Amplitudes 2022, Prague

Gravitational amplitudes from a color symmetry

(or: is gravity integrable?)

2208.XXXX

Alfredo Guevara

Society of Fellows
Harvard

BLACK HOLE INITIATIVE
HARVARD UNIVERSITY

Motivation

Diverse aspect of gravitational theories suggests that to some extent amplitudes enjoy integrable features.

- Integrability in AdS/CFT correlation functions [see yesterday’s] —> flat limit
- Double copy from (S)YM, Yangian symmetry etc
- Classical Gravity: Type D spacetimes (e.g. Kerr) have geodesic integrable motion, two body problem etc [see friday’s talks]
- **Self dual** (quantum) gravity, Penrose’s non-linear graviton and CP3 twistors, **Ward’s conjecture** (SD -> integrable hamiltonian)
- Asymptotic symmetries: Infinity of conserved charges in scattering problem

In this talk we will link some of these approaches to argue that perturbative scattering amplitudes in gravitational theories are universally controlled by an infinite set of conserved charges.

These charges organize themselves into a $w_{1+\infty}$ hierarchy recently observed in correlation functions of **celestial CFT.** [A.G, Himwich, Pate, Strominger; Strominger '20].

We also know that the hierarchy is promoted to **surface charges in asymptotically flat spacetimes** [Pranzetti, Raclauriu, Friedel '21]. In parallel, a gauge theory version of the hierarchy has recently emerged in the **twisted holography** approach [Costello, Paquette '21]

Looking forward to thursday talks!!

However, this talk concerns the usual momentum S-Matrix, we will show that the $w_{1+\infty}$ hierarchy can be made explicit. Here it emerges in the $N \to \infty$ limit of a colored SU(N) theory, thus is an avatar of **color kinematics** duality.

The simplest realization of $w_{1+\infty}$ is area preserving diffeomorphisms of the 2d torus/plane. Given coordinates $\mu_\alpha \, , \alpha = 1, 2$

$$w_{1+\infty} = \mathbb{C}(\mu_+ , \mu_-) \qquad \{\mu_\alpha , \mu_\beta\} = \epsilon_{\alpha\beta}$$

as such it is directly connected to the **kinematic algebras** of Monteiro and O'Connell [‘14]in the **self dual sector**.

Kinematic algebra from Quantum Groups

Historically, $w_{1+\infty}$ was first defined as a classical limit of a quantum algebra $\mathcal{W}_{1+\infty} := \lim \mathcal{W}_N$

- CFT operator algebra for infinite number of fields of spin $s = 1, 2, \ldots N$ [Kac;....; Prochazka]
- Has a Hopf algebra structure —> directly related to the "kinematic" Yangian $Y(\mathfrak{gl}_1)$ [Maulik, Okunkov; Tsymbaliuk;...]

Motivated by the interesting connections to integrability, here we will first introduce a quantum group at finite N, where $N \to \infty$ will play the role of classical limit . The suitable framework turns out to be the "momentum space" construction of J. Hoppe ['82].

Kinematic algebra from Quantum Groups

Consider two operators

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I} \qquad (*)$$

Classically they correspond to Penrose's twistor coordinates for a C^2 plane.

The C^2 plane is quantized into a torus $e^{N\mu_+} = e^{N\mu_-} = 1$, forming the **Heisenberg group**

$$e^{\mu_+} e^{\mu_-} = q\, e^{\mu_-} e^{\mu_+} \ , \qquad\qquad q = e^{4\pi i/N}$$

where q is the modular parameter of the torus. The classical limit is $N \to \infty, q \to 1$

after which the torus becomes the plane, and the structure (*) is a Poisson bracket

To connect with **momentum** amplitudes we introduce the quantized momentum in the dual torus

$$\lambda_\alpha \in Z_N \times Z_N$$

Then we define the momentum wavefunctions

$$G^{\lambda} := e^{[\lambda\mu]} = e^{\lambda_+\mu_- - \lambda_-\mu_+}$$

These N^2 generators form a basis of U(N), or SU(N) if we remove $G^{(0,0)}$. Indeed, Hoppe noted that even though the operators $\mu_\pm$ do not admit a finite dimensional representation, their exponentials do (the Heisenberg algebra)

The Heisenberg algebra implies the fundamental product

$$G^{\lambda_1} G^{\lambda_2} = q^{[\lambda_1 \lambda_2]/2} G^{\lambda_1+\lambda_2}$$
$$= (1 + 2\pi i[\lambda_1, \lambda_2]/N)\, G^{\lambda_1+\lambda_2} + \mathcal{O}(1/N^2)$$

To extract the classical limit we reescale by “hbar” $G^\lambda \to G^\lambda/N$, then

$$[G^{\lambda_1}, G^{\lambda_2}] = 4\pi i[\lambda_1 \lambda_2]\, G^{\lambda_1+\lambda_2}$$

which can be shown to agree with the Poisson bracket associated to $w_{1+\infty}$

So far we have explained how the large N limit of SU(N) provides a classical realization of $w_{1+\infty}$ as a color algebra of matrices.

Since $\lambda_\alpha \in Z_N \times Z_N \to \lambda_\alpha \in \mathbb{R}^2$ the resulting color algebra is in a sense **two dimensional,** it has recently appeared in the context of color-kinematics in connection with 2d integrable models/SDYM [Cheung, Mangan, Parra-Martinez, Shah; Armstrong-Williams, White, Wikeley ‘22 ,...]

Here we will construct 4d gravity by introducing an affine version of the algebra.

Affine Kac Moody algebra for $w_{1+\infty}$

We introduce weight ½ fields on an auxiliary CP^1:

$$\mu^{\pm} \to \mu^{\pm}(z) = \sum_n \mu^{\pm}_n / z^{n+1/2}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}} \mathbb{I}$$

Affine Kac Moody algebra for $w_{1+\infty}$

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}}\mathbb{I}$$

[Gravitational Goldstone modes]

Affine Kac Moody algebra for $w_{1+\infty}$

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}}\mathbb{I}$$

[Gravitational Goldstone modes]

$$G^\lambda = e^{[\lambda\mu]}$$

$$G^\lambda(z) =: e^{[\lambda\mu(z)]} :$$

[Graviton vertex operator]

Affine Kac Moody algebra for $w_{1+\infty}$

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}}\mathbb{I}$$

[Gravitational Goldstone modes]

$$G^\lambda = e^{[\lambda\mu]}$$

$$G^\lambda(z) =: e^{[\lambda\mu(z)]} :$$

[Graviton vertex operator]

$$G^{\lambda_1} G^{\lambda_2} = q^{[\lambda_1\lambda_2]/2} G^{\lambda_1+\lambda_2}$$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) = q^{[\lambda_1\lambda_2]/z_{12}} : e^{[\lambda_1\mu(z_1)]} e^{[\lambda_2\mu(z_2)]} :$$

[Quantum group]

Affine Kac Moody algebra for $w_{1+\infty}$

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}}\mathbb{I}$$

[Gravitational Goldstone modes]

$$G^\lambda = e^{[\lambda\mu]}$$

$$G^\lambda(z) =: e^{[\lambda\mu(z)]} :$$

[Graviton vertex operator]

$$G^{\lambda_1} G^{\lambda_2} = q^{[\lambda_1\lambda_2]/2} G^{\lambda_1+\lambda_2}$$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) = q^{[\lambda_1\lambda_2]/z_{12}} : e^{[\lambda_1\mu(z_1)]} e^{[\lambda_2\mu(z_2)]} :$$

[Quantum group]

$$[G^{\lambda_1}, G^{\lambda_2}] = 4\pi i[\lambda_1\lambda_2] G^{\lambda_1+\lambda_2}$$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) \sim \frac{[\lambda_1\lambda_2]}{z_{12}} G^{\lambda_1+\lambda_2}(z_2)$$

[OPE from classical limit]

Affine Kac Moody algebra for $w_{1+\infty}$

$$[\mu_+, \mu_-] = (4\pi i/N)\mathbb{I}$$

$$\mu_+(z_1)\mu_-(z_2) \sim \frac{4\pi i/N}{z_{12}}\mathbb{I}$$

[Gravitational Goldstone modes]

$$G^\lambda = e^{[\lambda\mu]}$$

$$G^\lambda(z) =: e^{[\lambda\mu(z)]} :$$

[Graviton vertex operator]

$$G^{\lambda_1} G^{\lambda_2} = q^{[\lambda_1\lambda_2]/2} G^{\lambda_1+\lambda_2}$$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) = q^{[\lambda_1\lambda_2]/z_{12}} : e^{[\lambda_1\mu(z_1)]} e^{[\lambda_2\mu(z_2)]} :$$

[Quantum group]

$$[G^{\lambda_1}, G^{\lambda_2}] = 4\pi i[\lambda_1\lambda_2] G^{\lambda_1+\lambda_2}$$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) \sim \frac{[\lambda_1\lambda_2]}{z_{12}} G^{\lambda_1+\lambda_2}(z_2)$$

2d color

2d ws + 2d color = 4d kinematics

[OPE from classical limit]

We thus have found a level 0 (tree-level) Kac Moody algebra of currents, for gauge group $w_{1+\infty}$

$$G^{\lambda_1}(z_1)G^{\lambda_2}(z_2) \sim \frac{[\lambda_1\lambda_2]}{z_{12}}G^{\lambda_1+\lambda_2}(z_2) \quad \Longleftrightarrow \quad j^a(z_1)j^b(z_2) \sim \frac{f^{ab}{}_c}{z_{12}}j^c(z_2)$$

We can further consider matter fields in the adjoint representation, which is universal (spin-indep)

$$\begin{aligned} G^{\lambda_1}(z_1)O^{\lambda_2}(z_2) &\sim \frac{[\lambda_1\lambda_2]}{z_{12}}O^{\lambda_1+\lambda_2}(z_2) \\ &= \frac{[\lambda_1\lambda_2]}{z_{12}}e^{[\lambda_1\frac{\partial}{\partial\lambda_2}]}O^{\lambda_2}(z_2) \\ &\sim \frac{(\epsilon_1\cdot k_2)^2}{k_1\cdot k_2}e^{\frac{F_1^{\mu\nu}J_{2\mu\nu}}{2\epsilon_1\cdot k_2}}O^{\lambda_2}(z_2) \end{aligned}$$

The latter form is nothing but an exponentiated form of the soft theorem for a general massless particle of helicity h:

$$G^+(k_1)O^h(k_2) \sim \frac{(\epsilon_1 \cdot k_2)^2}{k_1 \cdot k_2} e^{\frac{F_1^{\mu\nu} J_{2\mu\nu}}{2\epsilon_1 \cdot k_2}} O^h(k_2)$$

which has appeared in several contexts, including connections to BH physics. Indeed for **massive momenta**, a related construction in celestial holography gives the graviton-BH coupling [A.G., Himwich, Crawley, Strominger; to appear]

$$G^+(k_1)O^{\pm}_{\text{Kerr}}(p,a) \sim \frac{(\epsilon_1 \cdot p)^2}{k_1 \cdot p} e^{k_1 \cdot a} O^{\pm}_{\text{Kerr}}(p,a)$$

MHV sector from 2d Kac Moody

In our context, a direct application of the OPE provides MHV gravitational amplitudes in a recursive form, e.g.

$$\begin{aligned}\langle G^{\lambda}(z)G^{\lambda_1}(z_1)\cdots G^{\lambda_n}(z_n)\rangle &= \frac{1}{2\pi i}\oint \frac{dw}{w-z}\langle G^{\lambda}(w)G^{\lambda_1}(z_1)\cdots G^{\lambda_n}(z_n)\rangle \\ &= \sum_i \frac{[\lambda\lambda_i]}{z-z_i}e^{[\lambda\frac{\partial}{\partial\lambda_i}]}\langle G^{\lambda_1}(z_1)\cdots G^{\lambda_i}(z_i)\cdots G^{\lambda_n}(z_n)\rangle + \mathcal{A}_{\infty}(z)\end{aligned}$$

where $\mathcal{A}_{\infty}(z)$ is a polynomial term that can be determined from momentum conservation. The exponentiated expression matches the BCFW result (e.g. Hodge's recursion formula).

3pt MHV in (2,2) signature

To provide the 3pt MHV seed, we need to supplement the spectrum with a weight 0 auxiliary field $\Phi^{\lambda}(z)$ in (2,2) signature, such that

$$G^{\lambda_1}(z_1)\Phi^{\lambda_2}(z_2) \sim \delta(z_{12})\delta^2(\lambda_1+\lambda_2) \quad \Longleftrightarrow \quad j^a(z_1)\Phi^b(z_2) \sim \delta(z_{12})\delta^{ab}$$

Further relating $1/z_{12} \leftrightarrow \delta(z_{12})$ we observe that

$$\begin{aligned}\langle G^{\lambda_1}(z_1)G^{\lambda_2}(z_2)\Phi^{\lambda_3}(z_3)\rangle &= f^{\lambda_1\lambda_2\lambda_3}\delta(z_{12})\delta(z_{23}) \\ &= \delta^2(\lambda_1+\lambda_2+\lambda_3)\delta^2(z_1\lambda_1+z_2\lambda_2+z_3\lambda_3) \\ &\quad\times\left(\frac{[12]^3}{[13][23]}\right)^2\end{aligned}$$

*Note that momentum conservation is emergent in this picture.

Connection to soft physics: Multipole expansion

So far we have used the color algebra OPE to construct the momentum space S-Matrix, fixing the collinear singularities as well as the 3-point seed. The output takes the form of an exponentiated soft theorem, closely related to the ones of [He, Huang, Wen ‘14; Hamada, Shiu ‘19, A.G. ‘19]

We can recover the soft-mode results of celestial holography by performing a multipole expansion of the vertex operators:

$$G^{\lambda}(z) =: e^{[\lambda\mu(z)]} := \sum_n \frac{1}{n!}\lambda^{\alpha_1}\cdots\lambda^{\alpha_n} W_{\alpha_1\cdots\alpha_n}(z)$$

$$W_{\alpha_1\cdots\alpha_n}(z) =: \mu_{\alpha_1}(z)\cdots\mu_{\alpha_n}(z) :$$

A hierarchy of multipoles

$$W_\alpha(z_1)O^\lambda(z_2) \sim \frac{\lambda_\alpha}{z_{12}}O^\lambda(z_2)$$

[Translations]

A hierarchy of multipoles

$$W_\alpha(z_1)O^\lambda(z_2) \sim \frac{\lambda_\alpha}{z_{12}} O^\lambda(z_2)$$ [Translations]

$$W_{\alpha\beta}(z_1)O^\lambda(z_2) \sim \frac{\lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}}}{z_{12}} O^\lambda(z_2)$$ [Lorentz tr.]

A hierarchy of multipoles

$$W_\alpha(z_1)O^\lambda(z_2) \sim \frac{\lambda_\alpha}{z_{12}} O^\lambda(z_2)$$

[Translations]

$$W_{\alpha\beta}(z_1)O^\lambda(z_2) \sim \frac{\lambda_{(\alpha} \frac{\partial}{\partial \lambda^{\beta)}}}{z_{12}} O^\lambda(z_2)$$

[Lorentz tr.]

$$W_{\alpha\beta\gamma}(z_1)O^\lambda(z_2) \sim \frac{\lambda_{(\alpha} \frac{\partial}{\partial \lambda^{\beta}} \frac{\partial}{\partial \lambda^{\gamma)}}}{z_{12}} O^\lambda(z_2)$$

[Higher derivative, conformal?]

$$\cdots$$

A hierarchy of multipoles

$$W_{\alpha}(z_1)O^{\lambda}(z_2) \sim \frac{\lambda_{\alpha}}{z_{12}}O^{\lambda}(z_2)$$

[Translations]

$$W_{\alpha\beta}(z_1)O^{\lambda}(z_2) \sim \frac{\lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}}}{z_{12}}O^{\lambda}(z_2)$$

[Lorentz tr.]

$$W_{\alpha\beta\gamma}(z_1)O^{\lambda}(z_2) \sim \frac{\lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta}}\frac{\partial}{\partial\lambda^{\gamma)}}}{z_{12}}O^{\lambda}(z_2)$$

[Higher derivative, conformal?]

$$\cdots$$

$$W_{\alpha_1\cdots\alpha_n}(z_1)W_{\beta_1\cdots\beta_m}(z_2) \sim \frac{1}{z_{12}}\epsilon_{(\alpha_1(\beta_1}W_{\beta_2\cdots\beta_m)\alpha_2\cdots\alpha_m)}(z_2)$$

This recovers the hierarchy of charges introduced in [Strominger ‘21] corresponding the wedge-loop algebra of $w_{1+\infty}$. It consists of an infinite-dimensional extension of the Poincare group by higher-derivative operators.

Summary & outlook

- We have derived a kinematic algebra starting from a quantum group structure (Heisenberg algebra associated to SU(N)).
- Using (2,2) signature allow us to incorporate 4d kinematics directly from the 2d celestial sphere + 2d color.
- The classical limit directly leads to $w_{1+\infty}$ in momentum space. Scattering amplitudes transform under adjoint multiplets of this symmetry. Quantum/loop corrections to W are being addressed via different approaches []
- The symmetry provides an organizing principle for 1/r^n asymptotic charges. In classical gravity, the w-generators compute/generalize the known tower of Newman-Penrose charges.[Pranzetti, Raclariu, Friedel ; A.G.]

Thanks!

$$[\widehat{\mathrm{w}}^p_m, \widehat{\mathrm{w}}^q_n] = [m(q-1) - n(p-1)]\, \widehat{\mathrm{w}}^{p+q-2}_{m+n}.$$

$$1 - p \leq m \leq p - 1 \quad \text{with} \quad p = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots .$$

$$\{q^2_s(z), C(u', z')\} = \frac{\kappa^2}{8} \sum_{n=0}^{s} (-1)^{s+n} \frac{(n+1)(\Delta+2)_{s-n}}{(s-n)!} \partial^{1-s}_{u'} D^n_{z'} C(u', z') D^{s-n}_z \delta(z, z').$$