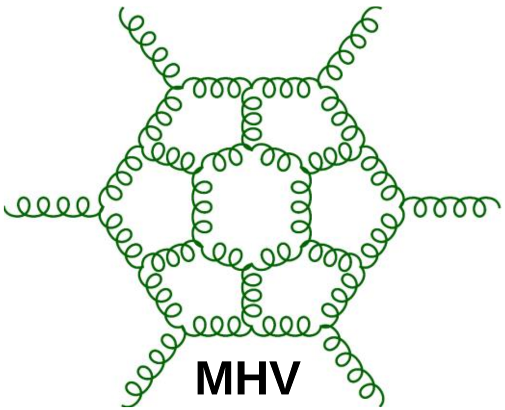
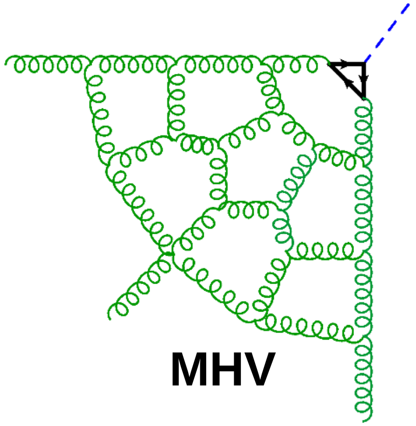
Antipodal Duality
and an eight loop application





Lance Dixon

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243, 2204.11901
LD, Y.-T. (Andy) Liu, to appear

Amplitudes 2022
Charles University, Prague, CZ
10 August 2022

Amplitudes 2022
August 8–12, 2022
Prague, Czech Republic

SLAC NATIONAL ACCELERATOR LABORATORY

Transcendental Structure

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L
- Weight ~ number of integrations, e.g.

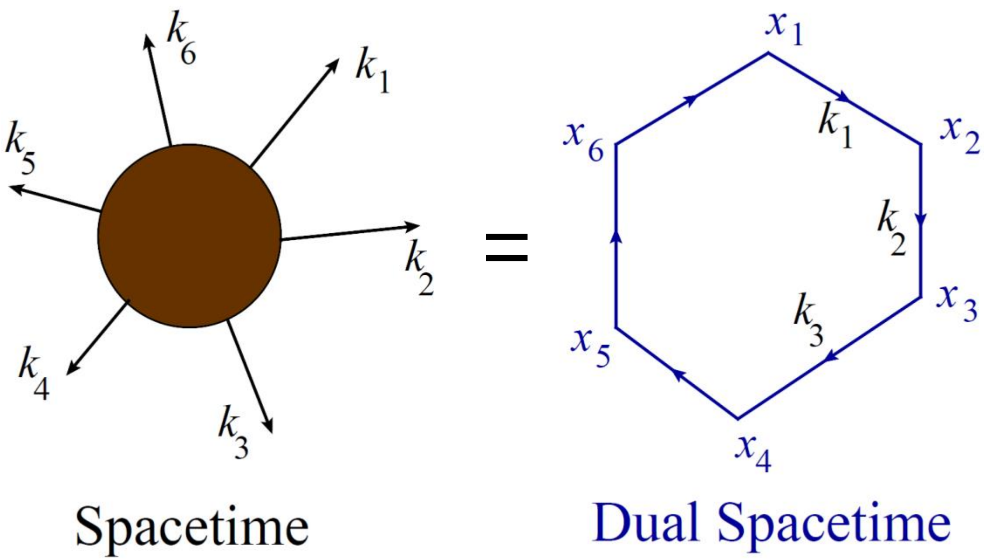
$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

In planar N=4 SYM:
Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality holds at both strong and weak coupling

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2\, x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

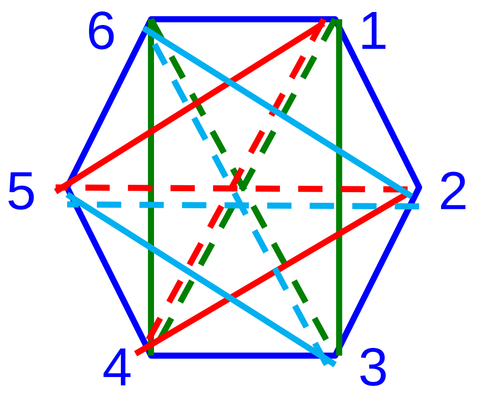
- Fixed, up to functions of invariant cross ratios:

$$\hat{u}_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

$n = 6$ → precisely 3 ratios:

$$\begin{cases} \hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \\ \hat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}} \\ \hat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}} \end{cases}$$



Hexagon function bootstrap

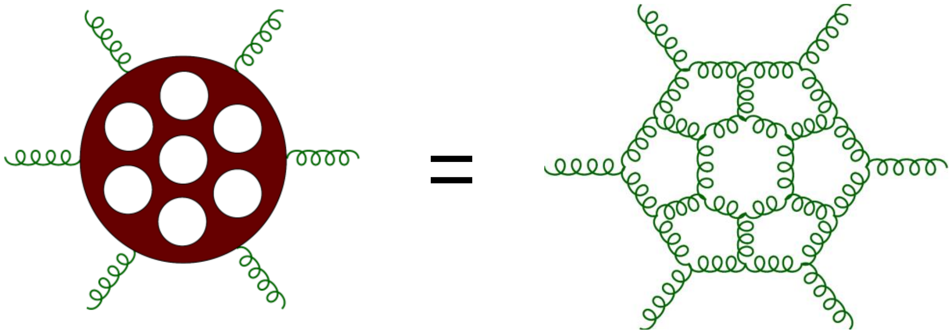
Loops

3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

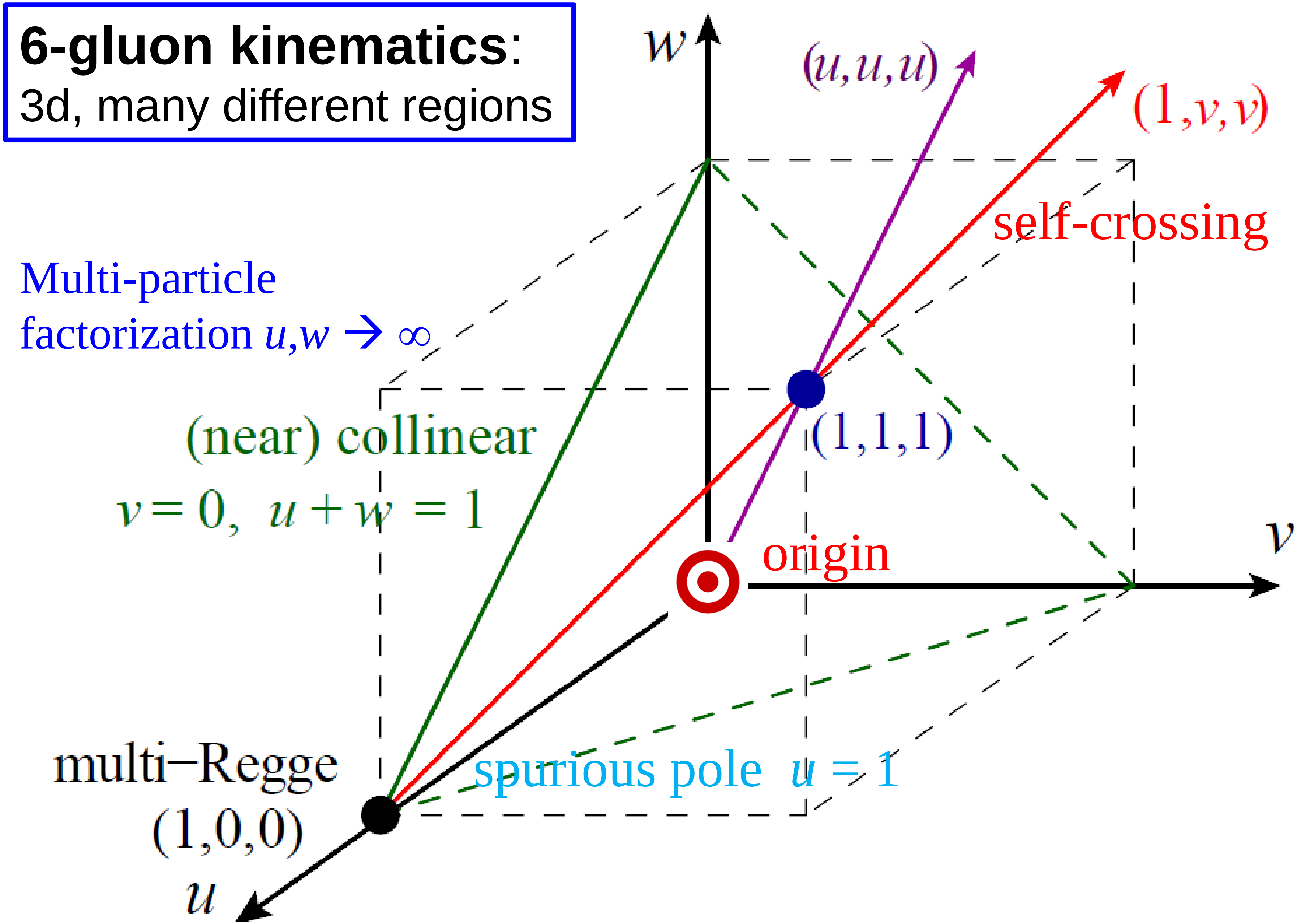
6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 22mm.nnnnn (NMHV 7 loop)

- Results are generalized polylogarithms with the same symbol alphabet to all loop orders!
- Same method used for “Higgs” form factor; see below



+ ~10^9 more

6-gluon kinematics:
3d, many different regions



Generalized polylogarithms

Chen, Goncharov, Brown,…

- Define as iterated integrals, e.g.

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

- Or define differentially: $dF = \sum_{s_k \in \mathcal{L}} F^{s_k}\, d \ln s_k$

- A Hopf algebra "co-acts" on space of polylogarithms,

$$\Delta: F \to F \otimes F$$

- Derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$
- so we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{L}$

Generalized polylogarithms (cont.)

- $\{n\text{-}1,1\}$ coaction can be applied iteratively
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_{k},s_j},

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k\in\mathcal{L}} F^{s_{k},s_j}\, d\ln s_k$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the symbol,

$$\mathcal{S}[F] = \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, d\ln s_{i_1} \ldots d\ln s_{i_n} \equiv \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1}\otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $\boldsymbol{n = 2L}$ is number of binary strings: $\boldsymbol{2^{2L}}$
- Alphabet: $\mathcal{L} = \{x, 1-x\}$
- $z_i = x \text{ if } w_i = 0, \quad z_i = 1 - x \text{ if } w_i = 1$

→ Symbol $\mathcal{S}[H_{\vec{w}}(x)] = (-1)^{\#1's} z_n \otimes z_{n-1} \otimes \cdots \otimes z_1$

- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Symbol alphabet for 6-gluon MHV amplitude

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

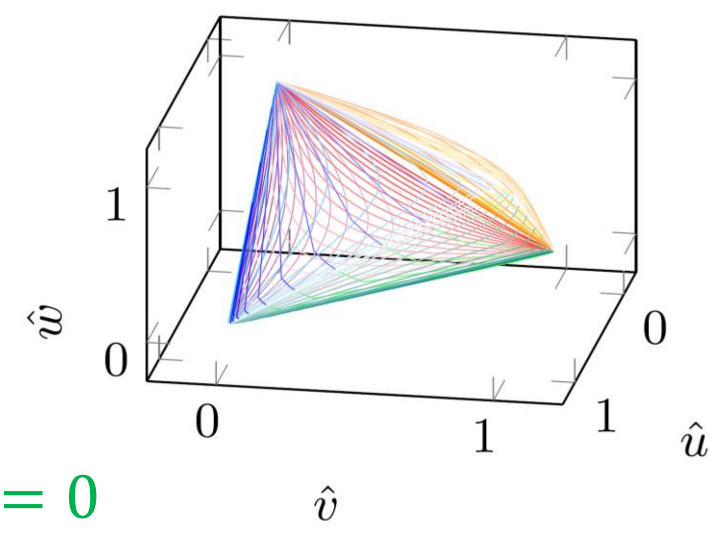
9 letters:

$$\mathcal{L}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$$

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

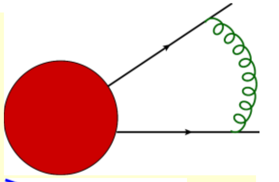
Importantly, there is a parity-preserving 2d surface where all three $\hat{y}_i \to 1$, so can delete from symbol, leaving only 6 letters:

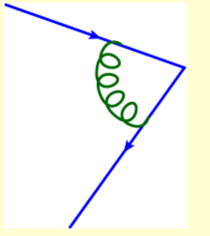
$$\hat{k}^\mu_{i+3} = -\hat{k}^\mu_i, i = 1,2,3$$



$$\Rightarrow \hat{\Delta}(\hat{u}, \hat{v}, \hat{w}) = (1-\hat{u}-\hat{v}-\hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

Removing Amplitude (or Form Factor) Infrared Divergences





- On-shell amplitudes IR divergent due to long-range gluons
- Polygonal Wilson loops UV divergent at cusps, anomalous dimension Γ_{cusp}

– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

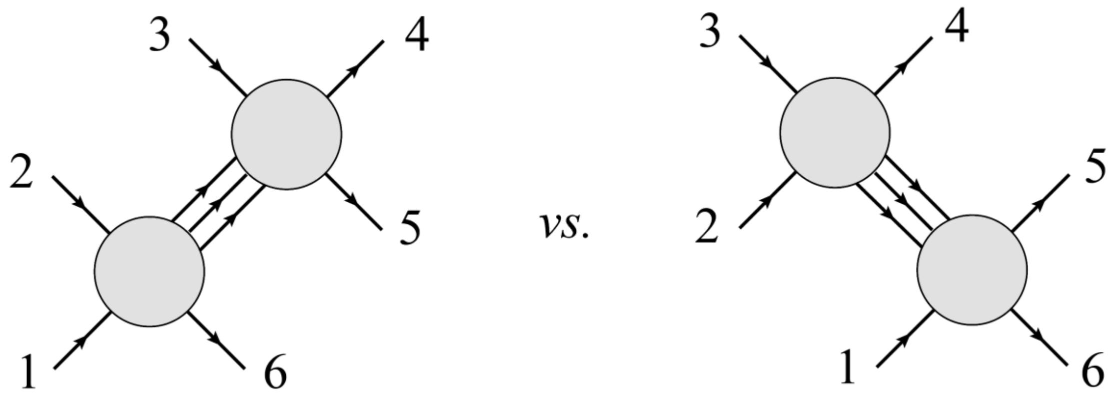
$$A_6(\hat{u}_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\mathrm{BDS-like}}(s_{i,i+1},\epsilon)} = \exp[\frac{\Gamma_{\mathrm{cusp}}}{4} A_6^{(1)} + R_6]$$

remainder function

Steinmann Relations

Steinmann (1960)

3-particle channels in amplitudes with $n \geq 6$ particles can cross threshold independently of any other invariants.
Most transparent in 3 → 3 scattering:



Can move s_{345} across 0 with all other invariants generic, and similarly for $s_{561} = s_{234}$. Furthermore, there is a region where both s_{345} and s_{561} can cross 0, and Steinmann →

$$\text{Disc}_{s_{234}}\ \text{Disc}_{s_{345}} \mathcal{A}_6(s_{ij}, s_{ijk}, \epsilon) = 0$$

Steinmann motivates basis change

- $\mathcal{L}_6 = \{\,\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$ → 1 for $\Delta = 0$

$\rightarrow \mathcal{L}_6' = \{\,\hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}}\}$

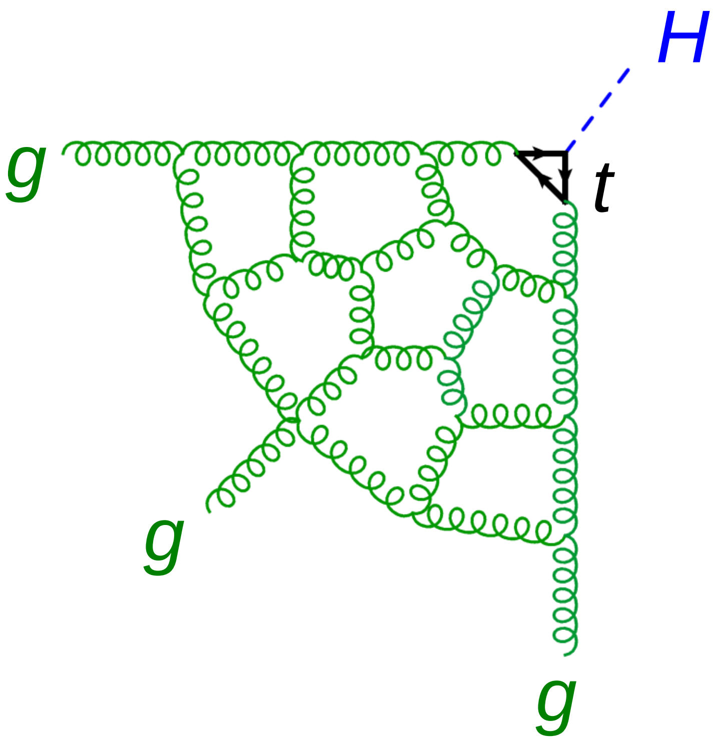
$$\hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}} = s_{234}^2 \times r(s_{i,i+1})$$
$$\hat{b} = \frac{\hat{v}}{\hat{w}\hat{u}} = s_{345}^2 \times \check{r}(s_{i,i+1})$$
$$\hat{c} = \frac{\hat{w}}{\hat{u}v} = s_{123}^2 \times \acute{r}(s_{i,i+1})$$

$$\mathrm{Disc}_{s_{234}}\,\mathrm{Disc}_{s_{345}}\mathcal{A}_6 = 0 \quad \Rightarrow \quad \mathrm{Disc}_{\hat{a}}\,\mathrm{Disc}_{\hat{b}}\,A_6 = 0$$

$$\Rightarrow \mathcal{S}[A_6] = \cdots \otimes \hat{a} \otimes \hat{b} \otimes \ldots + \ \ldots \qquad + \text{dihedral}$$

Bootstrap Goldilocks "Higgs" amplitude [planar N=4 form factor] to 8 loops

	Loops
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2012.12286,	3,4,5
2204.11901	6,7,8



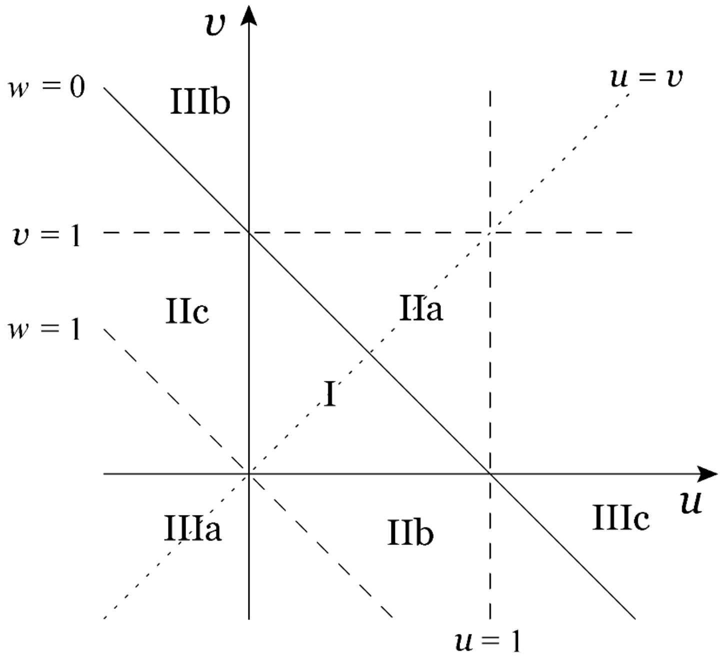
- Matrix elements of operator $G_{\mu\nu}^{a} G^{\mu\nu\, a}$ with n gluons in planar N=4 SYM
- *Hgg* form factor ($n = 2$) "too simple", no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* ($n = 3$) is "just right", depends on only 2 dimensionless ratios
- 8 loop results for function of 2 variables are a "data mine" for discovering e.g. antipodal duality

Hggg kinematics is two-dimensional

$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$



$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

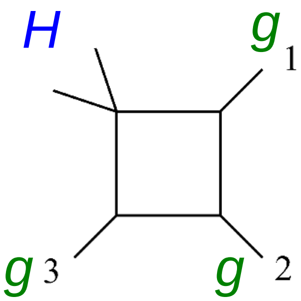
$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

One loop integrals/amplitudes



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right)+\mathrm{Li}_2\left(1-\frac{1}{v}\right)+\frac{1}{2}\ln^2\left(\frac{u}{v}\right)+\cdots$$

→ symbol $= u\otimes(1-u)+v\otimes(1-v)-u\otimes v-v\otimes u$

A two-loop story

- *Hggg* computed in QCD at 2 loops

  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM computed next (QMUL, a decade ago)

  Brandhuber, Travaglini, Yang, 1201.4170
- **Symbol alphabet:** $\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

- Highest weight part of QCD result **same as N=4 result!!**
- "Principle of maximal transcendentality"

  Kotikov, Lipatov, Velizhanin, hep-ph/0301021, hep-ph/0611204
- Does it hold here beyond two loops?
- Other operators: Ahmed et al., 1905.12770; Guo et al., 2205.12969

3-gluon form factor alphabet

- Motivated by 6 gluon case, switch to equivalent alphabet

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at 1 and 2 loops: just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b\otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b\otimes d\otimes d\otimes d + 2\, b\otimes b\otimes b\otimes d + \text{dihedral}$$

dihedral cycle: $a \to b \to c \to a, \quad d \to e \to f \to d$

dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

Antipodal duality

weak-weak duality

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \; x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

also corresponds to "twisted forward scattering":

$$\hat{k}_{i+n}^{\mu} = -\hat{k}_i^{\mu}, \qquad i = 1,2,\ldots,n \qquad\qquad (n = 3 \; here)$$

6-gluon alphabet and symbol map

- $\mathcal{L}_6 = \{\, \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → 1 for $\Delta = 0$

→ $\mathcal{L}_6' = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Kinematic map on letters:

$$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \qquad \textit{plus cyclic relations}$$

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\tfrac{1}{2})\hat{b} \otimes \hat{d} + \text{dihedral}$$

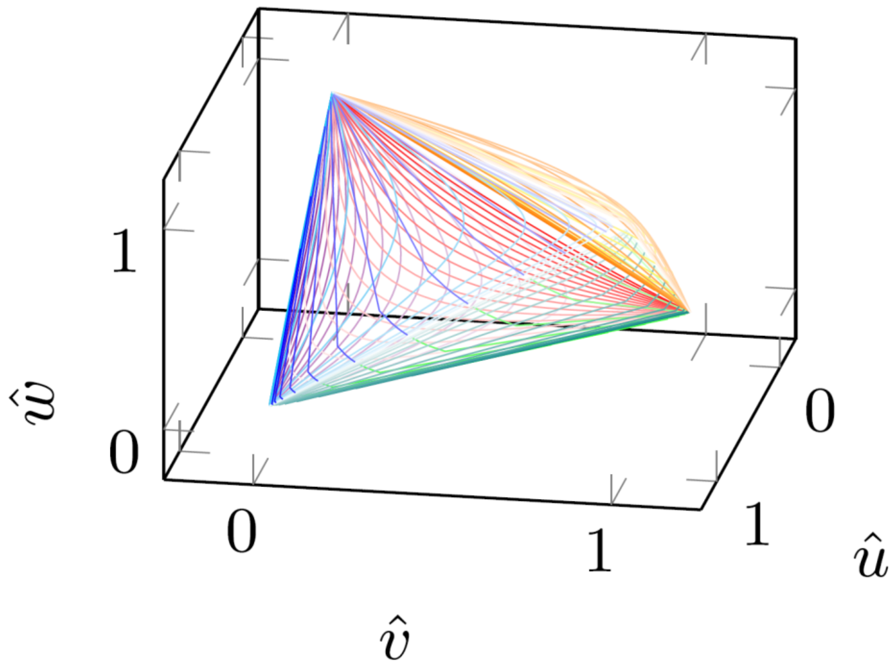
$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b} \otimes \hat{d} \otimes \hat{d} \otimes \hat{d} + \frac{1}{2}\hat{b} \otimes \hat{b} \otimes \hat{b} \otimes \hat{d} + \text{dihedral}$$

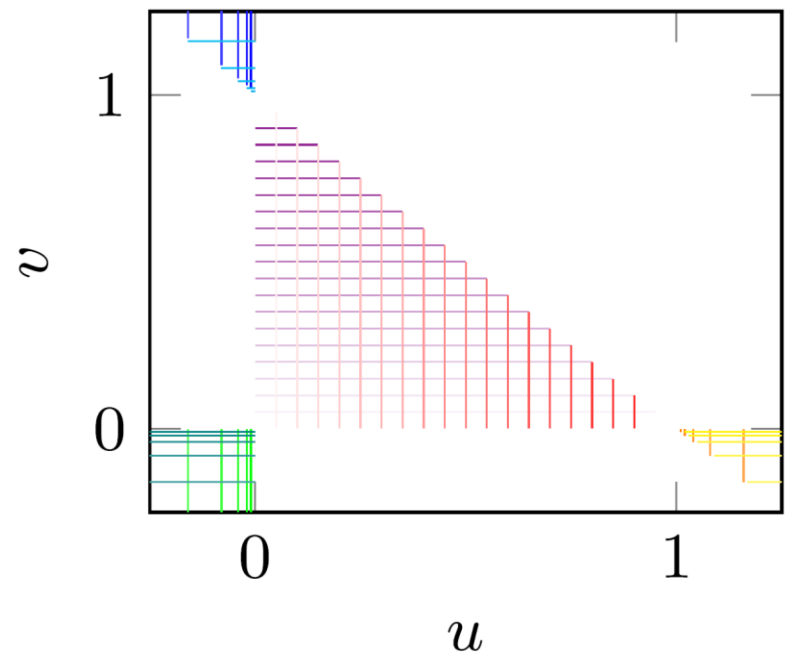
$$\ldots$$

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

- Works through 7 loops!

Map covers entire phase space for 3-gluon form factor



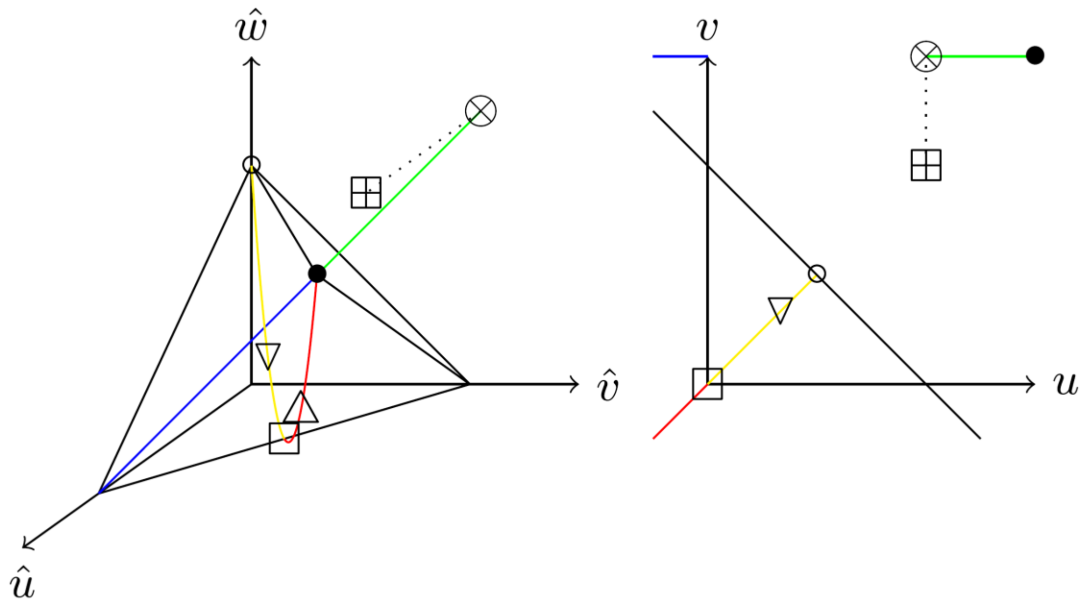


- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures



	$(\hat{u}, \hat{v}, \hat{w})$	(u, v, w)	functions
▽	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\sqrt[6]{1}$
□	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 0, 1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
●	$(1, 1, 1)$	$\lim_{u\to\infty}(u, u, 1-2u)$	MZVs
○	$(0, 0, 1)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	MZVs + logs
△	$(\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$	$(-1, -1, 3)$	$\sqrt[6]{1}$
⊞	(∞, ∞, ∞)	$(1, 1, -1)$	alternating sums
⊗	$\lim_{\hat{v}\to\infty}(1, \hat{v}, \hat{v})$	$\lim_{v\to\infty}(1, v, -v)$	MZVs
green–blue line	$(1, \hat{v}, \hat{v})$	$\lim_{v\to\infty}(u, v, 1-u-v)$	HPL$\{0, 1\}$
red–yellow line	$(\hat{u}, \hat{u}, (1-2\hat{u})^2)$	$(u, u, 1-2u)$	HPL$\{-1, 0, 1\}$

Simplest point

- $(\hat{u},\hat{v},\hat{w}) = (1,1,1) \Leftrightarrow u,v \to \infty$
- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$

$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in f-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$
- modulo $i\pi$ is best we can get from antipode map

Simplest form factor line is $v \to \infty$

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

$$\mathcal{L}' \to \{ \frac{1}{u},\ 1 - \frac{1}{u} \}$$

→ Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{u})$

$$F_3^{(1)}(v \to \infty) = 2H_{0,1} + 6\zeta_2$$

$$F_3^{(2)}(v \to \infty) = -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4$$

$$\begin{aligned} F_3^{(3)}(v \to \infty) = {} & 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\ & - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\ & - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2 \end{aligned}$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon MHV amplitude simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

- Let $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{\hat{v}})$

$$A_6^{(1)}(1,\hat{v},\hat{v}) = 2H_{0,1}$$

$$A_6^{(2)}(1,\hat{v},\hat{v}) = -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4$$

$$\begin{aligned} A_6^{(3)}(1,\hat{v},\hat{v}) = {} & 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\ & + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\ & + 42\zeta_4 H_{0,1} + 121\zeta_6 \end{aligned}$$

Exact map at symbol level, with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$,
if you also reverse order of symbol entries / HPL indices!!!
Works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

Antipodal duality "explains" adjacency relations for form factor

- Extended Steinmann relations for 6 gluon amplitude follow from causality (overlapping branch cuts → no double disc.):
- $\ldots \cancel{\otimes \hat{a} \otimes \hat{b}} \ldots \qquad + \text{dihedral}$ [ES, imposed]
- $\ldots \cancel{\otimes \hat{a} \otimes \hat{d}} \ldots \qquad + \text{dihedral}$ [follow from ES + first entry]

- The kinematic map on letters:

$$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \qquad \textit{plus dihedral images}$$

Takes the above conditions to the form factor restrictions:

$$\ldots \cancel{\otimes d \otimes e} \ldots \qquad + \text{dihedral}$$

$$\ldots \cancel{\otimes a \otimes d} \ldots \qquad + \text{dihedral}$$

- We observed these conditions empirically, but had no causality-based argument, prior to antipodal duality

Map in OPE parametrization

AmpOPE: Basso, Sever, Vieira, 1303.1396, …;
FFOPE: Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1+\hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1+T^2},$$

$$w = \frac{1}{(1+T^2)(1+S^{-2}(1+T^2))},$$

- Apply kinematic map →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

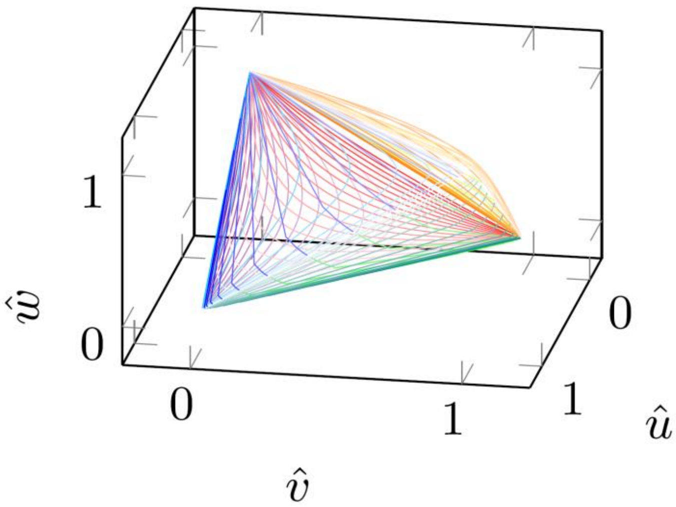
- Correspondence between

single flux tube excitations for the amplitude (T^1) and
double (or bound state) excitations for the form factor (T^2)

Exploit/test antipodal duality at 8 loops

LD, Y.-T. Liu, to appear

- Given form factor, antipodal duality determines symbol of MHV 6 gluon amplitude at 8 loops on $\Delta = 0$ surface.
- Lift symbol into bulk. Only 3 free parameters!
- 2 killed at origin, $(\hat{u}, \hat{v}, \hat{w}) \to (0,0,0)$
- last killed in process of lifting to full function level
- Need one OPE data point to kill one beyond-symbol ambiguity $\propto \zeta_8$



8 loop MHV 6-gluon amplitude at $(\hat{u}, \hat{v}, \hat{w}) = (1,1,1)$

LD, Y.-T. Liu, to appear

$$
\begin{aligned}
A_6^{(8)}(1,1,1) = {} & 9122624\, f_{9,7} + 11543472\, f_{7,9} + 5153280\, f_{11,5} + 19603536\, f_{5,11} + 23915376\, f_{3,13} \\
& + 371520\, f_{5,3,3,5} + 400320\, f_{3,3,5,5} + 400320\, f_{3,5,3,5} + 825216\, f_{3,3,3,7} \\
& - \zeta_2 \,(701856\, f_{7,7} + 1303232\, f_{9,5} + 430656\, f_{5,9} + 2061312\, f_{11,3} - 309696\, f_{3,11} \\
& \qquad + 160128\, f_{3,5,3,3} + 160128\, f_{3,3,5,3} + 117888\, f_{3,3,3,5} + 148608\, f_{5,3,3,3}) \\
& - \zeta_4 \,(3243888\, f_{5,7} + 3475296\, f_{7,5} + 3909696\, f_{9,3} + 3215472\, f_{3,9} + 353664\, f_{3,3,3,3}) \\
& - \zeta_6 \,(3612804\, f_{5,5} + 3791520\, f_{7,3} + 3409152\, f_{3,7}) - \zeta_8 \,(3720664\, f_{5,3} + 3456614\, f_{3,5}) \\
& - \frac{19560489}{5} \zeta_{10}\, f_{3,3} - \frac{512193667550809}{7639104} \zeta_{16}
\end{aligned}
$$

- Blue values successfully predicted by antipodal duality
- Result consistent with coaction principle at weight 16.

Antipodal “symmetry”

Y.-T. Liu, 2207.11815

- There’s a letter map for n-gluon MHV amplitudes, at least for $n = 6,7,8$, on their parity preserving surfaces, which maps the symbol into its own antipode: $$\mathcal{S}(R_{n,e}^{(2)}) = \frac{1}{4} S\left(\mathcal{S}(R_{n,e}^{(2)})\Big|_{\ln\phi_i \mapsto A_n^{ij} \ln\phi_j} \right)$$
- Not a map of the underlying variables.
- Doesn’t currently work past 2 loops.
- But it’s the first evidence for some kind of antipodal action beyond one loop and $n = 6$

Summary & Open Questions

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- Comparing the 6-gluon amplitude to the 3-gluon form factor, a strange new antipodal duality emerges, swapping the role of branch cuts and derivatives
- **What is the question to which antipodal duality is the answer?**
- Relation to flux tube representation?
- (How) does it hold at strong coupling??
- Does it hold at 8g-4gFF level???
- Meaning of antipodal symmetry?
- How much more can we exploit to learn more about both amplitudes and form factors?

Not the first antipodal duality?

Center for Research on Economic and Social Theory
Research Seminar in Quantitative Economics
Discussion Paper

GORMAN AND MUSGRAVE ARE
DUAL - AN ANTIPODAL THEOREM

T. Bergstrom and R. Cornes

June 1981

C-39

DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, Michigan 48109

Extra Slides

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[\left(\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon) \right) M^{1-\mathrm{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w) \right] \right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac… …y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\mathrm{loop}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} \equiv M^{1-\mathrm{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2} \sum_{i=1}^{3} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \ldots \left(1 - \frac{1}{v}\right) + \mathrm{Li}_2\left(1 - \frac{1}{w}\right) \Big]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div… …by:

$$\frac{\mathcal{F}_3^{\mathrm{BDS-like}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[\left(\frac{\Gamma_{\mathrm{cusp}}}{4} + \mathcal{O}(\epsilon) \right) M(L\epsilon) + C^{(L)} \right] \right\} \Rightarrow$$

$$\mathcal{E} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4} \mathcal{E}^{(1)} + R \right]$$

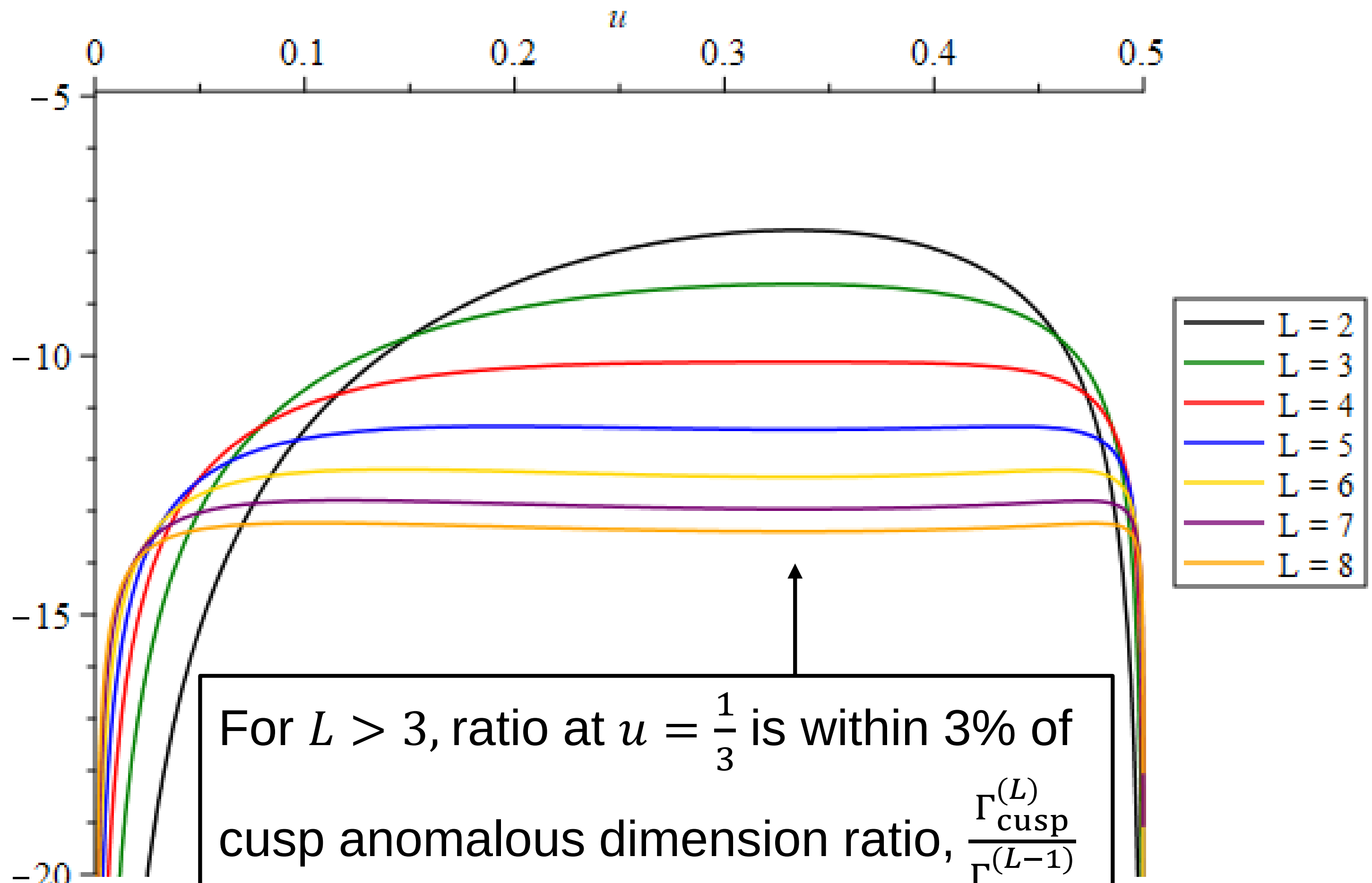
Finite radius of convergence

- Planar N=4 SYM has no renormalons ($\beta(g) = 0$) and no instantons ($e^{-1/g_{\rm YM}^2} = e^{-N_c/\lambda}$)
- Perturbative expansion can have finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling $g^2 \equiv \frac{N_c g_{\rm YM}^2}{16\pi^2} = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$

Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}} \to -16$
- Find same radius of convergence in high-loop-order behavior of amplitudes and form factors, in most kinematic regions.

Euclidean Region form factor numerics



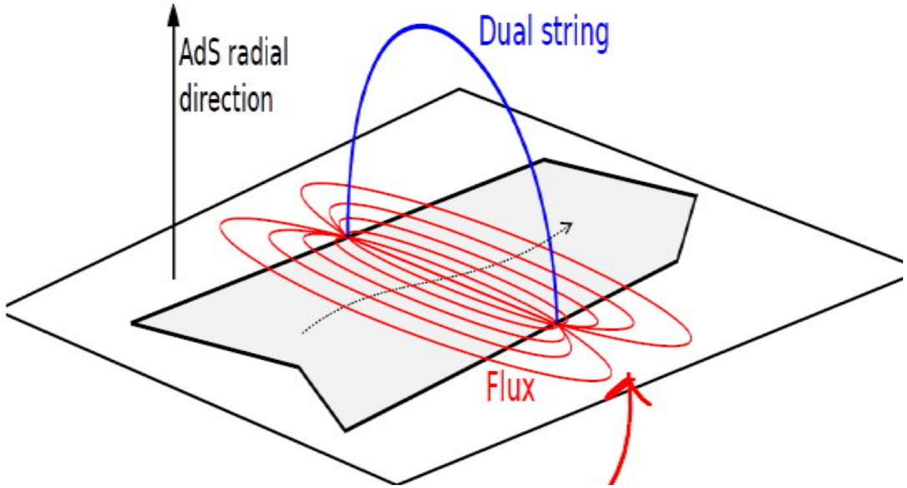
For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}}$

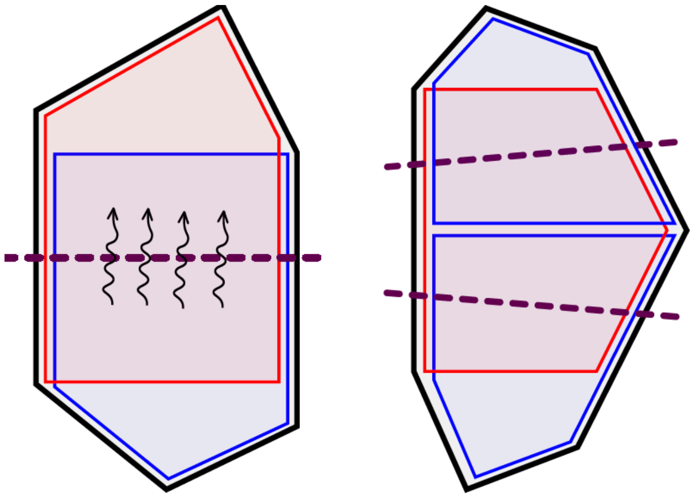
→ same finite radius of convergence…

$\mathcal{E} \equiv F_3$

Bootstrap boundary data:
Flux tubes at finite coupling

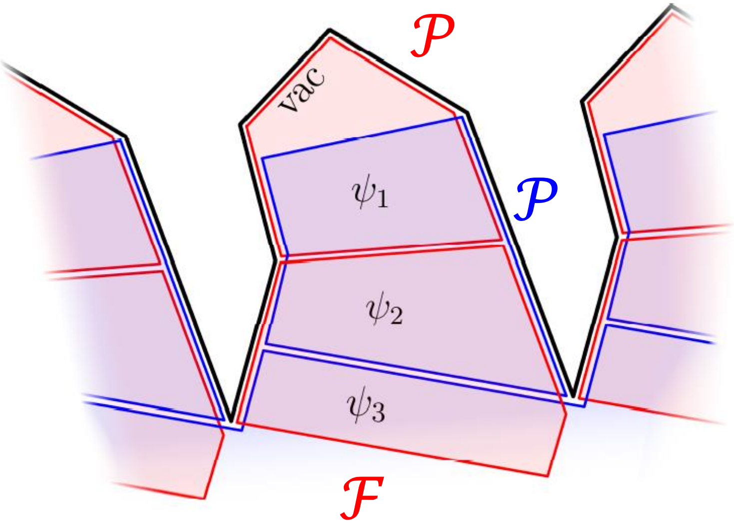
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

A New Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\rm hex} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{-\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \to 0$,

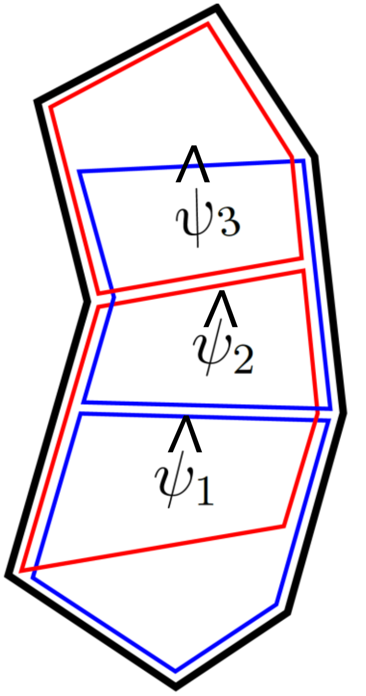
weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- 3-gluon form factor: $\quad \psi = helicity\ 0\ pairs\ of\ states$

$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

8-4 Kinematic Map in OPE Parametrization



- 8-point amplitude has D_8 dihedral symmetry; change it to D_4 of the form factor by requiring

$$\hat{T}_3 = \hat{T}_1, \quad \hat{S}_3 = \hat{S}_1, \quad \hat{F}_3 = \hat{F}_1$$

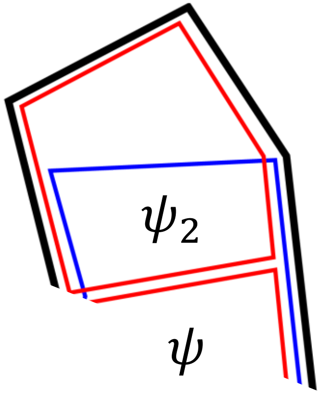
- To get $\mathcal{S}[R_8^{(2)}]$ to have only 8 final

entries, we also fix $\hat{F}_1 = \hat{F}_2 = 1$.

- The kinematic map becomes

$$\hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS},$$

$$\hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2 S_2}$$ and requires $F_2 = i$



8-gluon Amp ←→ 4-gluon FF

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, in progress

- We have a candidate kinematic map for a 4-dimensional surface (4-gluon FF is 5d).
- $\mathcal{S}[R_8^{(2)}]$ is known S. Caron-Huot, 1105.5606
- The kinematic+antipodal maps take it to a symbol with 40 letters, the first 8 of which are "right": $u_i = \frac{s_{i,i+1}}{s_{1234}}, \quad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$
- However, the candidate 2-loop 4-gluon form factor doesn't match the FFOPE (???)

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

evaluate to Riemann zeta values

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3}, \zeta_{7,3}, \zeta_{5,3,3}, \zeta_{9,3}, \zeta_{6,4,1,1}, \ldots$

Many “empirical” adjacency constraints

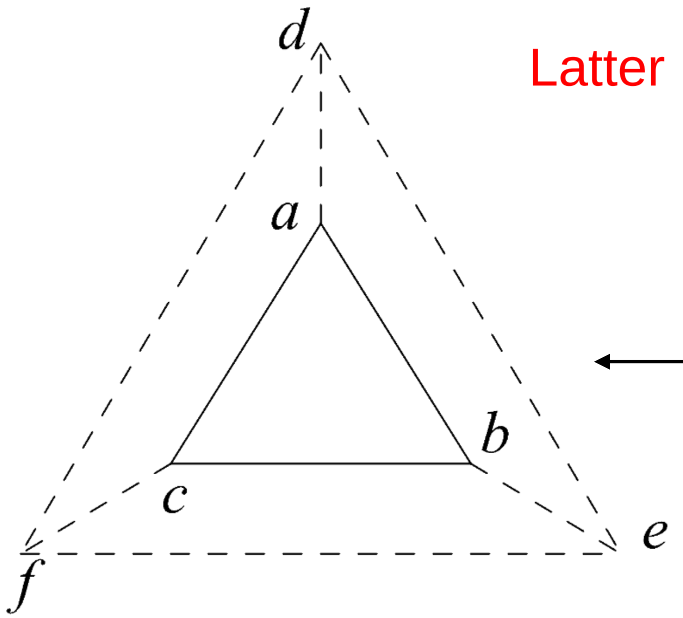
$$F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0$$

Hold for 2 loop QCD amplitudes too, planar and nonplanar!

LD, Mcleod, Wilhelm, 2012.12286

$$F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0$$

Latter NEW: Hold for planar N=4 SYM to 8 loops!



Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

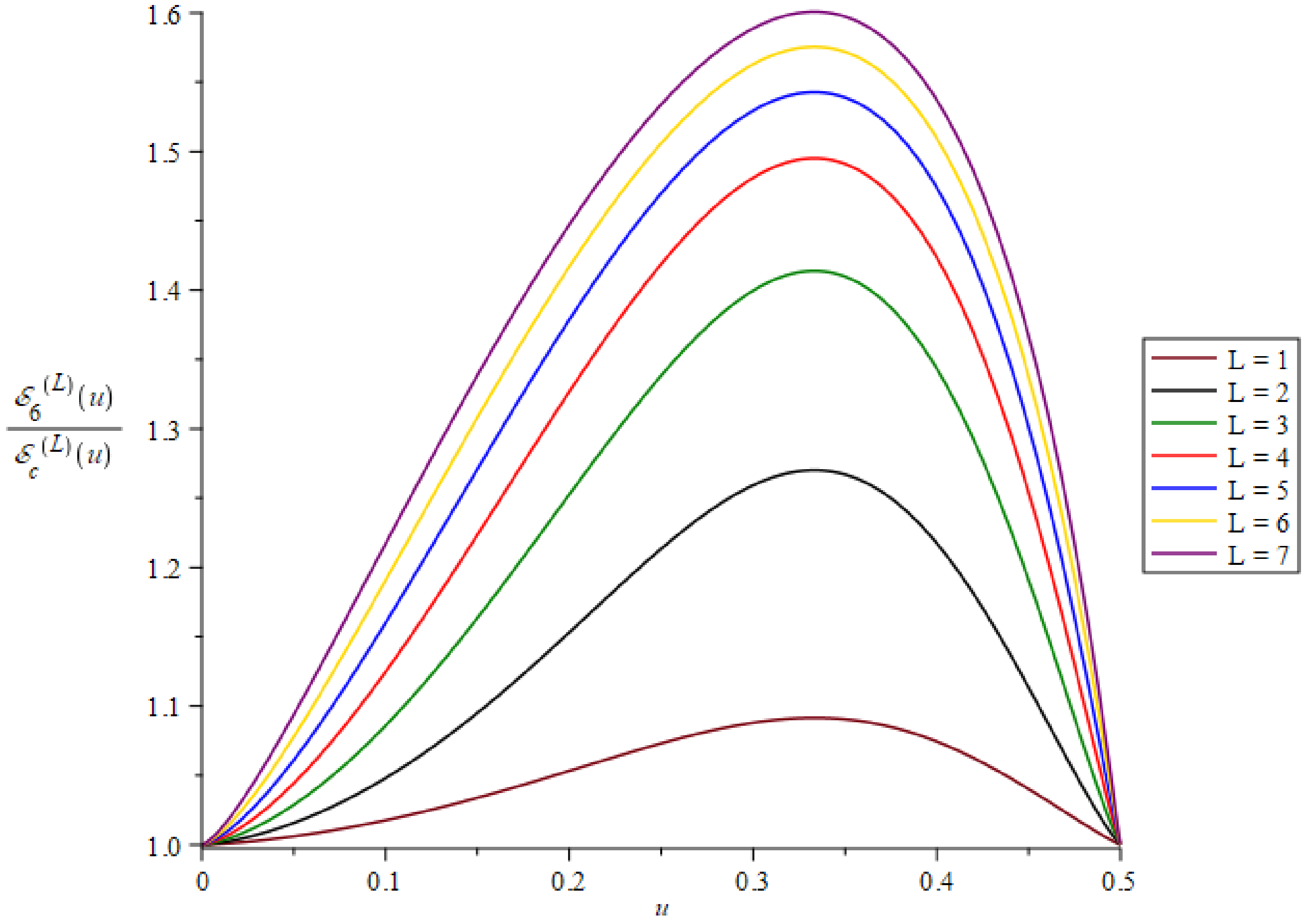
weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

Number of remaining parameters in form-factor ansatz after imposing constraints

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

Numerical implications of antipodal duality?



Origin at weak coupling

- Remarkably, MHV remainder R_6 and closely−related quantity $\ln \mathcal{E}$ are quadratic in logarithms through 7 loops CDDvHMP, 1903.10890

$$\ln \mathcal{E}(u_i) \approx -\frac{\Gamma_{\rm oct}}{24} \ln^2(u_1 u_2 u_3) - \frac{\Gamma_{\rm hex}}{24} \sum_{i=1}^{3} \ln^2 \frac{u_i}{u_{i+1}} + C_0$$

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$\Gamma_{\rm oct}$	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
$\Gamma_{\rm cusp}$	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$	$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$
$\Gamma_{\rm hex}$	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$	$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$

- Coefficients involve same BES kernel as for cusp, but “tilted” by angle α,

$$\Gamma_{\rm cusp} = \Gamma_{\alpha=\pi/4} \qquad \Gamma_{\rm oct} = \Gamma_{\alpha=0} \qquad \Gamma_{\rm hex} = \Gamma_{\alpha=\pi/3}$$

B. Basso, LD, G. Papathanasiou, 2001.05460