Two-loop QCD corrections to five-particle amplitudes with one massive leg

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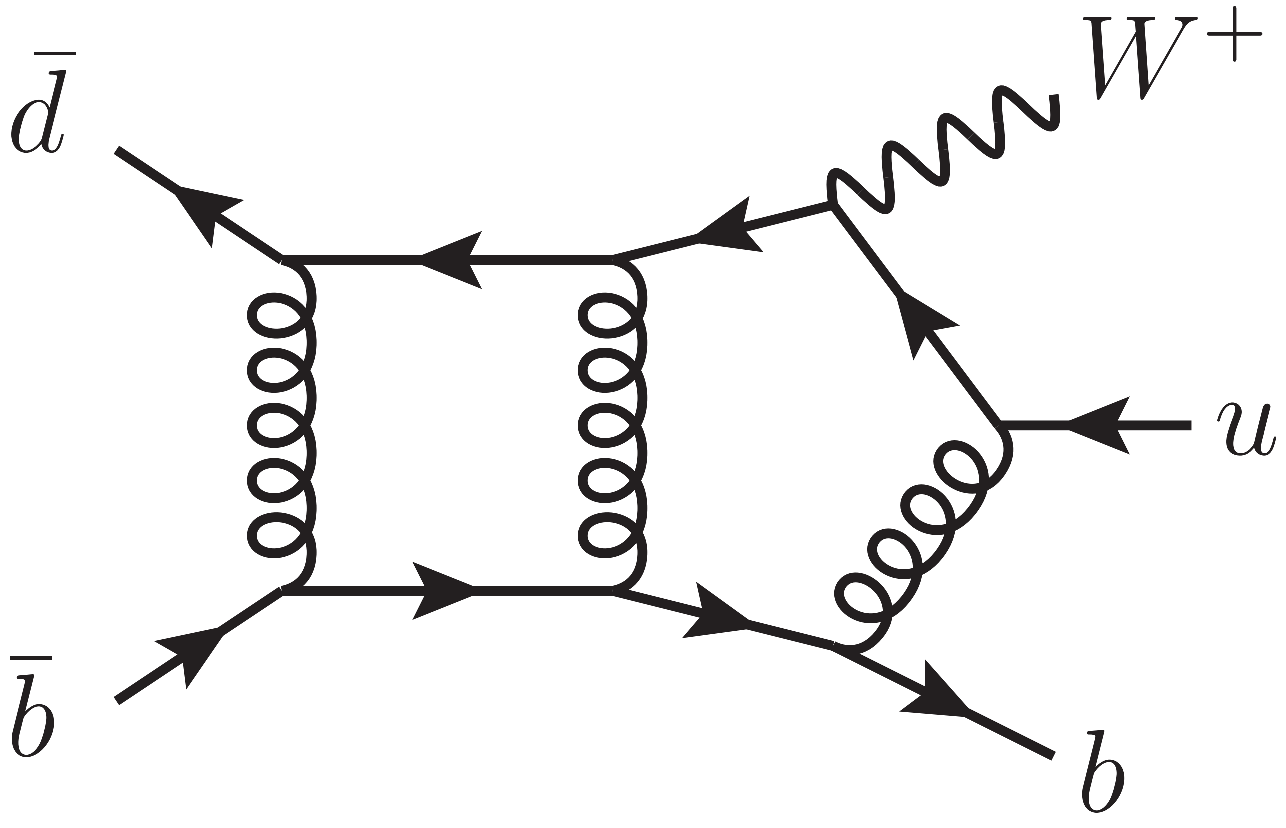
Outline

Recent advances for 2-loop 5-particle amplitudes with one external massive leg



- Basis of special functions: (one-mass) pentagon functions

  Dmitry Chicherin, Vasily Sotnikov, **SZ** 2110.10111

- $W(\to \ell\nu) + b\bar{b}$ production @ NNLO QCD

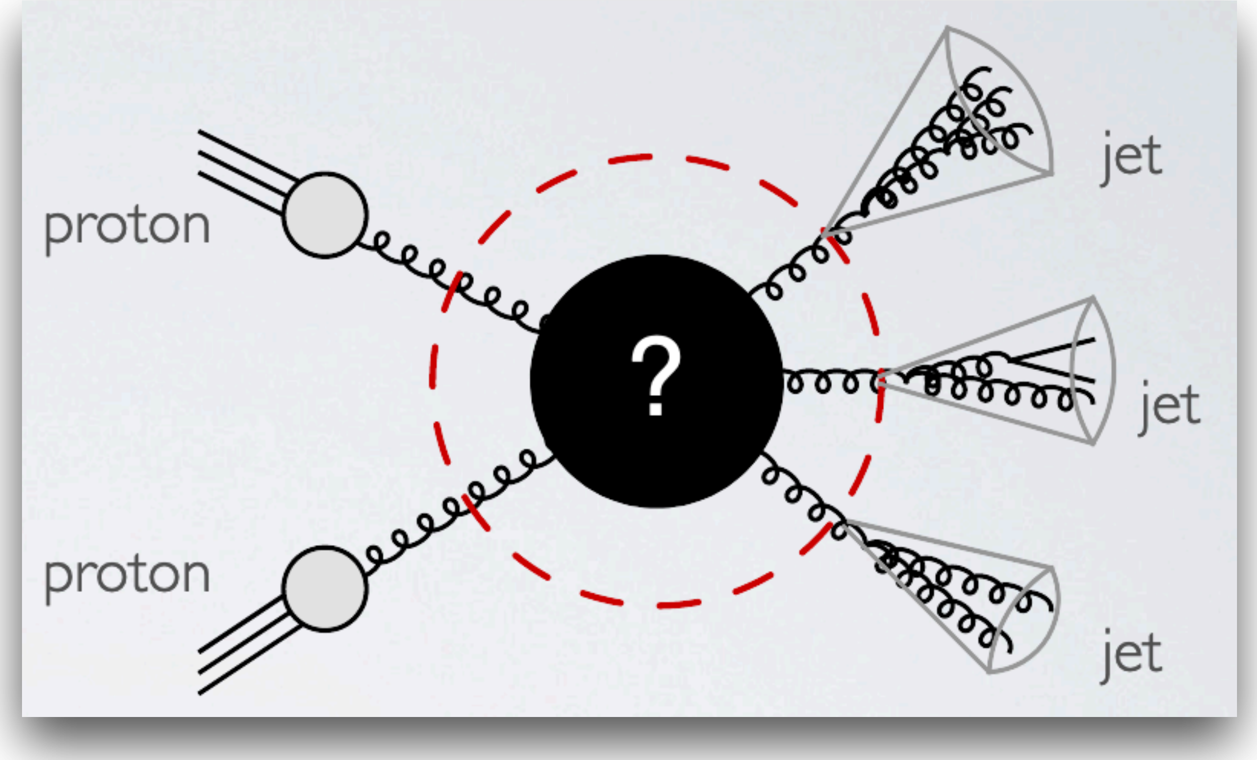
  Simon Badger, Heribertus Bayu Hartanto, **SZ** 2102.02516

  Heribertus Bayu Hartanto, Rene Poncelet, Andrei Popescu, **SZ** 2205.01687

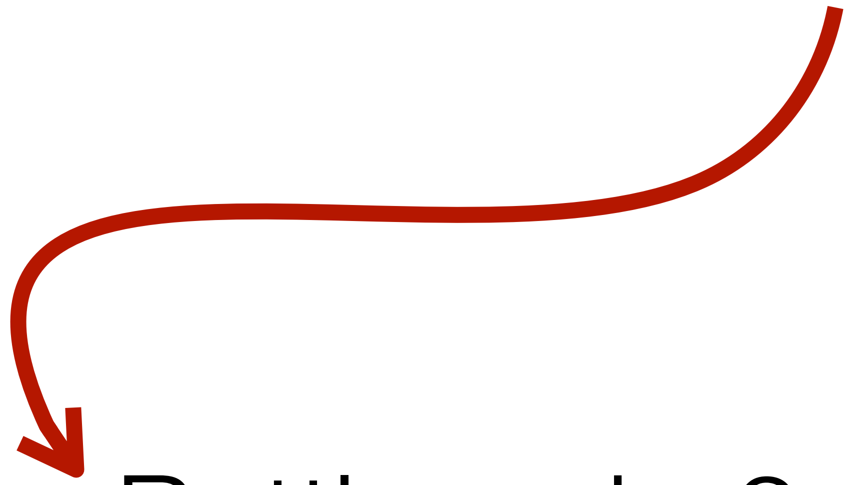
Urgent demand for NNLO QCD for LHC physics

Many observables probed at percent-level precision

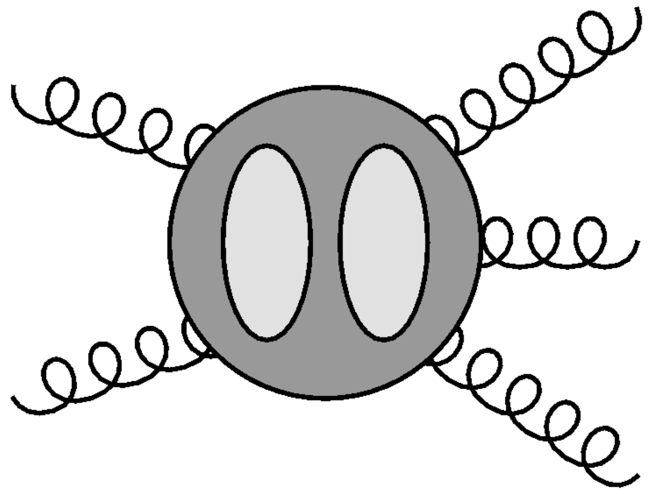
We **must** keep the theoretical uncertainties in line with the experimental ones



Current frontier: NNLO QCD corrections for $2 \to 3$ processes



Bottleneck: 2-loop 5-particle scattering amplitudes



Dramatic progress for massless 5-particle scattering

Feynman integrals

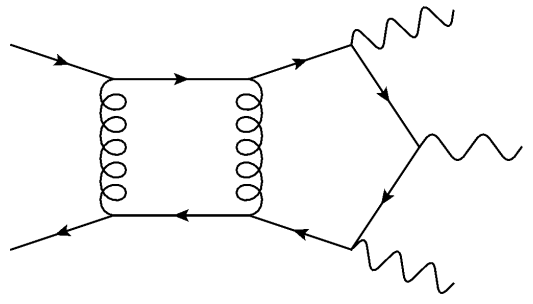
[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

Special function basis

[Gehrmann, Henn, Lo Presti ’18; Chicherin, Sotnikov ’20]

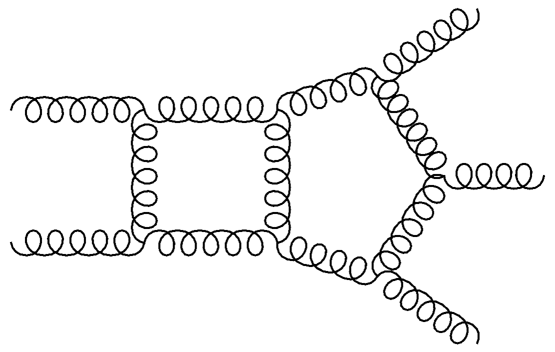
Scattering amplitudes

3γ
planar



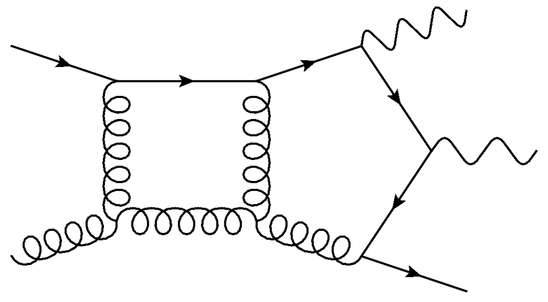
[Abreu, Page, Pascual, Sotnikov 2020;
Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$
planar

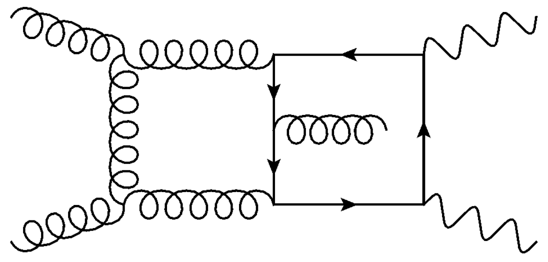


[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; Badger, Brönnum-Hansen, Bayu Hartanto, Peraro, Moodie, **SZ**, *to appear*]

$2\gamma + j$
full
colour



[Agarwal, Buccioni, von Manteuffel, Tancredi 2021 x2; Chawdhry, Czakon, Mitov, Poncelet 2021]



[Badger, Brönnum-Hansen, Chicherin, Gehrmann, B. Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

$d\sigma$ **@NNLO QCD:**

$pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]
$pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]
$pp \to 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

Five-particle scattering with one off-shell leg

$$pp \to \ V + 2j\,,\ V + b\bar{b}\,,\ H + 2j\,,\ H + b\bar{b}\,,\ V + \gamma j\,,\ V + \gamma\gamma \ldots$$

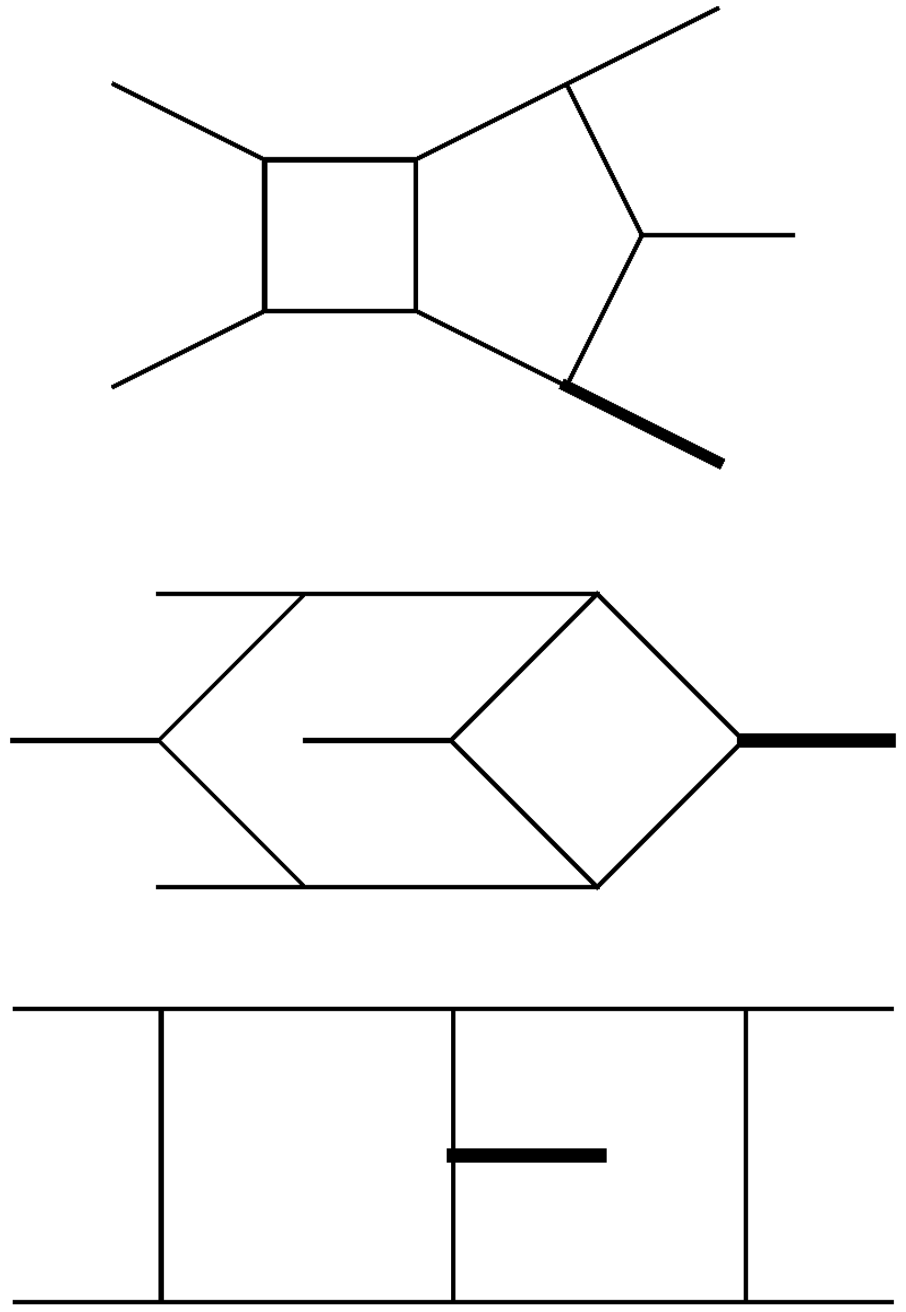
[from Les Houches 2021 “Precision wish-list”]

Rich potential phenomenology

Six independent variables & several square roots

Algebraic complexity of the amplitudes

Intricate analytic structure of the integrals



Phenomenology is very demanding

$$\textbf{amplitude} = \sum \textbf{Feynman diagrams}$$

?

$$\boxed{\textbf{amplitude} = \sum \textbf{rational coeffs} \times \textbf{special funcs}}$$

Analytic cancellation of UV/IR poles

Compact analytic expressions

Fast/stable evaluation across physical phase space

Amplitude workflow

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \textbf{Feynman diagram}_i$$

IBP reduction

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\, \textbf{MI}_i\left(\{p\},\epsilon\right)$$

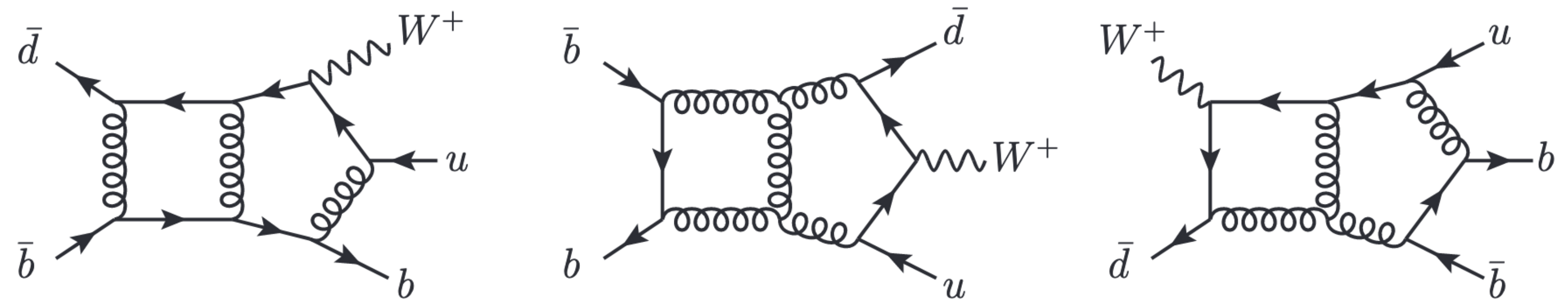
ϵ expansion

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_{w=-4}^{\infty} \epsilon^w \sum_i c_i^{(w)}\left(\{p\}\right)\, \textbf{mon}_{w,i}\,[f]$$

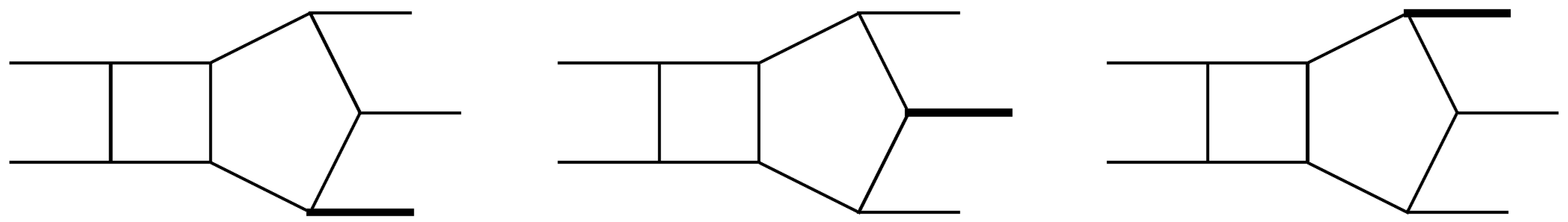
IR/UV subtraction

$$F^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\}\right)\, \textbf{mon}_i\,[f]$$

QGRAF



Master integrals



Special functions:

$\log(x)$, $\text{Li}_n(x)$, $G(a_1, \ldots; x)\ldots$

Amplitude workflow

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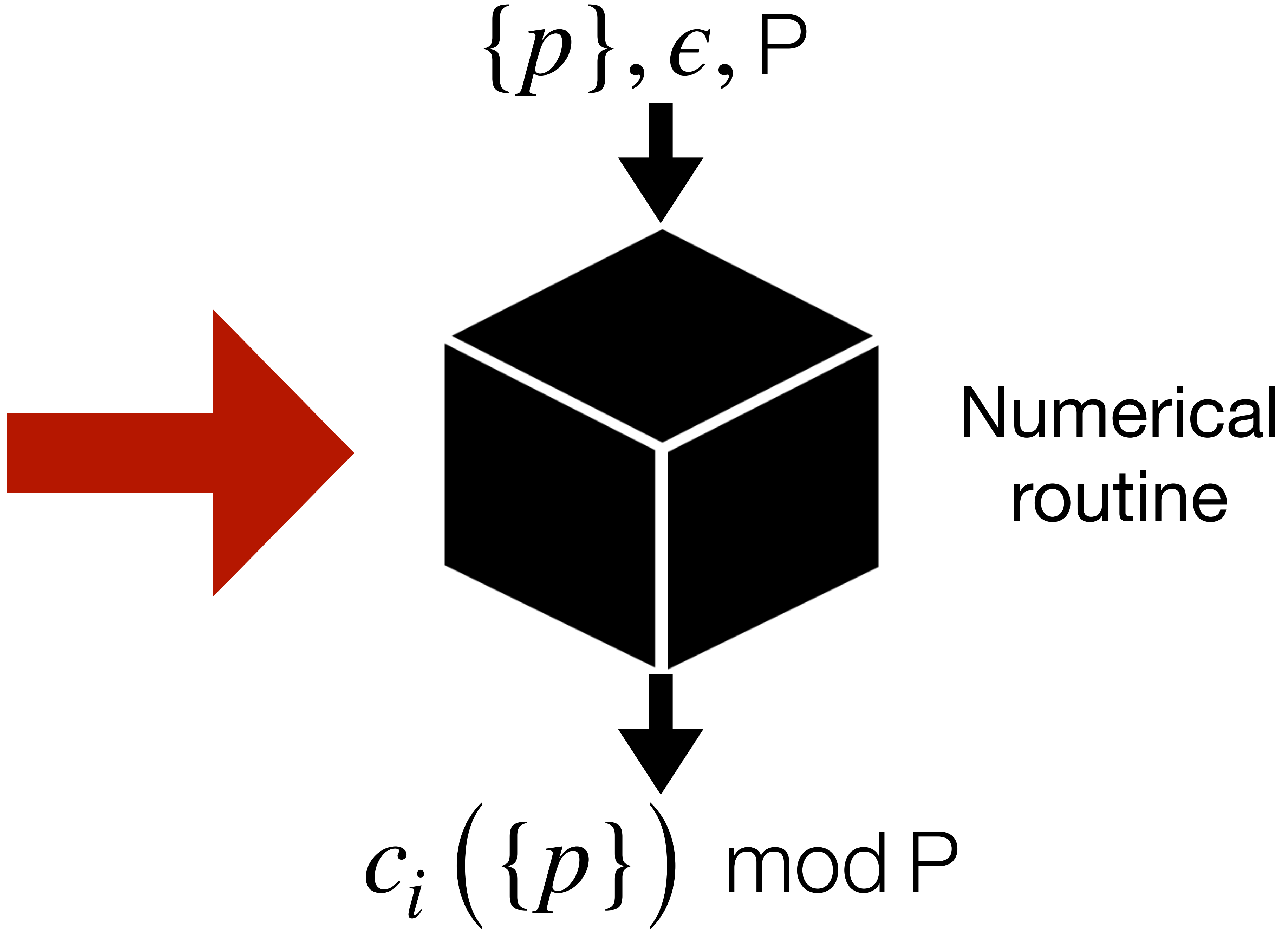


$\{p\}, \epsilon$, P

Numerical routine

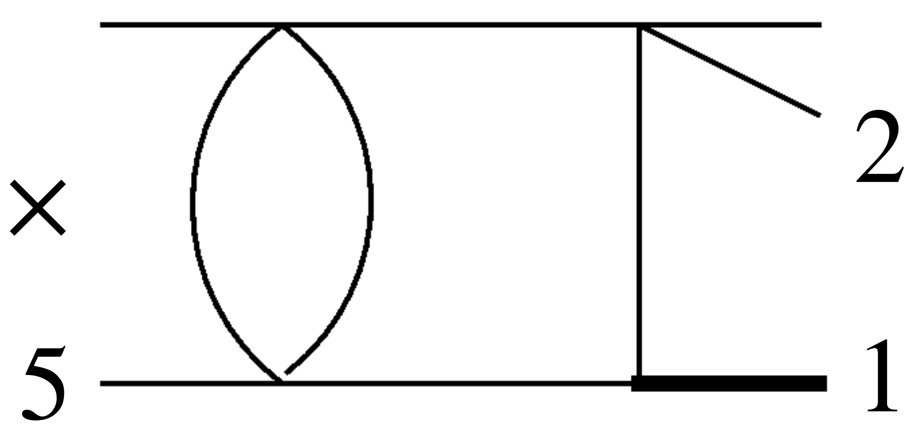
$c_i\left(\{p\}\right)$ mod P

Finite-field arithmetic + rational reconstruction in `FiniteFlow`
[Peraro 2019]

Master integrals

Special function basis

$$\epsilon^3(1-2\epsilon)\sqrt{\Delta_3^{(1)}}\times$$

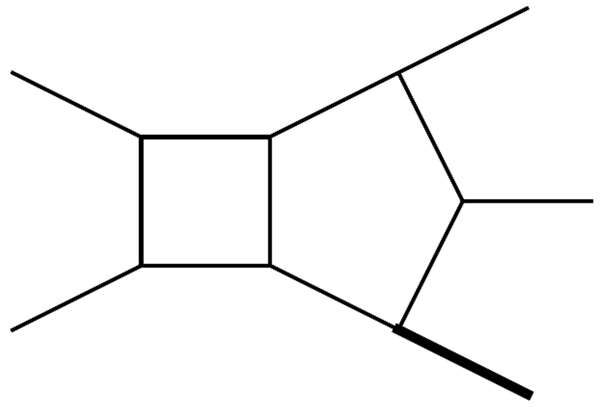
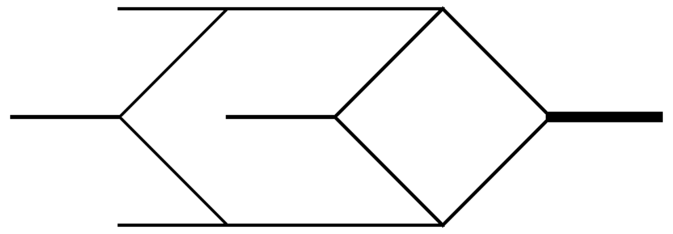
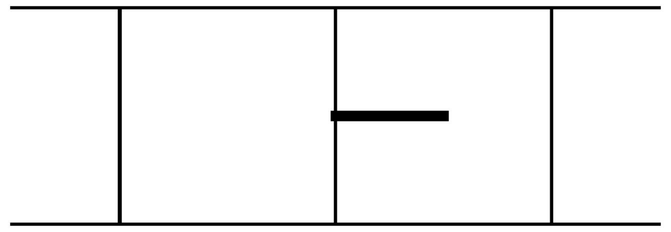


$$= \epsilon^2 f_{23}^{(2)} + \epsilon^3\left[\frac{1}{4}\left(f_1^{(1)} - f_6^{(1)}\right)f_{23}^{(2)} + \frac{1}{2}f_3^{(3)} - \frac{1}{2}f_{29}^{(3)}\right] + \epsilon^4 f_{47}^{(4)} + \mathcal{O}\left(\epsilon^5\right)$$

Canonical DEs for the master integrals

$$\boxed{d\,\overrightarrow{\mathbf{MI}}\left(\{p\},\epsilon\right) = \epsilon\left[\sum_i a_i\, d\log W_i\left(\{p\}\right)\right]\cdot\overrightarrow{\mathbf{MI}}\left(\{p\},\epsilon\right)}$$

[Henn 2013]

	Canonical DEs	**Multiple polylogs**	**Function basis**
	[Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020]	[Canko, Papadopoulos, Syrrakos 2020; Syrrakos 2020]	[Chicherin, Sotnikov, **SZ** 2021]
	[Abreu, Ita, Moriello, Page, Tschernow 2021]	[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022]*	
			

* MPLs + 1-fold integrations

Why a basis of special functions?

1. Get rid of all functional relations ⇒ analytic cancellations & simplifications

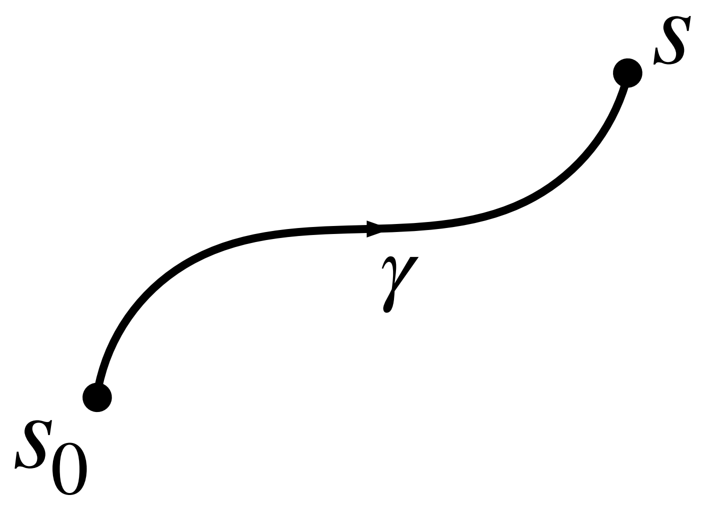
$$Li_2\,(z) + \frac{1}{2}\log^2(-z) + Li_2\left(\frac{1}{z}\right) + \frac{\pi^2}{6} = 0$$

➡ Simpler reconstruction & more compact expressions

2. Very efficient numerical evaluation

- Tailored representation in C++ library
- (Generalised power series expansion)

Chen’s iterated integrals



$$\left[w_{i_1},\ldots,w_{i_n}\right]_{s_0}(s) = \int_\gamma d\log w_{i_n}(s')\left[w_{i_1},\ldots,w_{i_{n-1}}\right]_{s_0}(s')$$

$n =$ transcendental weight

All functional relations become manifest in terms of iterated integrals

$$Li_2(z) + \frac{1}{2}\log^2(-z) + Li_2\left(\frac{1}{z}\right) + \frac{\pi^2}{6} = 0$$

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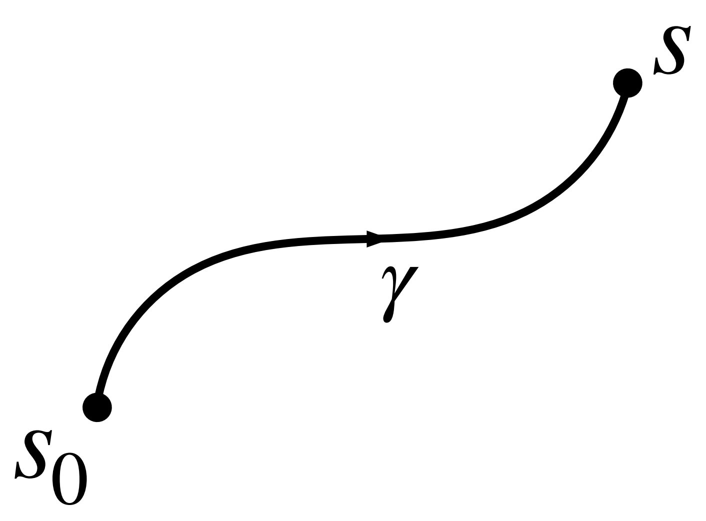

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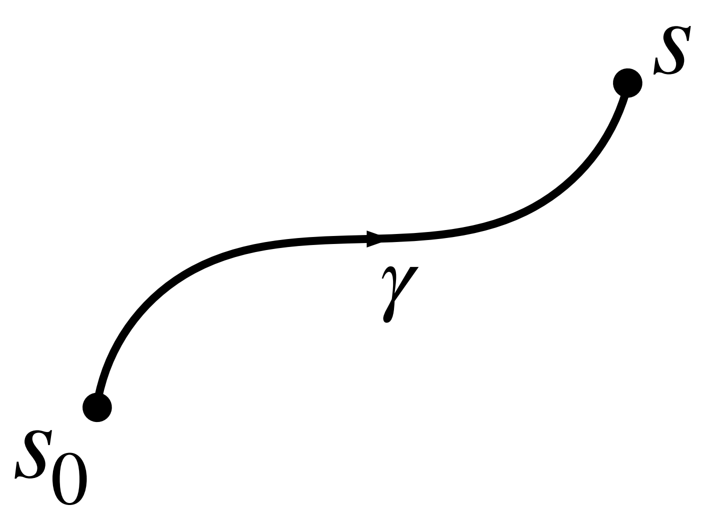

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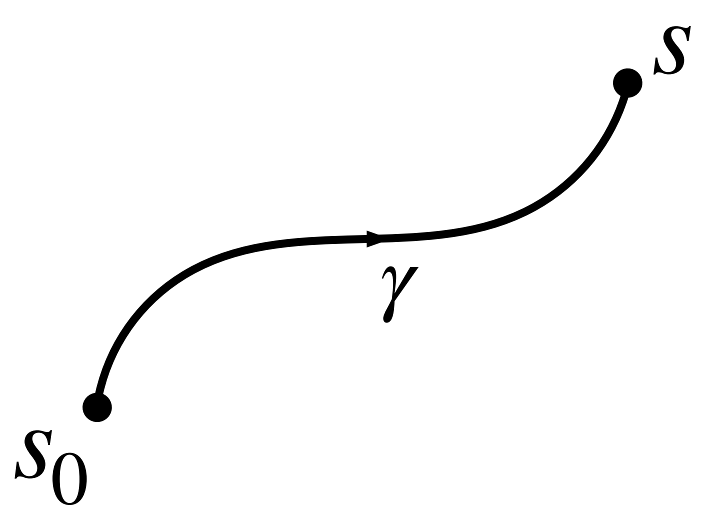

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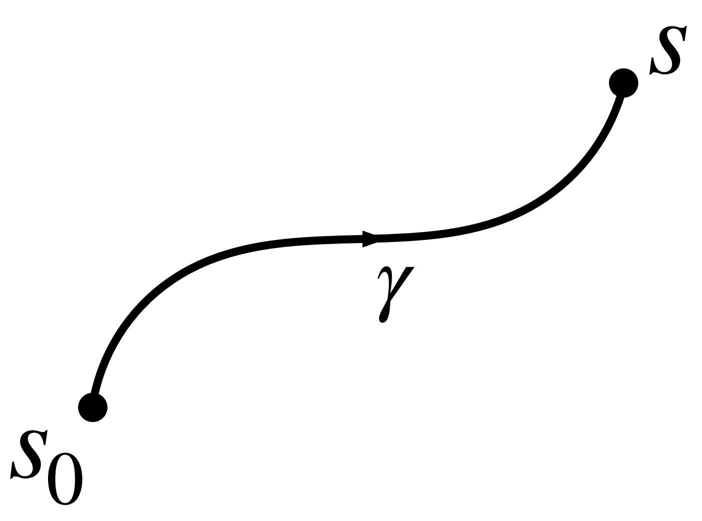

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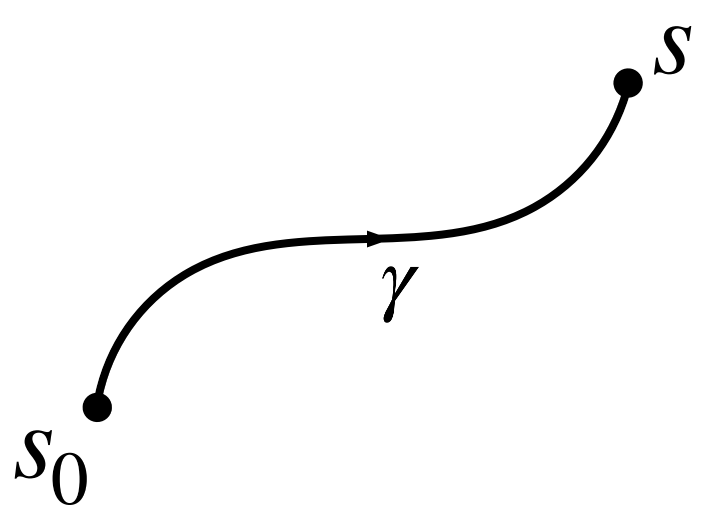

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$$\frac{1}{2}\log^2(-z) = \underline{[z,z]}_{-1}$$

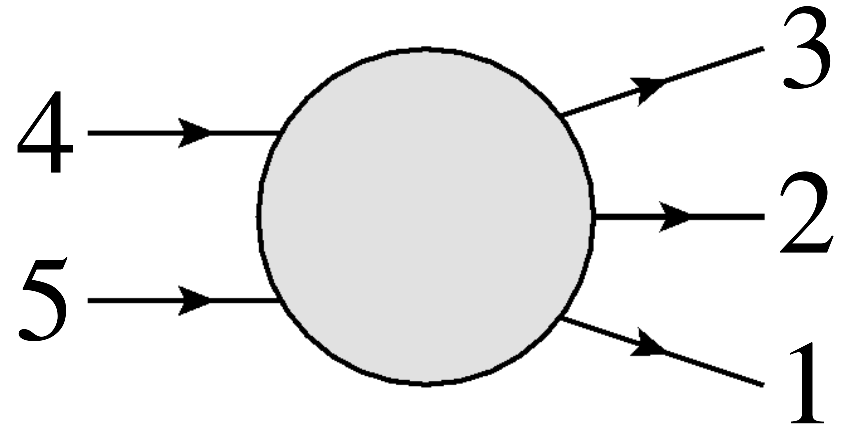
➥ Constructing a basis of special functions becomes a **linear algebra** problem

Efficient numerical evaluation

Weight 1 & 2: explicit expressions using *symbol* technology [Duhr, Gangl, Rhodes 2011]

$$f_2^{(1)} = \log\left(-s_{34}\right)$$

$$f_2^{(2)} = Li_2\left(\frac{s_{14}}{p_1^2}\right) + \log\left(-\frac{s_{14}}{p_1^2}\right)\log\left(1-\frac{s_{14}}{p_1^2}\right) + i\pi\log\left(s_{15}-s_{23}+s_{45}\right) - i\pi\log\left(p_1^2\right)$$



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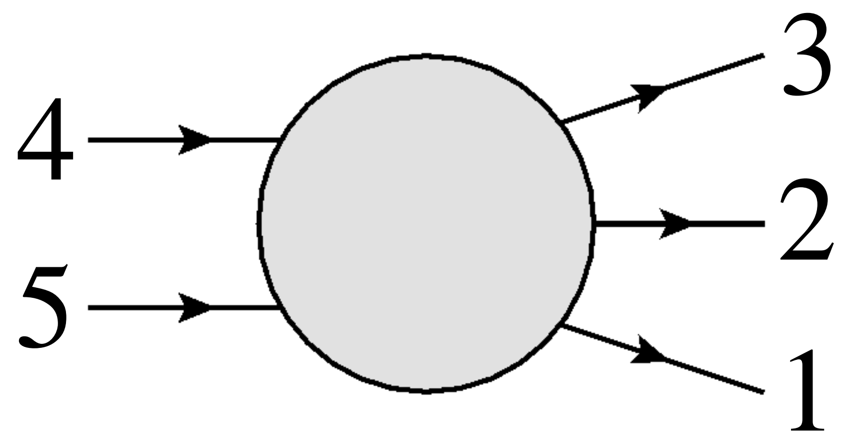
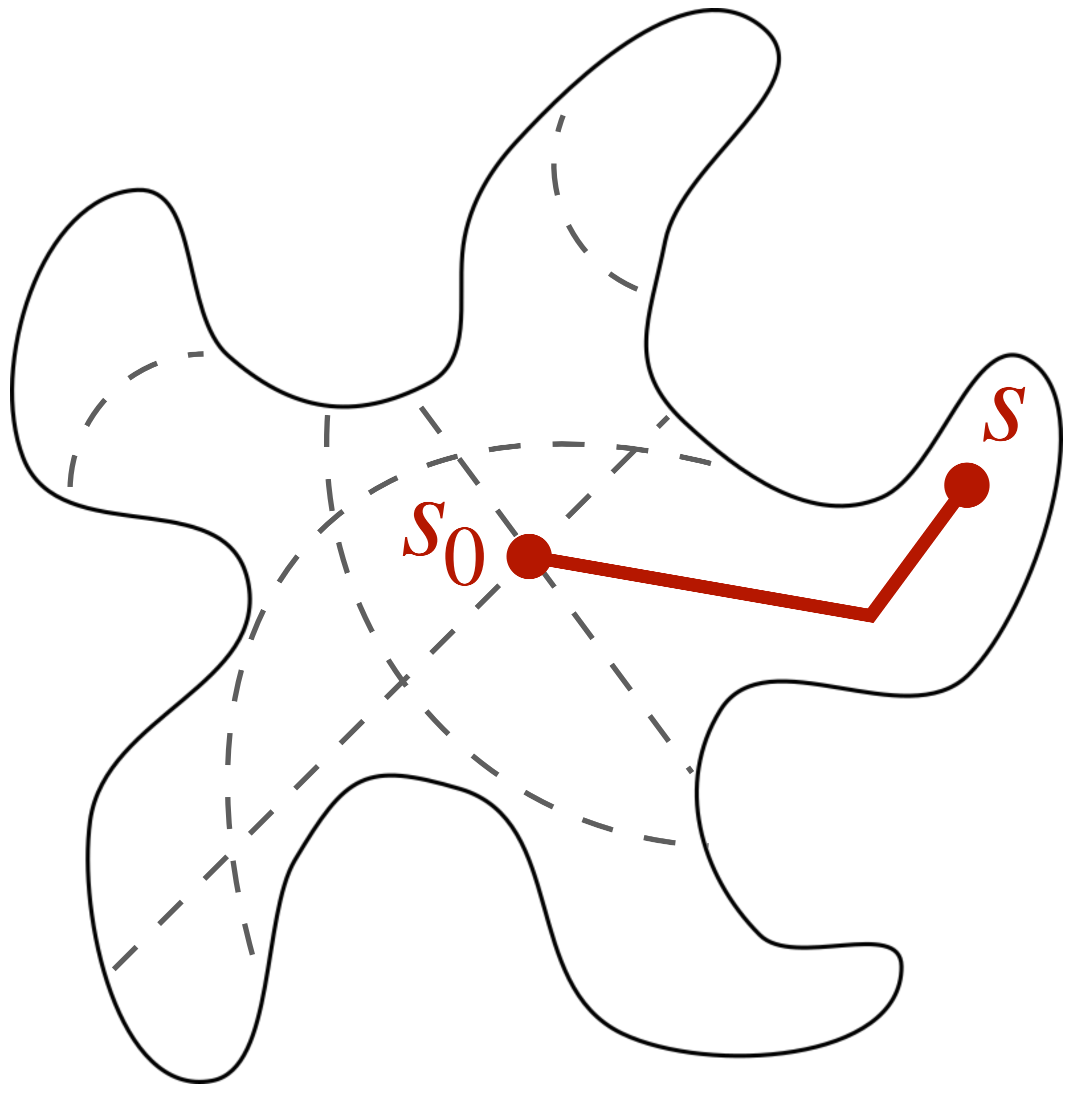


Weight 3 & 4: numerical integration of 1-fold integral representation with analytic integrands [Caron-Huot, Henn 2014]

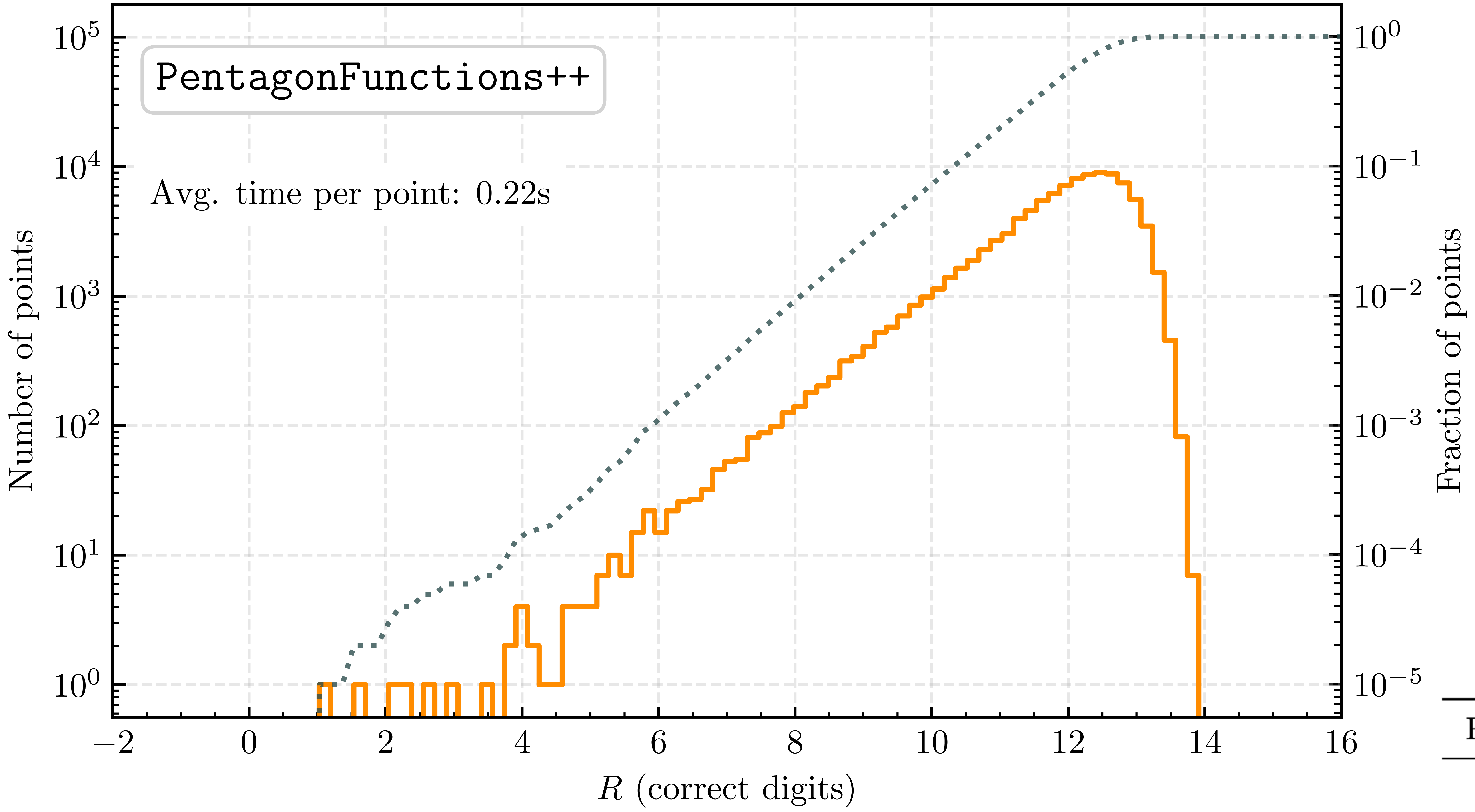
$$f^{(3)} \sim \int_0^1 dt \, \frac{d\log}{dt} \times f^{(2)} \qquad f^{(4)} \sim \int_0^1 dt \, \log \times \frac{d\log}{dt} \times f^{(2)}$$



s_{45} channel

Implemented in C++ library **PentagonFunctions++**

Efficient evaluation in the physical phase space

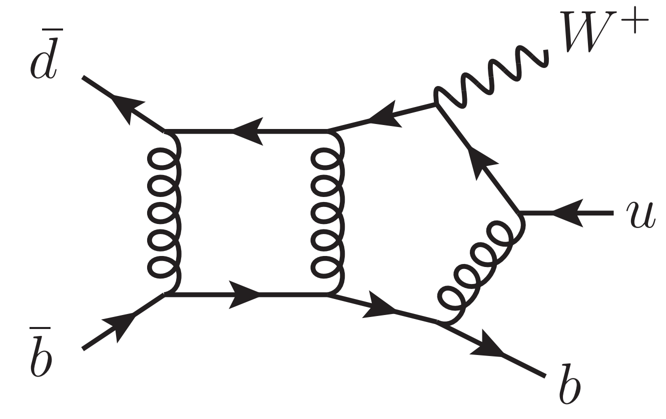


Precision	Correct digits	Timing (s)
double	12	0.19
quadruple	28	159
octuple	60	1695

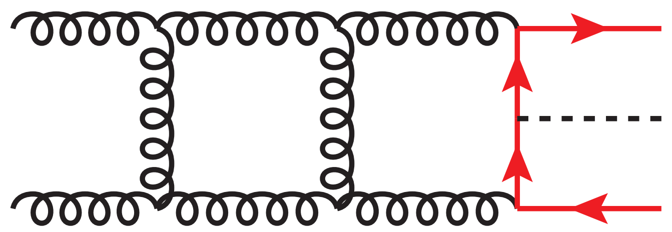
Ready for phenomenology

Scattering amplitudes

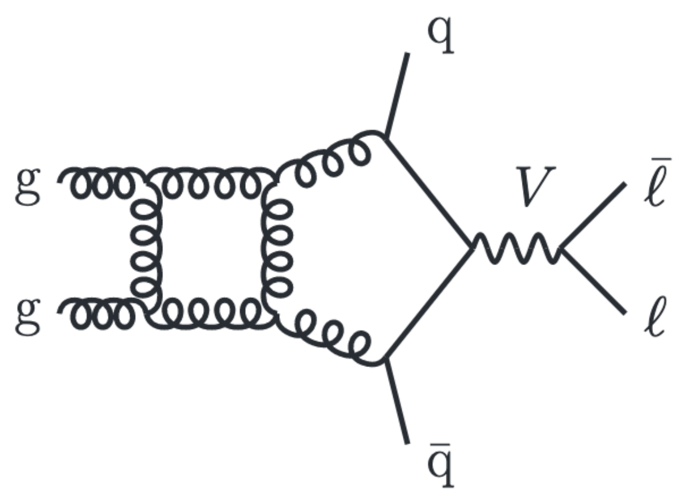
Spectacular progress for 2L 5pt 1-mass amplitudes @ leading colour



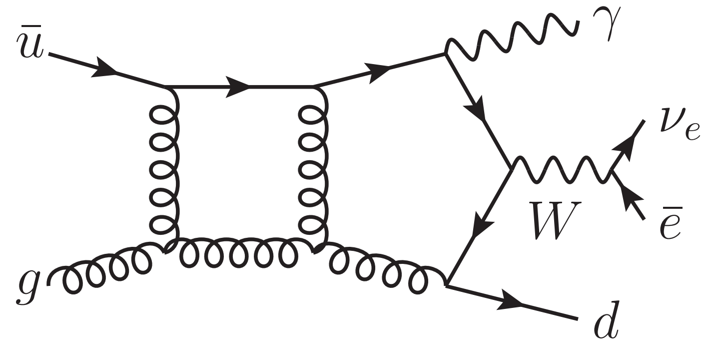
$pp \to Wb\bar{b}$ by Badger, Hartanto, **SZ** (2102.02516)



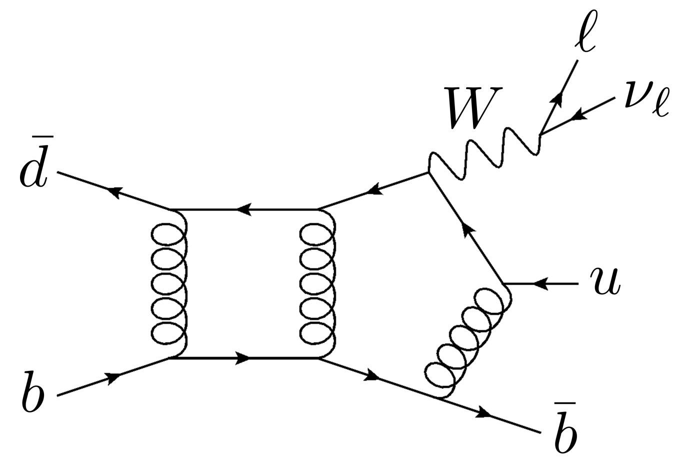
$pp \to Hb\bar{b}$ by Badger, Hartanto, Kryś, **SZ** (2107.14733)



$pp \to W(\to \ell\nu) + 2j$ by Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov (2110.07541)



$pp \to W(\to \ell\nu) + \gamma j$ by Badger, Hartanto, Kryś, **SZ** (2201.04075)



$pp \to W(\to \ell\nu) + b\bar{b}$ by Hartanto, Poncelet, Popescu, **SZ** (2205.01687)

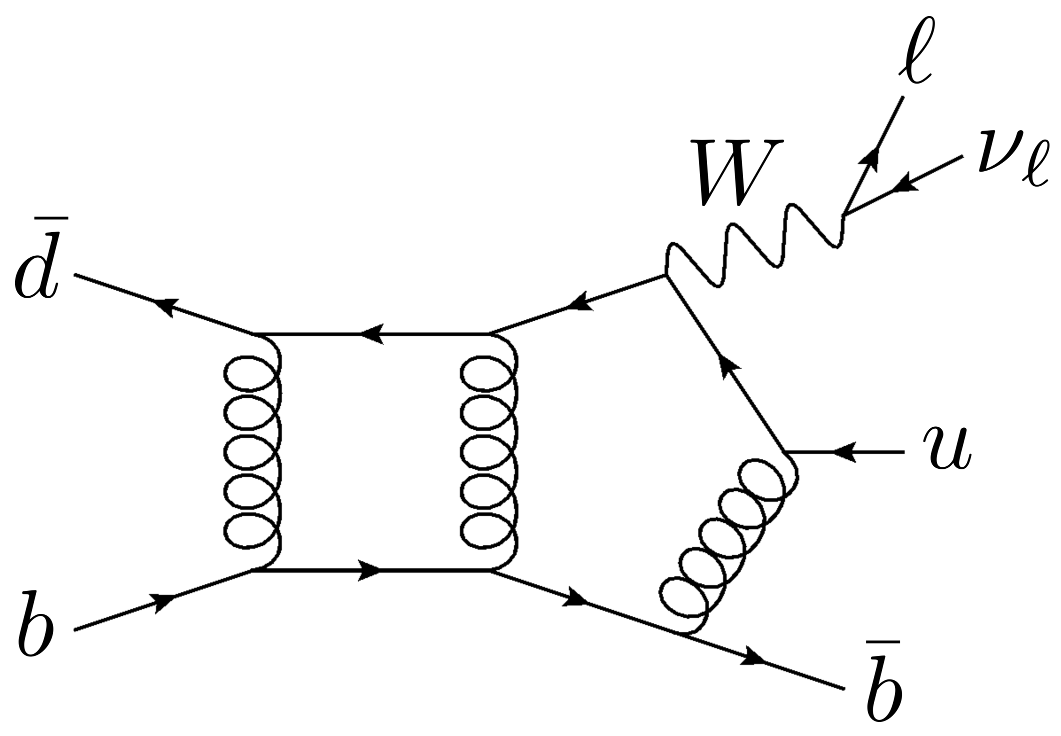
$W(\rightarrow \ell\nu) + b\bar{b}$ production at hadron colliders

Background to Higgs-strahlung $pp \rightarrow WH\,(H \rightarrow bb)$ and single top production $pp \rightarrow \bar{b}t\,(t \rightarrow Wb)$

Study b quark schemes: massless vs. massive

Large corrections and scale variation @ NLO [Febres Cordero, Reina, Wackeroth 2006, 2009; Badger, Campbell, Ellis 2010]

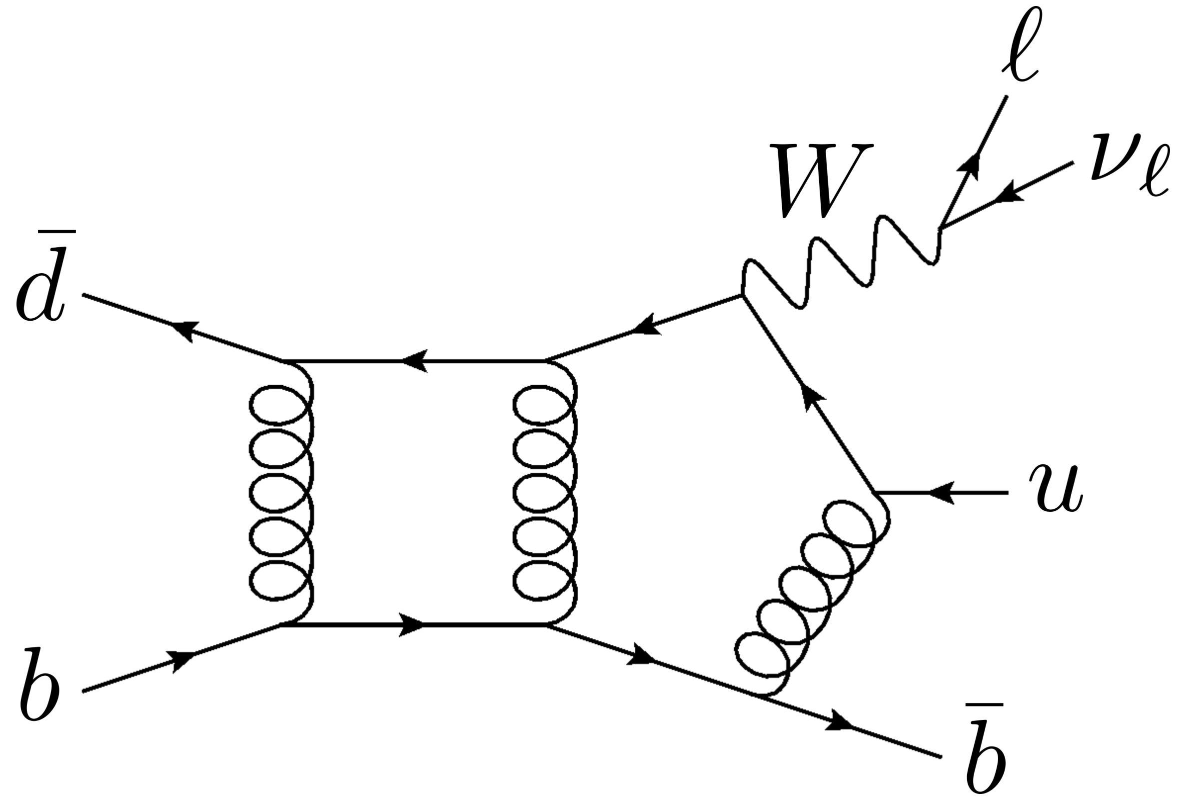
➪ NNLO QCD prediction needed



$W(\to \ell\nu) + b\bar{b}$ @ NNLO QCD

[Hartanto, Poncelet, Popescu, **SZ** 2022]

Leading colour approx for the 2-loop finite remainder



Massless b quark ⇒ flavour k_T jet algorithm ($R = 0.5$)

[Banfi, Salam, Zanderighi 2006]

`Stripper` (sector-improved residue subtraction scheme) + `OpenLoops2`

[Czakon 2010; Czakon, Heymes 2014]

[Buccioni, Lang, Lindert,
Maierhöfer, Pozzorini,
Zhang, Zoller 2019]

Setup follows CMS measurement [arXiv:1608.07561]

At least 2 b jets

Exactly 2 b jets, no other jets

	inclusive [fb]	$\mathcal{K}_{\mathrm{inc}}$	exclusive [fb]	$\mathcal{K}_{\mathrm{exc}}$
σ_{LO}	$213.2(1)^{+21.4\%}_{-16.1\%}$	-	$213.2(1)^{+21.4\%}_{-16.1\%}$	-
σ_{NLO}	$362.0(6)^{+13.7\%}_{-11.4\%}$	1.7	$249.8(4)^{+3.9(+27)\%}_{-6.0(-19)\%}$	1.17
σ_{NNLO}	$445(5)^{+6.7\%}_{-7.0\%}$	1.23	$267(3)^{+1.8(+11)\%}_{-2.5(-11)\%}$	1.067

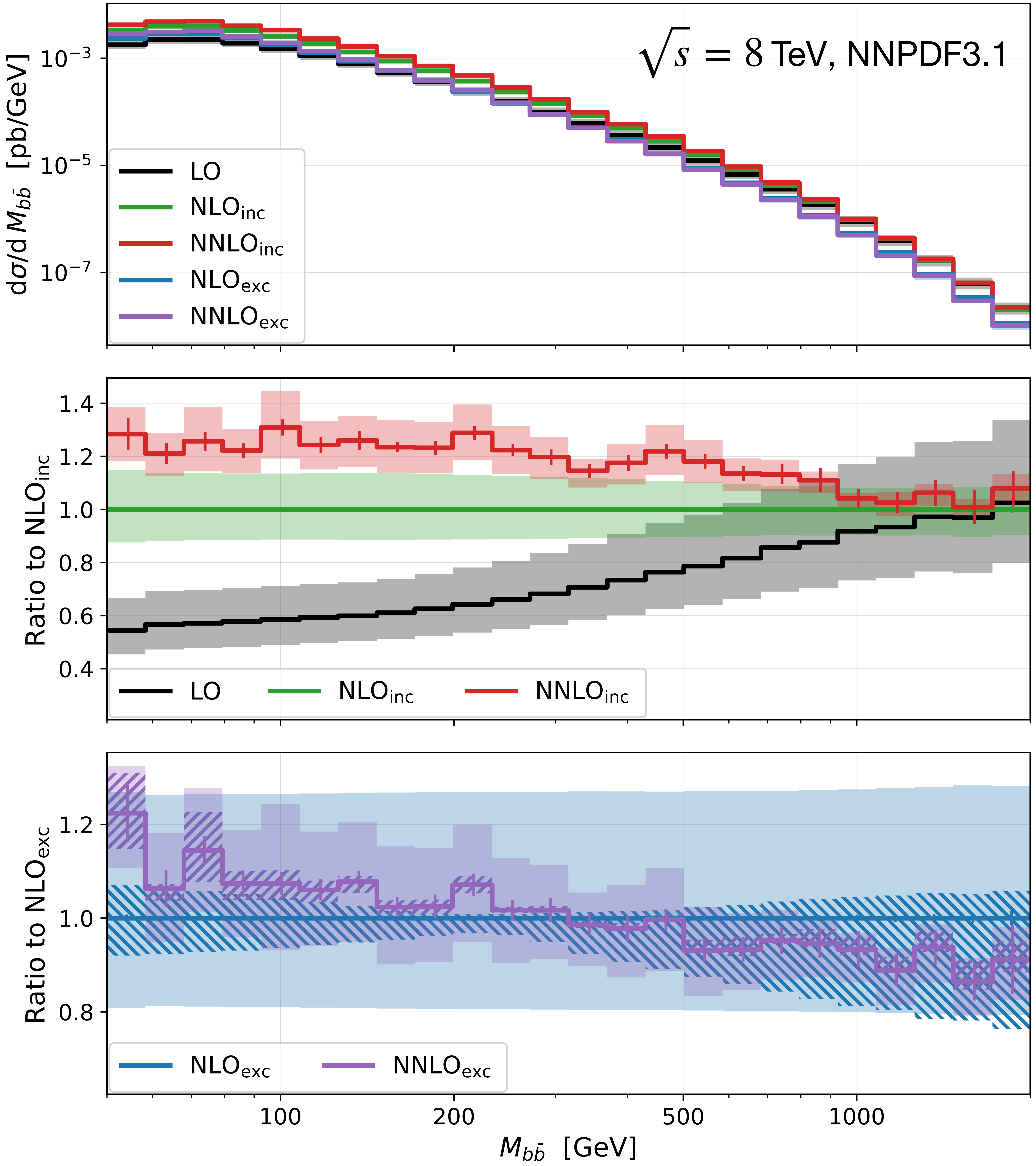
7-point scale variation

Uncorrelated prescription
[Stewart, Tackmann 2012]

Double virtual contributions: 5% (inc) and 10% (exc)

Perturbative convergence

Pentagon functions meet pheno demands



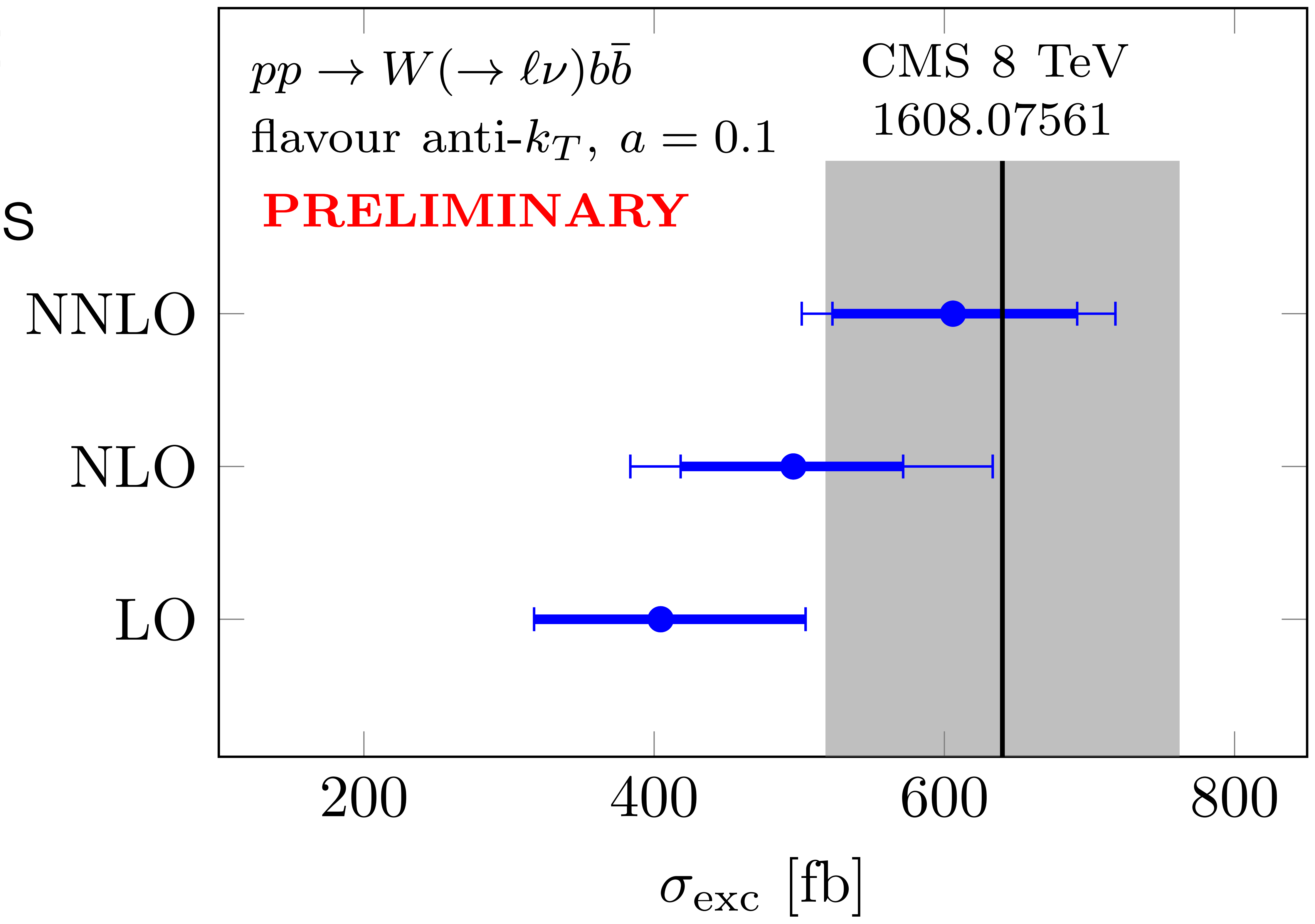
Improved agreement with CMS data

[work in progress with Hartanto, Poncelet, Popescu]

Use newly proposed **anti-kT flavoured jet algorithm** [Czakon, Mitov, Poncelet '22] to compare directly against the measurements

Hadronisation (0.81 ± 0.07) and DPI correction (0.06 ± 0.06 pb) factors included

Thick error bar → 7-pt scale variation, thin error bar → uncorrelated prescription



Conclusions

Function basis for all planar 2-loop 5-particle amplitudes with 1 mass (& C++ library)

[Chicherin, Sotnikov, **SZ** 2021]

Several 2-loop amplitudes computed at leading colour:

$$pp \to \boxed{W(\to \ell\nu) + b\bar{b}}, \ Hb\bar{b}\,, \ W(\to \ell\nu) + \gamma j, W(\to \ell\nu) + 2j$$

First NNLO QCD prediction for a 2 → 3 process with a massive external particle

[Hartanto, Poncelet, Popescu, **SZ** 2022]

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[Hartanto, Poncelet, Popescu, **SZ** 2022]

Thank you!

Back-up slides

$W(\to \ell\nu) + b\bar{b}$ @ NNLO QCD

[Hartanto, Poncelet, Popescu, **SZ** 2022]

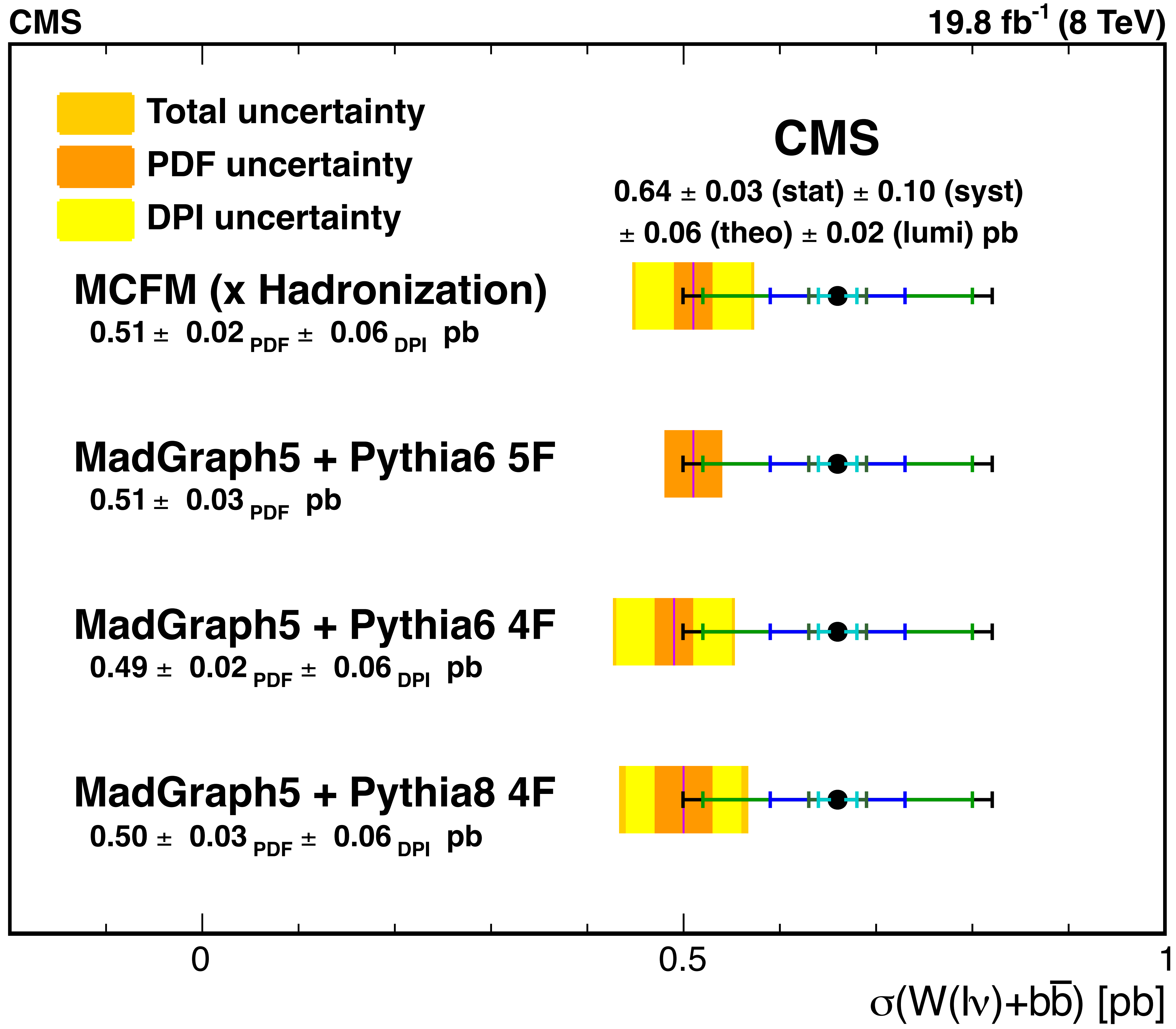
Event-selection criteria for jets and charged leptons from CMS 2017:

$$p_{\mathrm{T},\ell} > 30\,\mathrm{GeV}\,, \quad |\eta_\ell| < 2.1\,,$$

$$p_{\mathrm{T},j} > 25\,\mathrm{GeV}\,, \quad |\eta_j| < 2.4$$

Central renormalisation and factorisation scales:

$$\mu_R = \mu_F = E_T(\nu\ell) + p_T(b_1) + p_T(b_2)$$



CMS arXiv:1608.07561

Flavoured anti-kT jet algorithm

[Czakon, Mitov, Poncelet ’22]

The standard anti-kT distance is multiplied by a damping function $\mathcal{S}_{ij}$ if both pseudo-jets i and j have the same non-zero flavour of opposite sign

$$\mathcal{S}_{ij} = 1 - \Theta(1-x)\,\cos\left(\frac{\pi}{2}x\right) \qquad x = \frac{1}{a}\frac{k_{T,i}^2 + k_{T,j}^2}{2\,k_{T,max}^2}$$

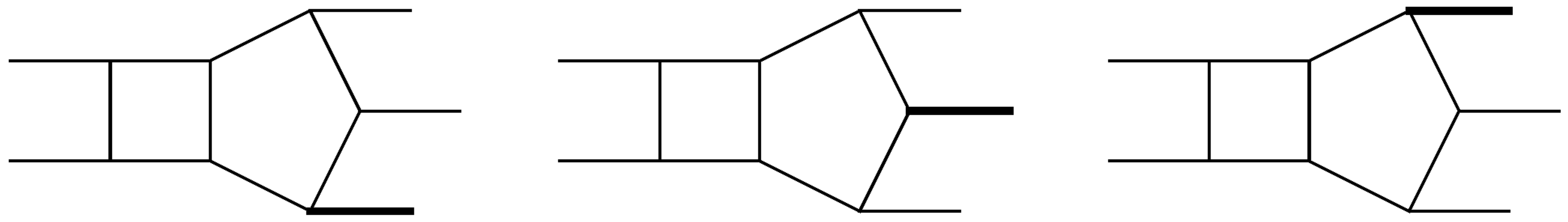
Tunable “softness” parameter

Planar alphabet

[Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020]

$$d\,\overrightarrow{\mathbf{MI}}(s,\epsilon) = \epsilon\, d\tilde{A}\,(s)\cdot \overrightarrow{\mathbf{MI}}\,(s,\epsilon)$$

$$\tilde{A}\,(s) = \sum_i a_i \log w_i\,(s)$$



+ permutations

$$s = \left(p_1^2\,, s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\,, tr_5\right)$$

$$tr_5 = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

156 letters, 4 square roots

(Massless: 31 letters, 1 square root)

$$W_1 = p_1^2\,,$$

$$\{W_2,\ldots,W_5\} = \{\sigma\,(s_{12}) : \sigma \in S_4/S_3[3,4,5]\}\,,$$

$$\{W_{118},\ldots,W_{123}\} = \left\{\sigma\left(\frac{s_{12}+s_{13}+\sqrt{\Delta_3^{(1)}}}{s_{12}+s_{13}-\sqrt{\Delta_3^{(1)}}}\right) : \sigma \in S_4/(S_2[2,3]\times S_2[4,5])\right\}$$

$$\Delta_3^{(1)} = (p_1^2)^2 + s_{23}^2 + s_{45}^2 - 2p_1^2 s_{23} - 2s_{23}s_{45} - 2s_{45}p_1^2$$

Numerical evaluation through generalised series expansion

Apply generalised series expansion method directly to the special functions

$$d\vec{f} = \epsilon \, d\tilde{B} \cdot \vec{f}$$

$$\vec{f} = \begin{pmatrix} \epsilon^4 f_i^{(4)} \\ \epsilon^3 f_i^{(3)} \\ \epsilon^2 f_i^{(2)} \\ \epsilon^1 f_i^{(1)} \\ 1 \end{pmatrix}$$

Much simpler than the DEs for the master integrals

Generalised series expansion implemented in **DiffExp** [Hidding 2020]

Evaluation in any kinematic region

Classical polylogarithms: $\dfrac{dLi_n(z)}{dz} = \dfrac{Li_{n-1}(z)}{z}$, $\quad Li_1(z) = -\log(1-z)$

$$d\begin{pmatrix} \epsilon^2 Li_2(z) \\ \epsilon^2 Li_2(1-z) \\ \epsilon \log z \\ \epsilon \log(1-z) \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & 0 & 0 & -d\log z & 0 \\ 0 & 0 & -d\log(1-z) & 0 & 0 \\ 0 & 0 & 0 & 0 & d\log(z) \\ 0 & 0 & 0 & 0 & d\log(1-z) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \epsilon^2 Li_2(z) \\ \epsilon^2 Li_2(1-z) \\ \epsilon \log z \\ \epsilon \log(1-z) \\ 1 \end{pmatrix}$$

$$\vec{f}\left(\frac{1}{2}\right) = \begin{pmatrix} \epsilon^2\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right) \\ \epsilon^2\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right) \\ -\epsilon \log 2 \\ -\epsilon \log 2 \\ 1 \end{pmatrix}$$

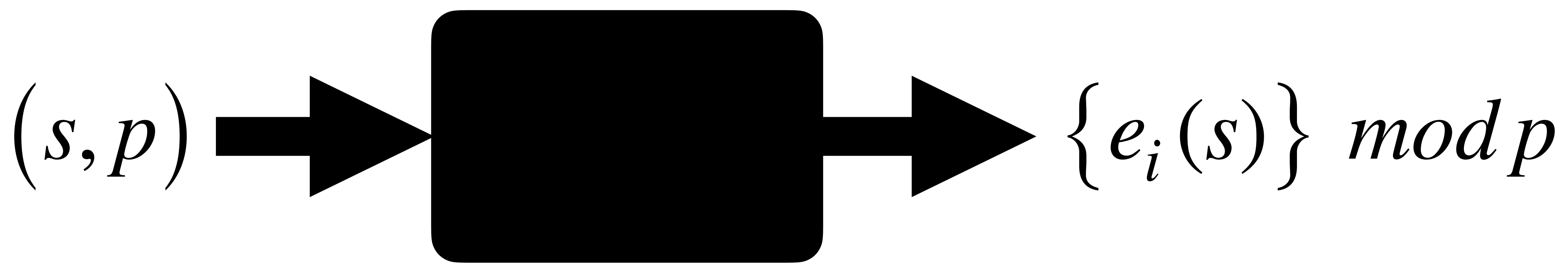
$$Li_2(z) = -\left[1-z, z\right]_{1/2} + \log(2)\left[z\right]_{1/2} + \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$$

$$\log(1-z) = \left[1-z\right]_{1/2} - \log(2)$$

Rational coefficients

$$\boxed{F^{(2)}(s,\epsilon) = \sum_i e_i(s)\ \text{mon}_i\left(\vec{f}\right)}$$

Black box interpolation



$$\left(s,p\right) \longrightarrow \blacksquare \longrightarrow \left\{e_i(s)\right\}\ mod\, p$$

Reconstructed in $\dfrac{\#\ \text{points}\ \times\ \text{eval. time}}{\#\ \text{cores}}$ ⌛

How to reduce the number of required sample points?

1. Make a good ansatz
2. Choose good variables

The denominators factorise into letters

[Abreu, Dormans, Febres Cordero, Ita, Page 2018]

$$\boxed{F^{(2)}\left(\{p\},\epsilon\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left(\vec{f}\right)}$$

$$e_i\,(s) = \frac{N_i(s)}{D_i(s)}$$

$$D_i = \prod_j w_j^{a_j}$$

Determined by reconstructing the coefficients on univariate phase-space slices

$$\Rightarrow \quad e_i\,(s) = \frac{N_i(s)}{D_i(s)}$$

Degrees known

Entirely known ✓

Can we use this information to construct an ansatz?

Univariate partial fraction decomposition

$$e(x,y) = \frac{-2x^4 - 4x^3y + 5x^2y^2 - xy^3 + 4y^4}{(x-y)y^2(x^2+y^2)} = -\frac{2x}{y^2} - \frac{6}{y} + \frac{1}{x-y} + \frac{3y}{x^2+y^2}$$

Construct ansatz based on the knowledge of degrees and denominators

$$e(x,y) = \frac{q_1(x)}{y^2} + \frac{q_2(x)}{y} + \frac{q_3(x)}{x-y} + \frac{q_4(x) + q_5(x)y}{x^2+y^2}$$

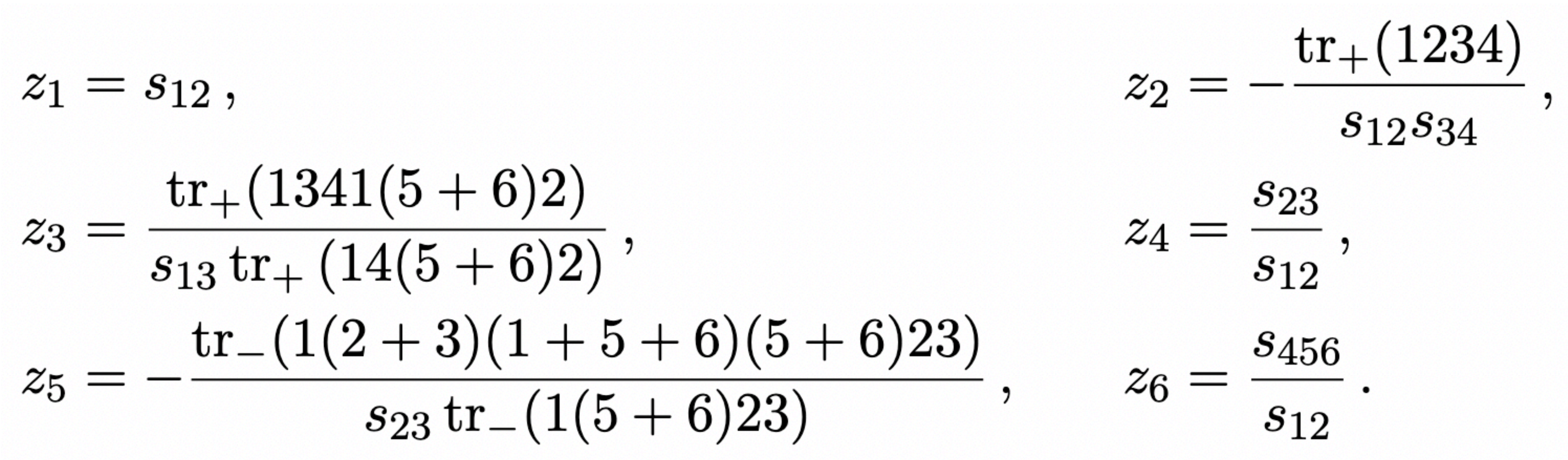
Linear fit to reconstruct the coefficients $q_i(x)$ →

- Lower degrees
- One fewer variables

Variable chosen by experimenting at one loop

Kinematic variables

- scalar invariants s_{ij} and tr_5
- momentum twistors
- spinor brackets $\langle ij \rangle,\ [ij]$
- other?



$$z_1 = s_{12}\,, \qquad z_2 = -\frac{\mathrm{tr}_+(1234)}{s_{12}s_{34}}\,,$$

$$z_3 = \frac{\mathrm{tr}_+(1341(5+6)2)}{s_{13}\,\mathrm{tr}_+\,(14(5+6)2)}\,, \qquad z_4 = \frac{s_{23}}{s_{12}}\,,$$

$$z_5 = -\frac{\mathrm{tr}_-(1(2+3)(1+5+6)(5+6)23)}{s_{23}\,\mathrm{tr}_-(1(5+6)23)}\,, \qquad z_6 = \frac{s_{456}}{s_{12}}\,.$$

Tune parameterisation to helicity configuration

For each helicity configuration, try all permutations of the momentum twistor variables @ 1 loop, and choose the one leading to the simplest expressions

→ **Huge** simplifications at 2 loops for $pp \to W\gamma j$ [Badger, Bayu Hartanto, Kryś, **SZ** 2022]

Solving the canonical DEs in terms of iterated integrals is trivial

$$\begin{cases} d\left[w_{i_1}, \ldots, w_{i_n}\right]_{s_0}(s) = d\log w_{i_n}(s)\,\left[w_{i_1}, \ldots, w_{i_{n-1}}\right]_{s_0}(s) \\ \left[w_{i_1}, \ldots, w_{i_n}\right]_{s_0}(s_0) = 0 \end{cases}$$

Chen's iterated integrals

Canonical DEs

$$\begin{cases} d\,\overrightarrow{\mathbf{MI}}^{(w)}(s) = \sum_i a_i\, d\log w_i(s)\; \overrightarrow{\mathbf{MI}}^{(w-1)}(s) \\ \overrightarrow{\mathbf{MI}}^{(w)}(s_0) = \boxed{\overrightarrow{\mathbf{MI}}_0^{(w)}} \end{cases}$$

Insensitive to square roots!

MPL-expressions + `PSLQ` algorithm

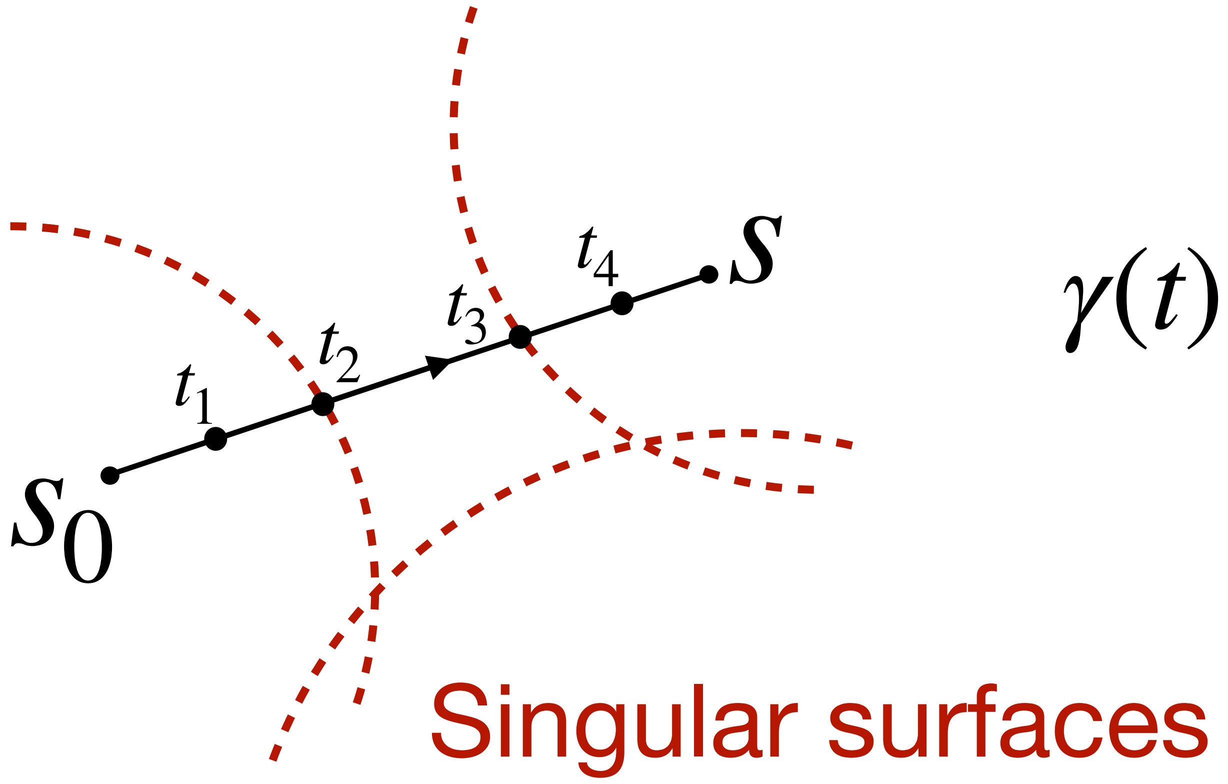
Series solution of the DEs

[Moriello 2019]

Integrate DEs along 1-dim. path γ

$$\overrightarrow{\mathbf{MI}}(t,\epsilon) := \overrightarrow{\mathbf{MI}}\left(s=\gamma(t),\epsilon\right)$$

$$\frac{d}{dt}\overrightarrow{\mathbf{MI}}(t,\epsilon) = \epsilon\, A(t)\cdot\overrightarrow{\mathbf{MI}}(t,\epsilon)$$



$$\gamma(t) = s_0 + t(s-s_0)$$

Generalised series solution around any point t_k

$$\overrightarrow{\mathbf{MI}}^{(w)}(t) = \sum_{j_1\geq 0}\sum_{j_2=0}^{w}\overrightarrow{c}_{j_1,j_2}(t-t_k)^{\frac{j_1}{2}}\log^{j_2}(t-t_k)$$

Compute solutions at various t_k and match them $\overrightarrow{\mathbf{MI}}(0,\epsilon)\longrightarrow\overrightarrow{\mathbf{MI}}(1,\epsilon)$

Boundary values from the MPL expressions

[Canko, Papadopoulos, Syrrakos 2020; Syrrakos 2020]

3000-digit precision using `GiNaC` [Vollinga, Weinzierl 2004]

`PSLQ` algorithm to find basis of transcendental constants

$$\begin{aligned} G(0,1;1) &= -1.644934067\ldots \\ G(3/2,2;1) &= 0.4060916335\ldots \\ G(3/2,1;1) &= 1.436746367\ldots \end{aligned} \quad \longrightarrow \quad 3\,G(0,1;1) + 4\,G(3/2,1;1) - 2\,G(3/2,2;1) = 0$$