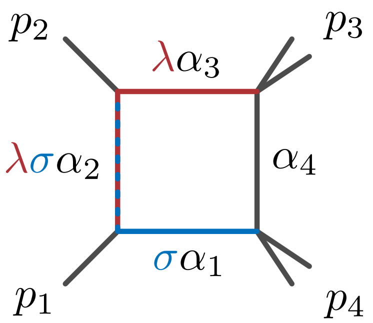
Set $\epsilon = 0$

$$= -\frac{1}{\epsilon}\int \frac{\mathrm{d}^4\alpha}{\mathrm{GL}(1)^2}\,\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{2\epsilon}}{(s\alpha_1\alpha_3 + t\alpha_2\alpha_4 + p_1^2\alpha_1\alpha_2 + p_3^2\alpha_3\alpha_4 + \cancel{p_4^2\alpha_4\alpha_1})^{2-\epsilon}} + \mathcal{O}(\epsilon^0)$$

$$= -\frac{1}{\epsilon}\int_0^\infty \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{(s\alpha_1 + t\alpha_2 + p_1^2\alpha_1\alpha_2 + p_3^2)^2} + \mathcal{O}(\epsilon^0)$$

$$= -\frac{1}{\epsilon}\,\frac{\log\left(\frac{st}{p_1^2p_3^2}\right)}{st - p_1^2p_3^2} + \mathcal{O}(\epsilon^0)$$

Example: soft-collinear divergence in D=4-2ϵ



Fix $\alpha_2 = \alpha_3 = \alpha_4 = 1$

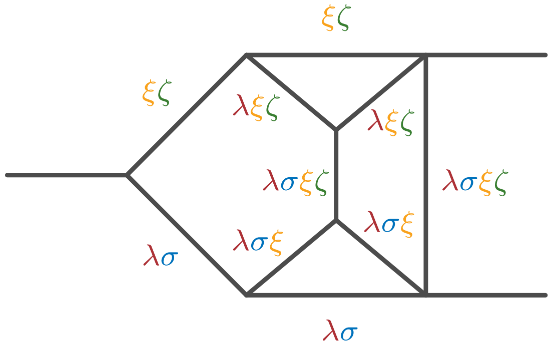
Set $\epsilon = 0$

$$= \frac{1}{\epsilon^2} \int \frac{\mathrm{d}^4\alpha}{\mathrm{GL}(1)^3} \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{2\epsilon}}{(s\alpha_1\alpha_3 + t\alpha_2\alpha_4 + \cancel{p_3^2\alpha_3\alpha_4} + \cancel{p_4^2\alpha_4\alpha_1})^{2-\epsilon}} + \mathcal{O}(\epsilon^{-1})$$

$$= \frac{1}{\epsilon^2} \int_0^\infty \frac{\mathrm{d}\alpha_1}{(s\alpha_1+t)^2} + \mathcal{O}(\epsilon^{-1})$$

$$= \frac{1}{\epsilon^2}\frac{1}{st} + \mathcal{O}(\epsilon^{-1})$$

Example: cusp anomalous dimension at four loops



$$= \frac{1}{144\epsilon^4} \int \frac{\mathrm{d}^{10}\alpha}{\mathrm{GL}(1)^5} \frac{1}{s^2} \big[(\alpha_1\alpha_5+\alpha_3\alpha_5+\alpha_1\alpha_7)(\alpha_2\alpha_6+\alpha_4\alpha_6+\alpha_2\alpha_8)\,\alpha_9 + (\alpha_1+\alpha_3)(\alpha_2+\alpha_4)(\alpha_5+\alpha_7)(\alpha_6+\alpha_8)\,\alpha_{10} \big]^{-2} + \mathcal{O}(\epsilon^{-3})$$

$$= \frac{\pi^4}{5184 s^2 \epsilon^4} + \mathcal{O}(\epsilon^{-3})$$

In agreement with [Henn, Smirnov, Smirnov, Steinhauser '16-20]

Rich literature on closely related topics

- Sector decomposition & expansion by regions

[Binoth, Heinrich, Bogner, Weinzierl, Kaneko, Ueda, Borowka, Jones, Kerner, Schlenk, Zirke, Jahn, Langer, Magerya, Poldaru, Villa, Pak, Smirnov, Smirnov, Ananthanarayan, Ramanan, Sarkar, Semenova, ...]

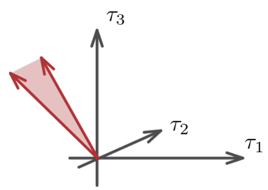
- Feynman integrals as GKZ hypergeometric functions

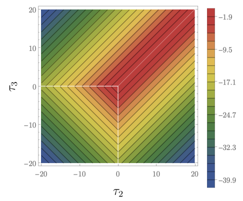
[enormous literature in the 20^{th} century, de la Cruz, Klausen, Tellander, Helmer, Feng, Chang, Chen, Zhang, Abreu, Britto, Duhr, Gardi, Matthew, Mastrolia, Telen, SM, ...]

[poster by Datta]

We want to exploit the extra combinatorics of Symanzik polynomials

2. Divergences as redundancies



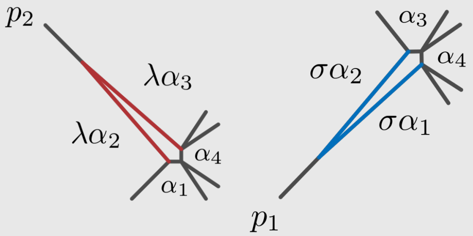


Divergence is a GL(1)$^{k+1}$ redundancy

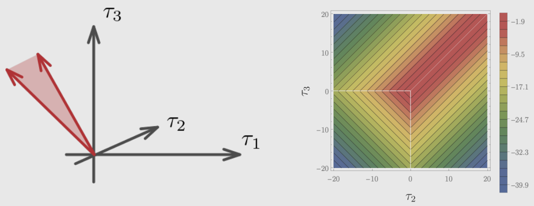
computed by fixing $k+1$ Schwinger parameters

(cf. fixing gauge in gauge theory)

1. Worldline interpretation



2. Divergences as redundancies



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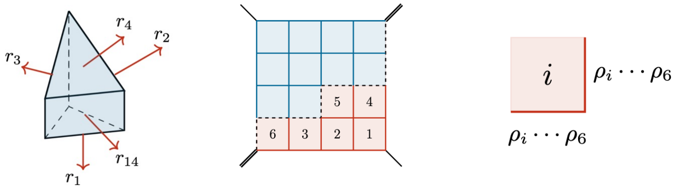
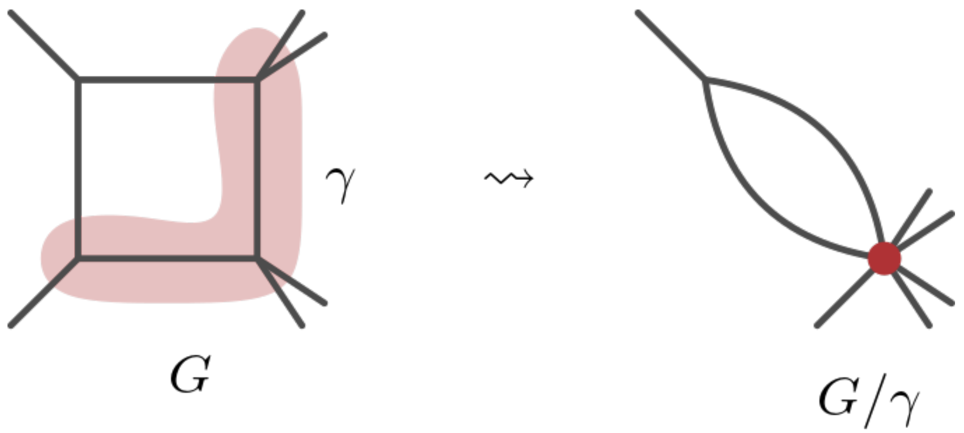


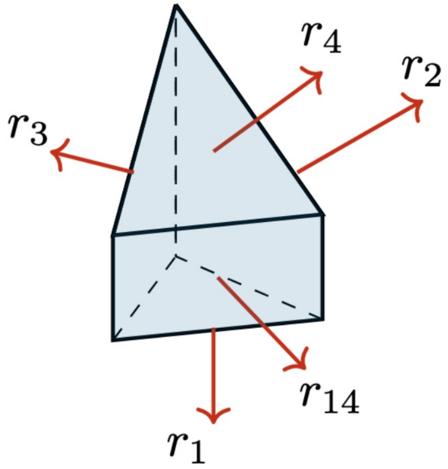
No need to do it "by hand" for every Feynman diagram



$$\alpha_{e\in\gamma} = \lambda\hat{\alpha}_e$$
$$\lambda \to 0$$

Rays generated by subdiagrams γ

Feynman polytopes



- Where are the divergences?
- What type are they?
- How do they nest/overlap?

$$\mathrm{Trop}(\gamma) = \begin{cases} -d_\gamma & \text{if} \quad \mathcal{F}_{G/\gamma} \neq 0 \\ d_{G/\gamma} & \text{if} \quad \mathcal{F}_{G/\gamma} = 0 \end{cases}$$

UV-like

IR-like

Previously studied in the UV case with generic kinematics

- Blow-ups of the Schwinger parameter space
  [Bloch, Esnault, Kreimer, Brown, Schultka ‘05-19]

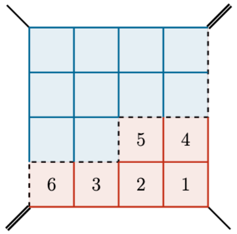
- Numerical computations of finite integrals
  [Panzer, Borinsky ‘19-20]

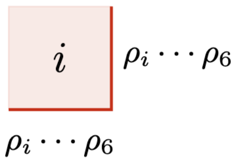
IR singularities need non-generic kinematics

(Feynman polytopes are in general *not* generalized permutohedra)

Lots of applications to IR divergences, including all-loop results

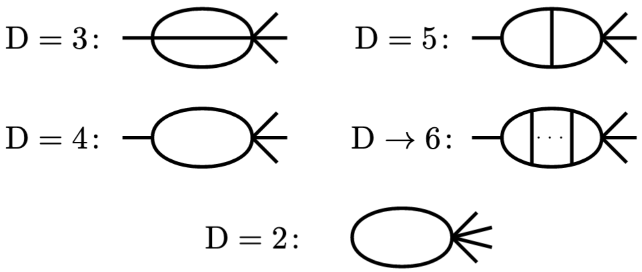
(details in [Arkani-Hamed, Hillman, SM ‘22])

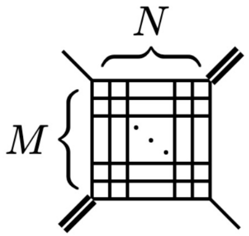




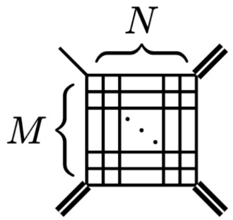
$$\partial \mathbf{F}_G \supset \begin{cases} \mathbf{U}_\gamma \times \mathbf{F}_{G/\gamma} & \text{on} \quad a_\gamma = \mathrm{L}_\gamma, \\ \mathbf{F}_\gamma \times \mathbf{U}_{G/\gamma} & \text{on} \quad a_\gamma = \mathrm{L}_\gamma + 1. \end{cases}$$

$$(1+c)(\mathrm{L}_{\gamma_1} + \mathrm{L}_{\gamma_2}) + (c)_{\gamma_1} + (c)_{\gamma_2} \geqslant (1+c)(\mathrm{L}_{\gamma_1 \cup \gamma_2} + \mathrm{L}_{\gamma_1 \cap \gamma_2}) + (c)_{\gamma_1 \cup \gamma_2} + (c)_{\gamma_1 \cap \gamma_2}.$$

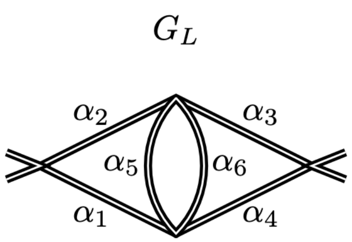


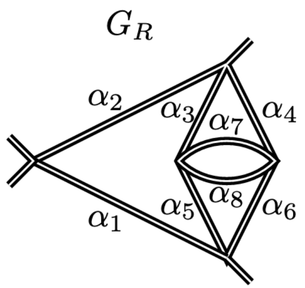


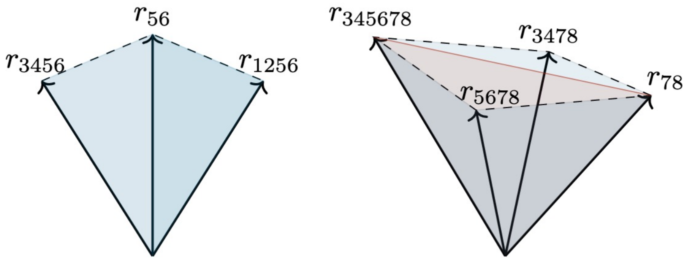
$$= \frac{2^{MN}}{(MN)!} \frac{C_{M,N}}{\varepsilon^{MN} s^N t^M} + \ldots,$$



$$= \frac{F_{M,N}(s,t,p_i^2)}{(MN)!} \frac{C_{M,N}}{\varepsilon^{MN}} + \ldots,$$







$$C_{M,N} = (MN)! \frac{\mathrm{sf}(M-1)\mathrm{sf}(N-1)}{\mathrm{sf}(M+N-1)}$$

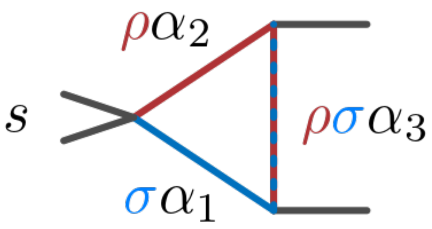
$$\mathrm{sf}(N) = 0!1!2!\cdots N!$$

If there's time:

Glimpse at the Laurent expansion in dimensional regularization, D=4-2ϵ

(inspired by [Brown, Kreimer '11] in BPHZ renormalization and [Brown, Dupont '19] in string theory amplitudes)

Simplest example

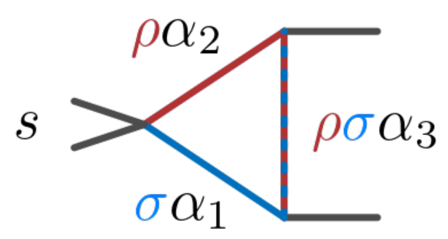


$$= \frac{\Gamma(1+\epsilon)}{(-s)^{1+\epsilon}} \left\{ \frac{1}{\epsilon^2} + \int \frac{\mathrm{d}^3\alpha}{\mathrm{GL}(1)} \left[\left(\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}\right)^{-\epsilon} \mathrm{d}\log\left(\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}\right) - \left(\frac{\alpha_1}{\alpha_1+\alpha_3}\right)^{-\epsilon} \mathrm{d}\log\left(\frac{\alpha_1}{\alpha_1+\alpha_3}\right)\right] \right.$$

$$\left. \wedge \left[\left(\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}\right)^{-\epsilon} \mathrm{d}\log\left(\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}\right) - \left(\frac{\alpha_2}{\alpha_2+\alpha_3}\right)^{-\epsilon} \mathrm{d}\log\left(\frac{\alpha_2}{\alpha_2+\alpha_3}\right)\right] \right\}$$

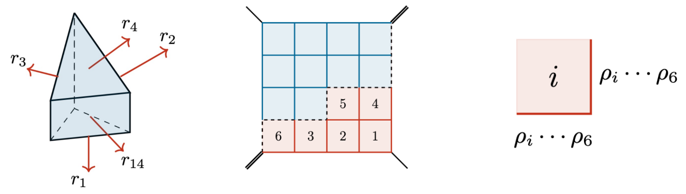
All integrals manifestly finite

Series[%, {ϵ,0,0}]



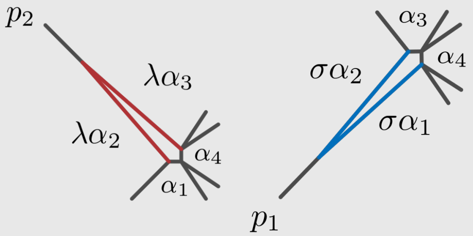
$$= \frac{\Gamma(1+\epsilon)}{(-s)^{1+\epsilon}} \left\{ \frac{1}{\epsilon^2} + \underbrace{\int_0^\infty \mathrm{d}\log\left(\frac{\alpha_1+1}{\alpha_1+\alpha_2+1}\right) \wedge \mathrm{d}\log\left(\frac{\alpha_2+1}{\alpha_1+\alpha_2+1}\right)}_{-\zeta_2} + \mathcal{O}(\epsilon) \right\}$$

3. Computational scheme

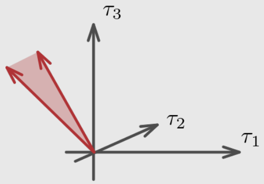


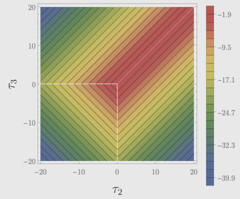
Combinatorics of UV/IR divergences is summarized by Feynman polytopes

1. Worldline interpretation

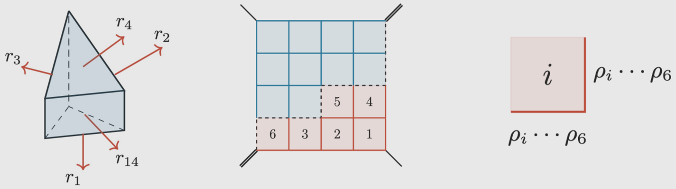


2. Divergences as redundancies





3. Computational scheme



Thank you!