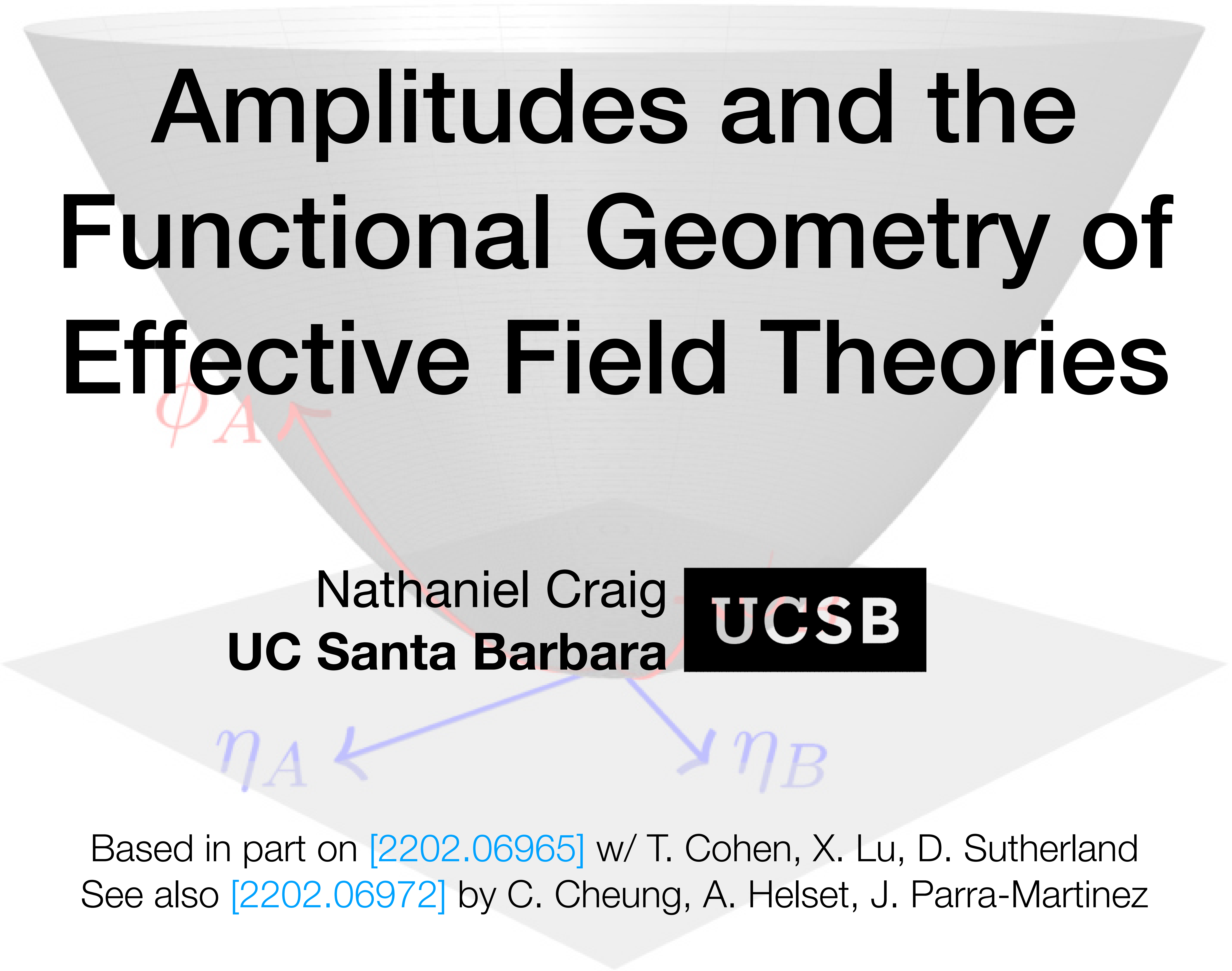
Amplitudes and the Functional Geometry of Effective Field Theories



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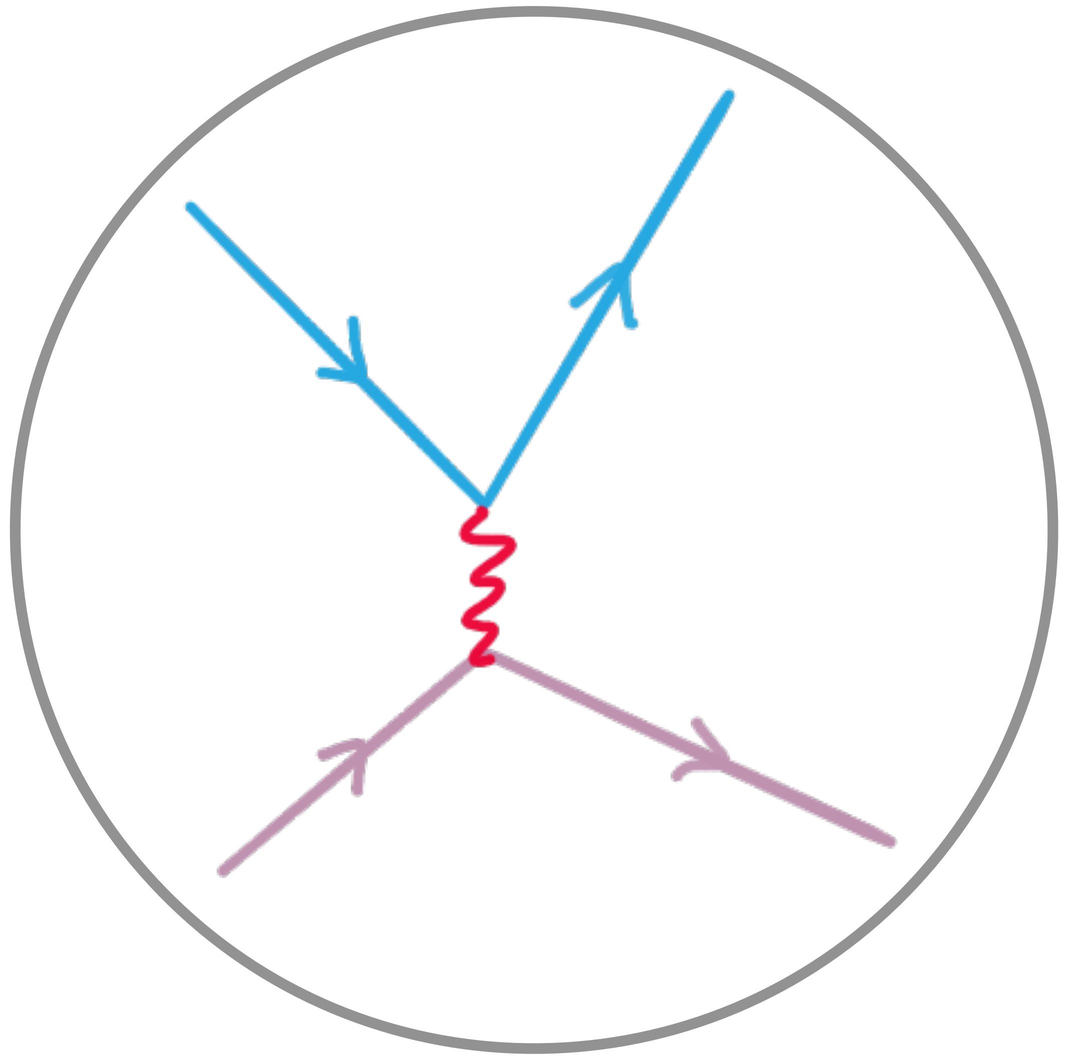
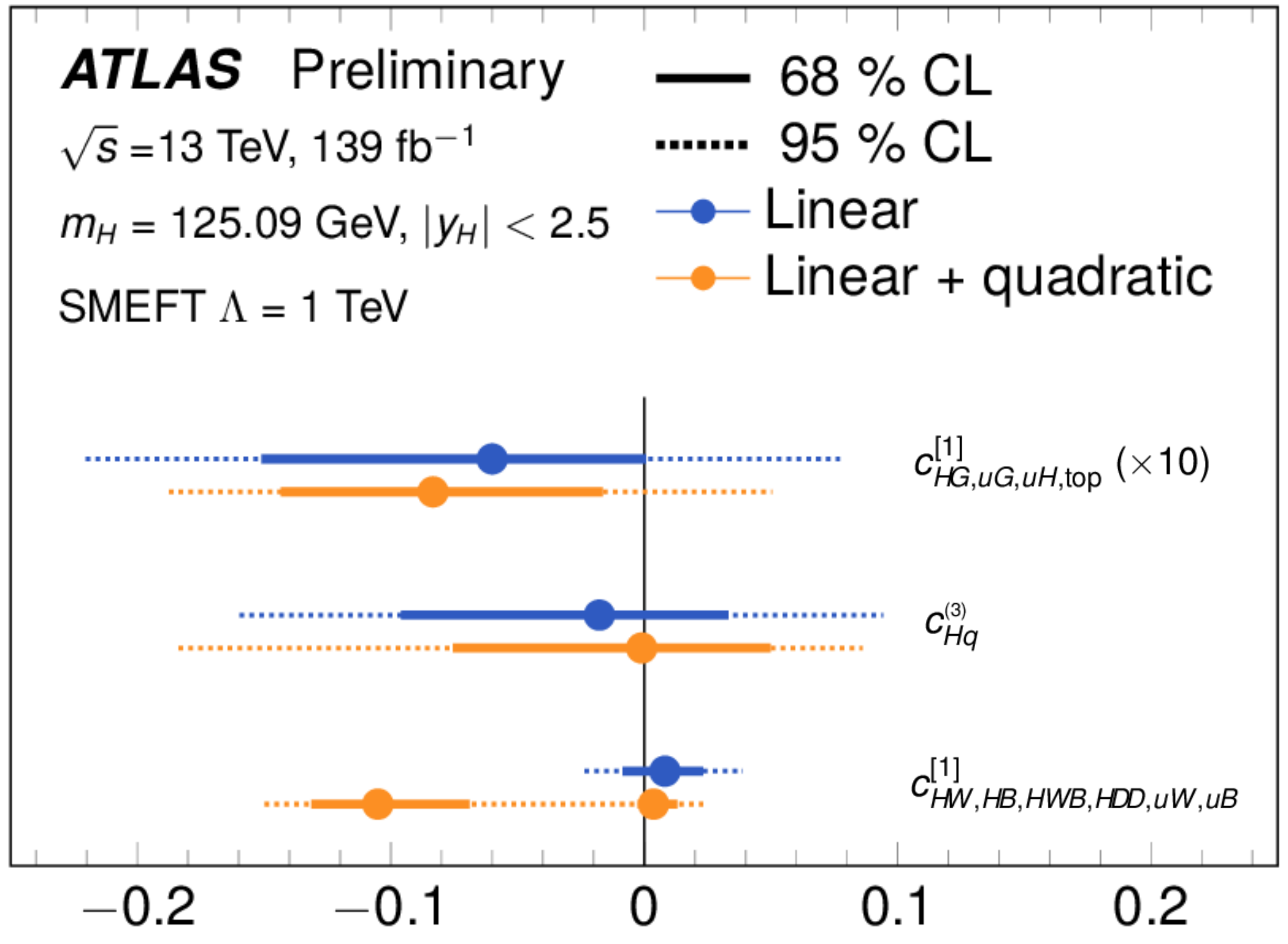
Based in part on [2202.06965] w/ T. Cohen, X. Lu, D. Sutherland
See also [2202.06972] by C. Cheung, A. Helset, J. Parra-Martinez



$$\mathcal{L}_{\rm SM} + \sum_i \frac{c_i}{\Lambda^{\Delta_i - 4}} \mathcal{O}_i$$



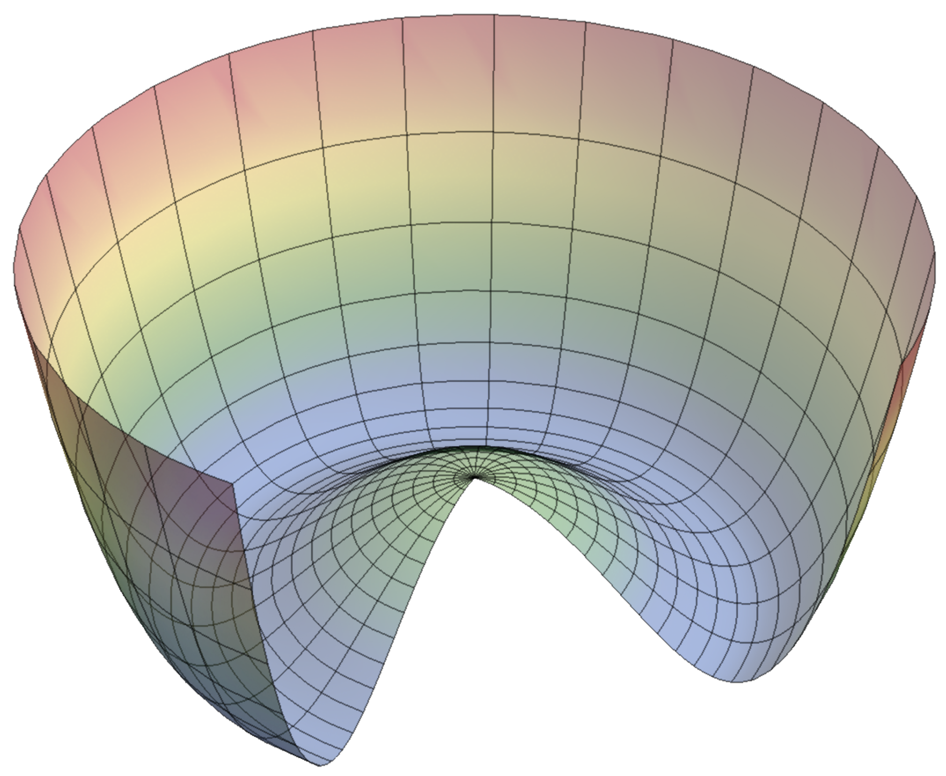
Geometry & EFT

Field space geometry for EFT

2-deriv. terms define metric on the scalar manifold:

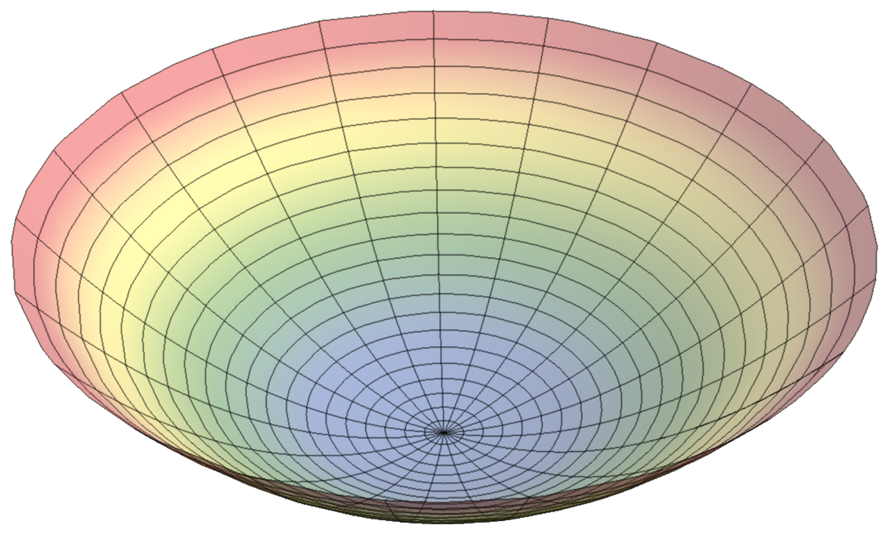
$$\mathcal{L} = -V + \frac{1}{2} g_{ab}[\phi](\partial_\mu \phi^a)(\partial^\mu \phi^b)$$

Fields ϕ are coordinates on space of field values ("target space").
Non-derivative field redefs are just coordinate transforms.



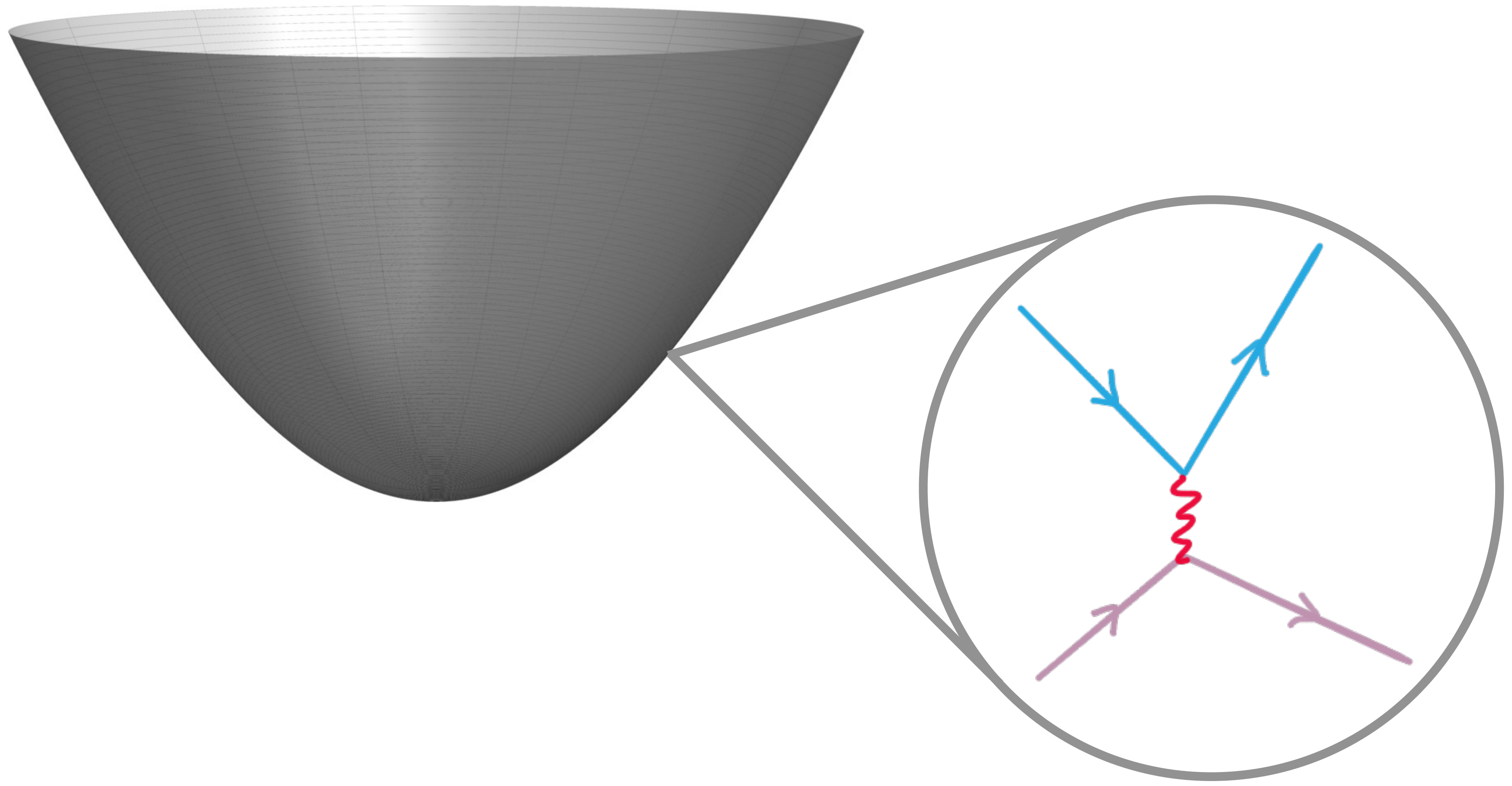
Use geometric quantities to connect Lagrangians and amplitudes, geometric invariants to classify EFTs, easily capture field redefs.

Long history (primarily) applied to nonlinear sigma models, e.g.
[Honerkamp '72; Tataru '75; Alvarez-Gaume, Freedman, Mukhi '81, …]



Application to EFTs of the SM [Alonso, Jenkins, Manohar '15 & '16; Helset, Martin, Trott '20; Cohen, NC, Lu, Sutherland '20;…]

Measuring Geometry



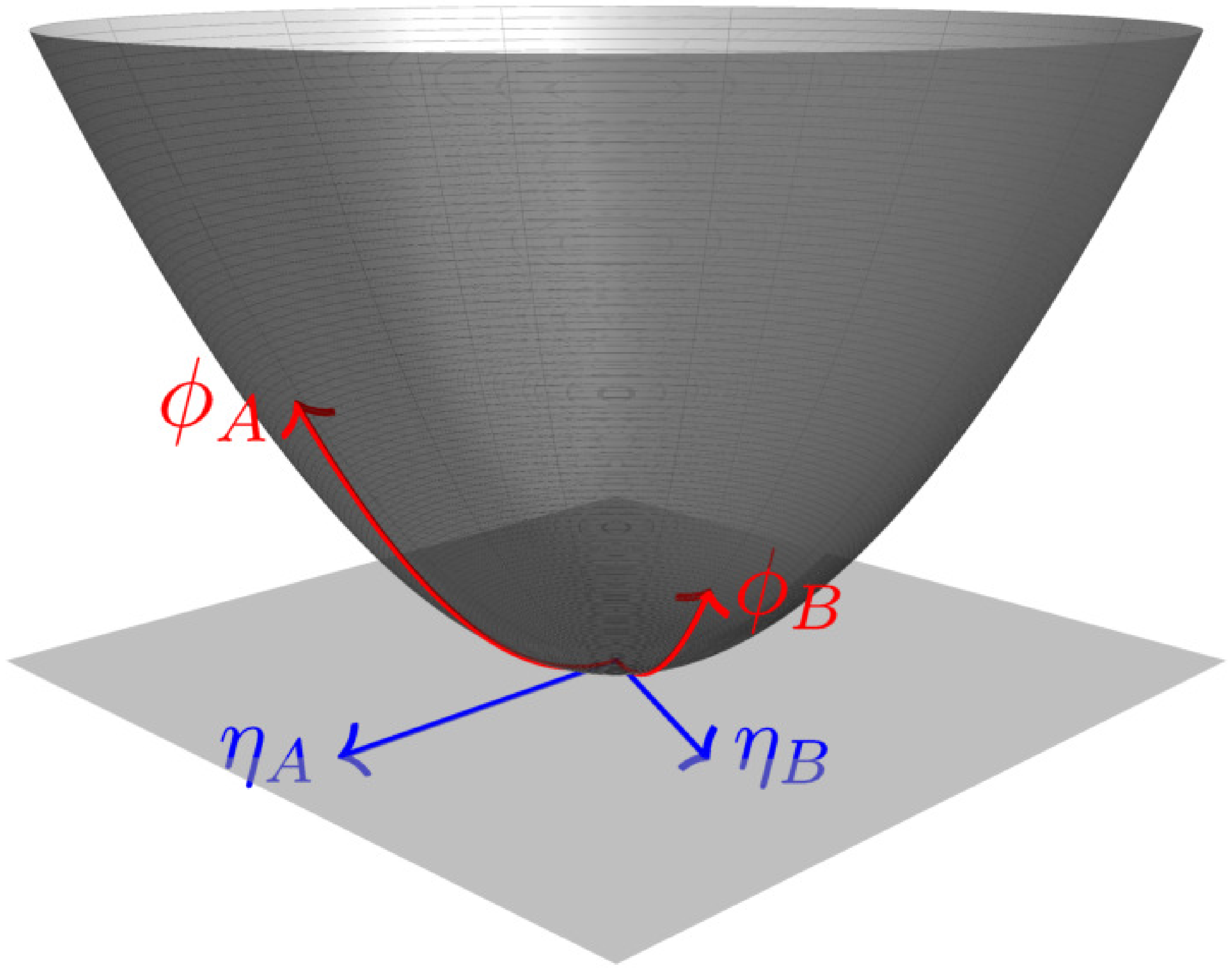
Amplitudes can be written in terms of geometric quantities on scalar manifold, e.g. for Higgs EFT [Alonso, Jenkins, Manohar '15; Nagai, Tanabashi, Tsumura, Uchida '19; Cohen, NC, Lu, Sutherland '21; Cheung, Helset, Parra-Martinez '21]

$$\mathcal{A}(\pi_i \pi_j \to hh) = -\delta_{ij} \mathcal{K}_h(h = \pi_k = 0) E^2 + \dots$$

Connection is transparent in **normal coordinates**

$$\phi^i = \eta^i - \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k + (\frac{1}{3}\Gamma^i_{jk}\Gamma^j_{lm} - \frac{1}{6}\Gamma^i_{kl,m})\eta^k\eta^l\eta^m + \mathrm{O}(\eta^4)$$

$$\mathcal{L}_\eta = \frac{1}{2}\partial\eta^i\partial\eta^j \left(g_{ij} - \frac{1}{3}(R_{ikjl} + R_{jkil})\eta^k\eta^l + \mathrm{O}(\eta^3)\right)$$

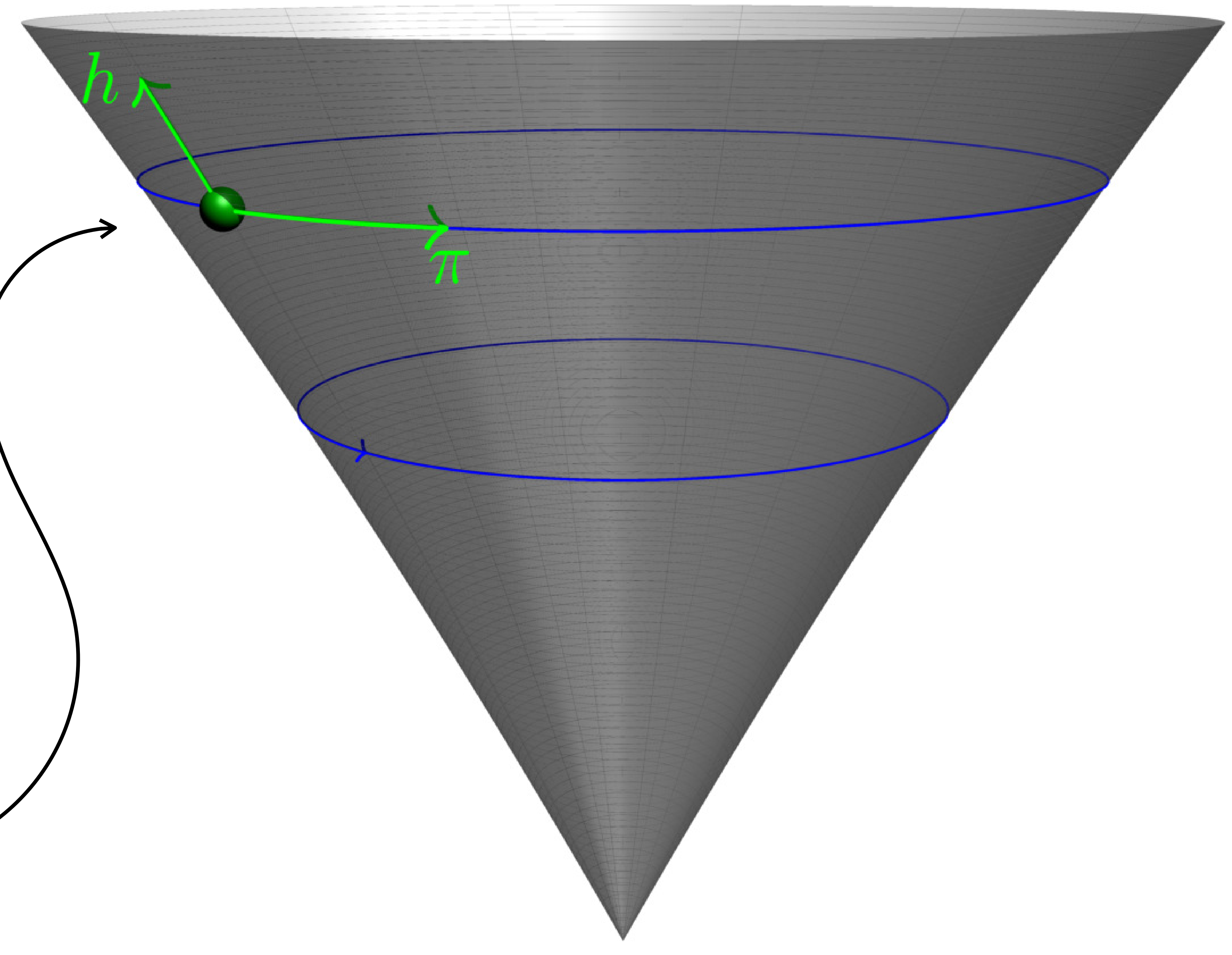


Geometry and Unitarity

Parts of $2 \to n > 2$ amplitudes that grow with energy are derivatives of sectional curvatures [Cohen, NC, Lu, Sutherland '21]

$$\mathcal{A}\left(\pi_i \pi_j \to h^n\right) = -E^2\, \delta_{ij}\, \partial_h^{n-2} \mathcal{K}_h|_{h=0} + \mathcal{O}(E^0)$$

Higher-point amplitudes reconstruct coefficients in the Taylor expansion of geometric invariants on the EFT manifold.



It will be apparent in high-point amplitudes measured **here** if something unusual is happening (say) over **there**

Applying unitarity bound to $\pi\pi \to h^n$

$$E \lesssim 4\pi \times \left| \frac{\partial_h^{n-2} \mathcal{K}_h}{n!} \right|_{h=0}^{-\frac{1}{n}} (n!)^{\frac{1}{n}} \approx \begin{cases} 4\pi\, |\mathcal{K}_h|_{h=0}^{-\frac{1}{2}} & n = 2 \\ 4\pi v_\star (n!)^{\frac{1}{n}} & n = \text{‘a few’} \end{cases}$$

What's the problem?

Target space geometric picture only contains information up to 2 derivatives (no positivity bounds ☹️)

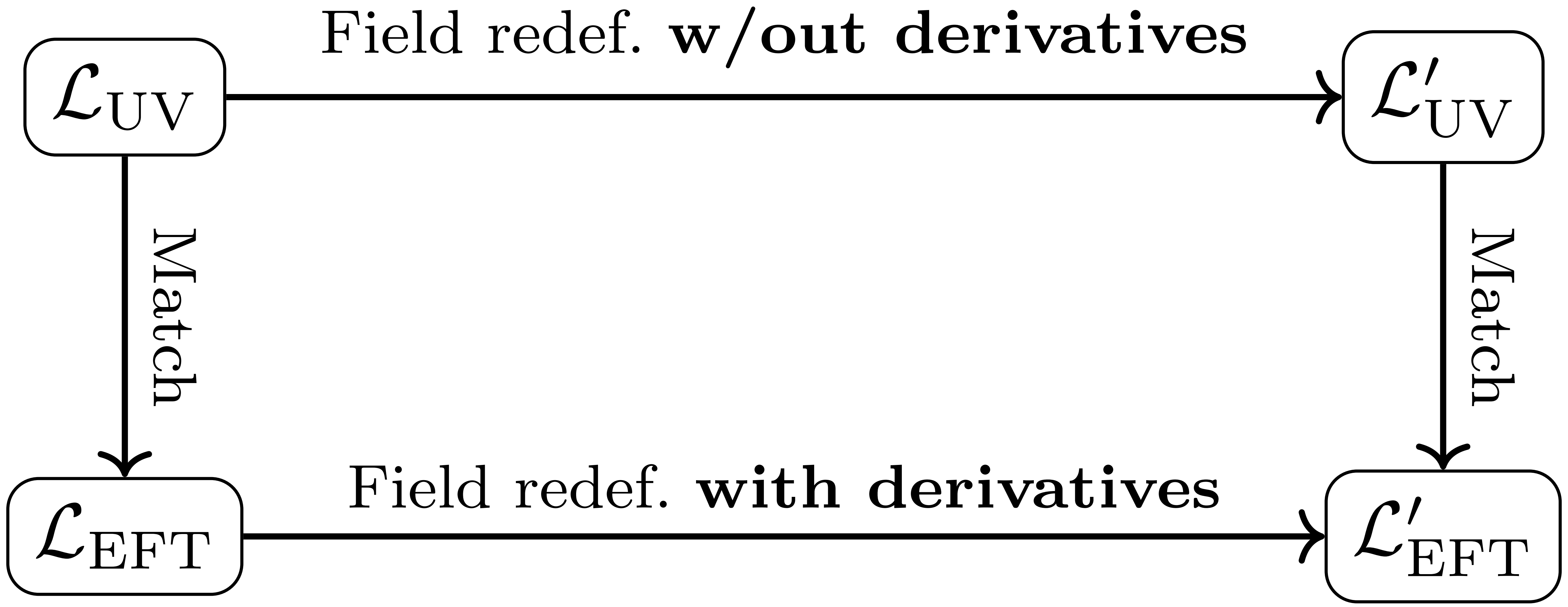
Even worse, "invariants" are susceptible to *derivative* field redefinitions:

$$\mathcal{L} = -V(\tilde{\phi}) + \frac{1}{2} g_{\alpha\beta}(\tilde{\phi}) \partial\tilde{\phi}^\alpha \partial\tilde{\phi}^\beta + \mathcal{O}(\partial^4) \quad \Rightarrow \quad \mathcal{L} = -V(\phi) + \frac{1}{2}\left(g_{\alpha\beta}(\phi) - V_{,\gamma}(\phi) h^\gamma_{\alpha\beta}(\phi)\right) \partial\phi^\alpha \partial\phi^\beta + \mathcal{O}(\partial^4)$$

$$\tilde{\phi}^\alpha = \phi^\alpha + \frac{1}{2} h^\alpha_{\gamma_1\gamma_2}(\phi)(\partial\phi^{\gamma_1}\partial\phi^{\gamma_2})$$

"Just don't do that" won't suffice.

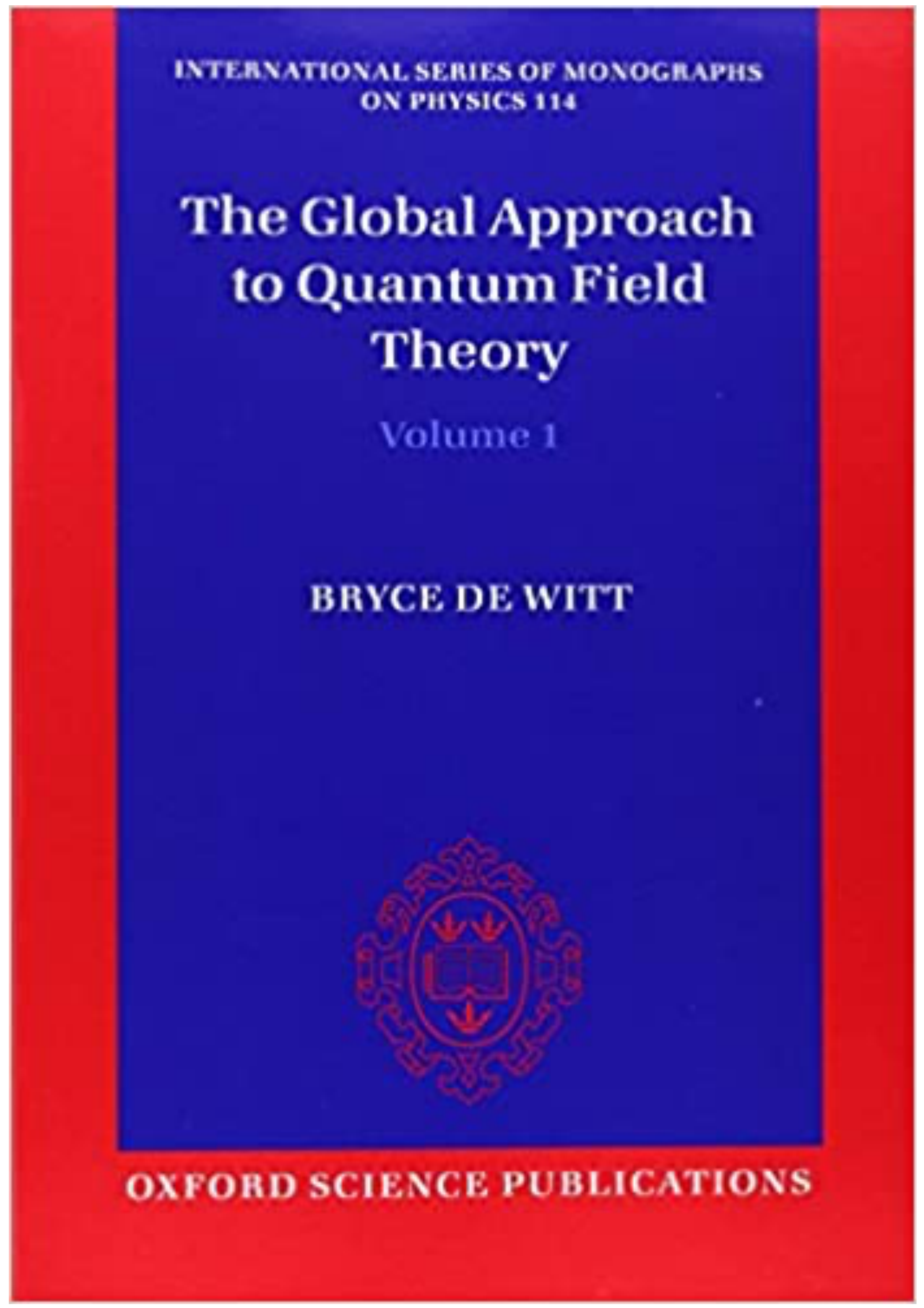
1. Derivative redefinitions often arise in basis transformations & using equations of motion.
2. Inevitable IR consequence of non-derivative field redefinitions in the UV.



Functional Geometry

The main idea [Cohen, NC, Lu, Sutherland '22]: go off-shell, and switch from studying fields at one spacetime point (target space) to fields at every spacetime point (configuration space).

$$S[\phi] = \int \mathrm{d}^4x \left(-V(\phi) + \frac{1}{2} g_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta + \ldots \right)$$



As we learned, this is not unlike an approach advocated by DeWitt; let's use his absurdly condensed notation:

$$\phi^\alpha(x) \to \phi^x \qquad \int d^4x\, \phi^\alpha(x) J_\alpha(x) \to \phi^x J_x$$

$$\underbrace{e^{i(\Gamma[\phi] + J_x \phi^x)}}_{\text{1PI effective action}} = \underbrace{e^{iW[J]}}_{\text{generating functional}} = \int \mathcal{D}\eta\, e^{i(S[\eta] + J_x \eta^x)}$$

Functional Geometry

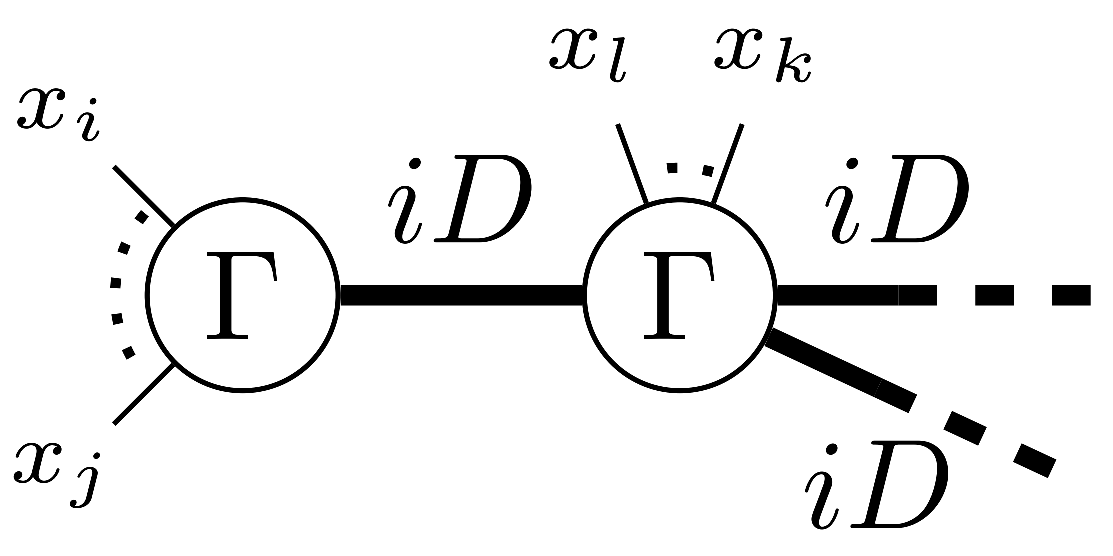
The interesting objects will be the *amputated correlators* $\mathscr{M}$

$$\mathcal{M}_{x_1\cdots x_n} \equiv -\left(-iD^{-1}_{x_1 y_1}\right)\cdots\left(-iD^{-1}_{x_n y_n}\right)\frac{\delta^n W[J]}{\delta J_{y_1}\cdots\delta J_{y_n}}$$

The sum of tree graphs assembled from

- *k*-point 1PI vertices $-i\dfrac{\delta^k(-\Gamma)}{\delta\phi^{x_1}\cdots\delta\phi^{x_k}}$
- propagators ***iD*** $(-iD^{-1}_{xy}) = \dfrac{\delta^2(-\Gamma)}{\delta\phi^x\delta\phi^y}$

$$= \sum_{\text{graphs}}$$



These are "almost amplitudes"; obtain amplitudes $\mathscr{A}$ by setting $J_x = 0$

and tacking on external wavefunctions $\prod_i (\epsilon_i^\mu) e^{ip_i\cdot x_i}$

$$\left[\prod_{i=1}^{n}(\epsilon_i^\mu)e^{ip_i x_i}\right]\mathcal{M}_{x_1\cdots x_n}\Big|_{J=0} = -(2\pi)^4\delta^4\left(\sum_i p_i\right)Z_\eta^{-n/2}\mathcal{A}$$

Functional Geometry

Define configuration space generalizations of the familiar "target space" geometric quantities:

"Metric"

$$iD^{-1}_{xy} = \frac{\delta^2(-\Gamma)}{\delta\phi^x \delta\phi^y}$$

"Connection"

$$G^y_{x_1 x_2} = iD^{yz}\, \frac{\delta^3(-\Gamma)}{\delta\phi^z \delta\phi^{x_1} \delta\phi^{x_2}}$$

Reduce to familiar target space geometric objects when restricted to constant field configurations

E.g. for $\mathcal{L} = -V + \frac{1}{2} g_{ab}(\partial_\mu \phi^a)\left(\partial^\mu \phi^b\right)$

$$\int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{ip_1 x_1} e^{-ip_2 x_2} \left[-iD^{-1}_{ab}(x_1, x_2)\Big|_{\partial_\mu \phi_i = 0} \right] = (2\pi)^4 \delta^4(p_1 - p_2)\left(-p_2^2\, g_{ab} + V_{,ab}\right)$$

Off-shell Recursion

These identifications also make sense because, remarkably, we can write an *n+1*-leg correlator in terms of an *n*-leg correlator by acting with "covariant derivative" $\nabla \sim \partial - \Gamma$

$$\mathcal{M}_{x_1\cdots x_n x} = \nabla_x \mathcal{M}_{x_1\cdots x_n} = \frac{\delta}{\delta\phi^x}\mathcal{M}_{x_1\cdots x_n} - \sum_{i=1}^{n} G^y_{x x_i}\mathcal{M}_{x_1\cdots \hat{x}_i y\cdots x_n}$$

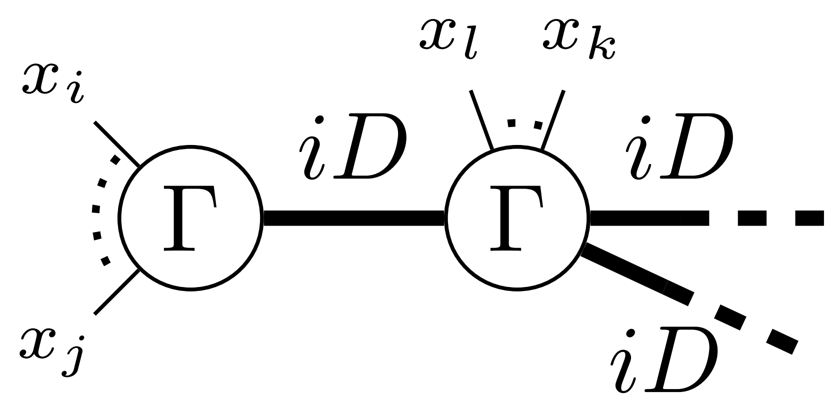
(take the string $x_1 \cdots x_i \cdots x_n$ and replace x_i w/ y)

Functional derivative w/ "connection" generates parallel transport on field space manifold.

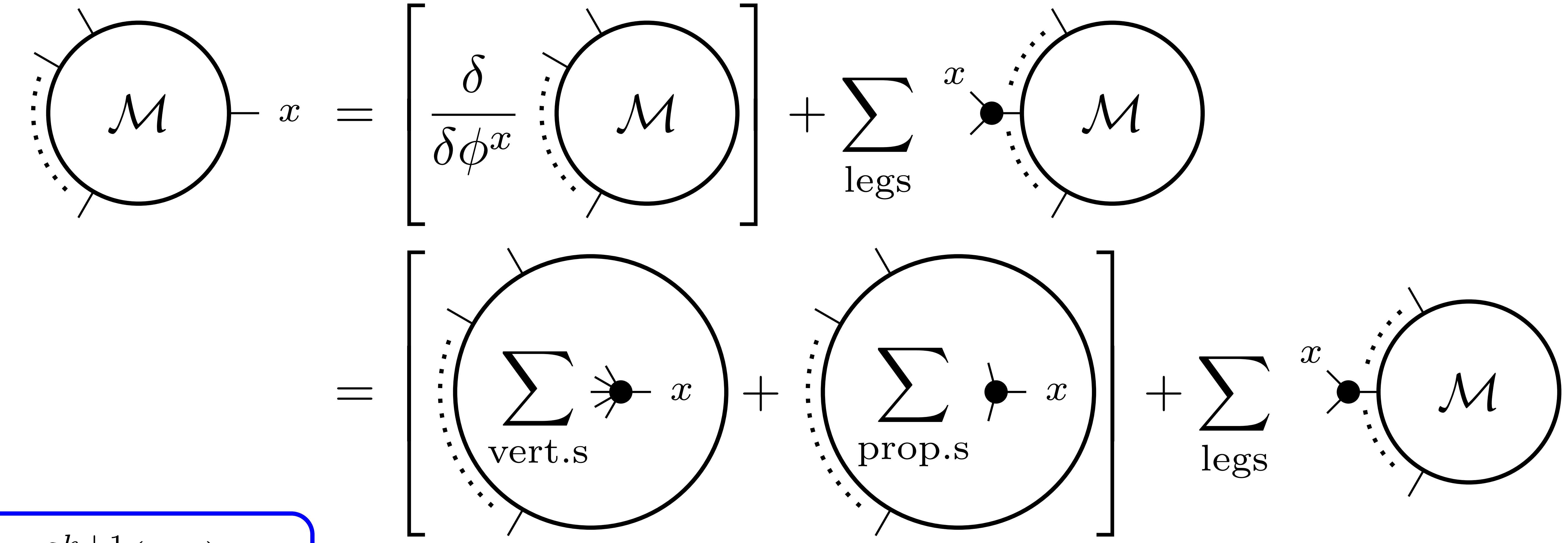
A new form of off-shell recursion! Can be used to obtain Berends-Giele when $J = 0$ is enforced.

Off-shell Recursion

$$\mathcal{M}_{x_1\cdots x_n} = \sum_{\text{graphs}}$$



$$\mathcal{M}_{x_1\cdots x_n x} = \nabla_x \mathcal{M}_{x_1\cdots x_n} = \frac{\delta}{\delta\phi^x}\mathcal{M}_{x_1\cdots x_n} - \sum_{i=1}^{n} G^y_{x x_i}\mathcal{M}_{x_1\cdots\hat{x}_i y\cdots x_n}$$



$$\frac{\delta}{\delta\phi^x}\left[-i\frac{\delta^k(-\Gamma)}{\delta\phi^{y_1}\cdots\delta\phi^{y_k}}\right] = -i\frac{\delta^{k+1}(-\Gamma)}{\delta\phi^{y_1}\cdots\delta\phi^{y_k}\,\delta\phi^x}$$

$$\frac{\delta}{\delta\phi^x}D^{y_1y_2} = D^{y_1z_1}\left[-i\frac{\delta^3(-\Gamma)}{\delta\phi^{z_1}\,\delta\phi^x\,\delta\phi^{z_2}}\right]D^{z_2y_2}$$

$$-G^y_{xx_i}\mathcal{M}_{x_1\cdots\hat{x}_i y\cdots x_n} = -i\frac{\delta^3(-\Gamma)}{\delta\phi^x\,\delta\phi^{x_i}\,\delta\phi^z}D^{zy}\mathcal{M}_{x_1\cdots\hat{x}_i y\cdots x_n}$$

On-shell Covariance

Does this "functional geometry" accommodate derivative field redefinitions?

Consider a general field redefinition (possibly w/ derivatives) $\phi(x) \to \tilde{\phi}(x)$

(express functionally as $\phi[\tilde{\phi}]$)

At tree level, the (effective) action transforms as a scalar, $\tilde{\Gamma}[\tilde{\phi}] = \tilde{S}[\tilde{\phi}] = S\big[\phi[\tilde{\phi}]\big] = \Gamma\big[\phi[\tilde{\phi}]\big]$

Can use this to show that the amputated correlators (also "metric", "connection") are covariant up to evanescent terms vanishing on shell:

$$\widetilde{\mathcal{M}}_{x_1\cdots x_n} = \left(\frac{\delta\phi^{y_1}}{\delta\tilde{\phi}^{x_1}}\cdots\frac{\delta\phi^{y_n}}{\delta\tilde{\phi}^{x_n}}\right)\mathcal{M}_{y_1\cdots y_n} + a_{x_1\cdots x_n y_1}\frac{\delta(-\Gamma)}{\delta\phi^{y_1}}$$

$$+\sum_{i=1}^{n} b_{x_1\cdots\hat{x}_i\cdots x_n y_1}\frac{\delta\phi^{y_2}}{\delta\tilde{\phi}^{x_i}}\frac{\delta^2(-\Gamma)}{\delta\phi^{y_1}\delta\phi^{y_2}}$$

on shell

$$\frac{\delta(-\Gamma)}{\delta\phi^{y_1}} = J_{y_1} = 0$$

$$\left.\frac{\delta^2(-\Gamma)}{\delta\phi^{y_1}\delta\phi^{y_2}}\right|_{J=0} = -iD^{-1}_{y_1y_2}\Big|_{J=0} = 0$$

$\Rightarrow$ manifests invariance of amplitudes under derivative field redefinitions

Another approach

"**Geometry-Kinematics Duality**" [Cheung, Helset, Parra-Martinez '22]:
Treat momentum like a flavor index; an arbitrary theory of massless bosons is classically equivalent to an NLSM with momentum-dependent target space metric.

Map NLSM flavor multiplet to a single scalar: $\phi^i \quad \to \quad \phi(p)$

Target space metric maps to a "kinematic metric": $g_{ij} \quad \to \quad g(p_1, p_2)\delta(p_{12})$

Index sums map to momentum integrals: $\sum_i \to \int_p = \int \frac{d^D p}{(2\pi)^D}$

See Julio's talk on Friday

Conclusions

- Field space geometry a powerful bridge between UV theories, EFTs, and amplitudes, but conventional geometry ("target space") fails past 2 derivatives.

- Functional geometry ("configuration space") provides a generalization that

  1. reduces to conventional target space geometry in appropriate limits,

  2. gives rise to new off-shell recursion relating amputated correlators,

  3. manifests on-shell covariance under general field redefinitions.

- Intriguing relation to "geometry-kinematics duality" (see Julio's talk)

- Much to learn from a geometric picture incorporating higher-derivative terms…

Thank you!