Symbology for elliptic Feynman integrals and scattering amplitudes

Matthias Wilhelm, Niels Bohr Institute

Amplitudes '22, Prague

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[2106.14902] with A. Kristensson and C. Zhang
[2206.08378] with C. Zhang
[22xx.xxxxx] with A. Spiering, C. Zhang, R. Morales and Q. Yang

The Niels Bohr International Academy

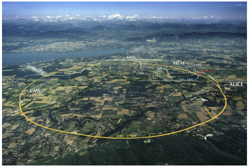
VILLUM FONDEN

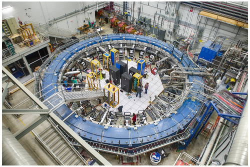
Motivation

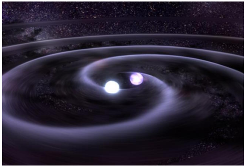
What numbers and functions occur in QFT?

What identities do they satisfy?

How can we exploit this knowledge to extend our calculational reach?

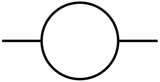




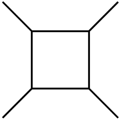


Goncharov polylogarithms and the symbol

At one loop: Polylogarithms



$$\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right),$$



$$\overset{D=4}{\sim} \mathrm{Li}_2(\dots)+\dots$$

Goncharov polylogarithms and the symbol

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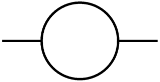
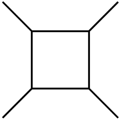
 $\overset{D=2}{\sim} \log\left(\frac{1+2\frac{m^2}{s}+\sqrt{1+4\frac{m^2}{s}}}{1+2\frac{m^2}{s}-\sqrt{1+4\frac{m^2}{s}}}\right)$,  $\overset{D=4}{\sim} \mathrm{Li}_2(\ldots)+\ldots$

More generally: Goncharov polylogarithms [Chen (1977)], [Goncharov (1995)], ...

$$G(a_1,\ldots,a_n;x)=\int_0^x \frac{dt}{t-a_1}G(a_2,\ldots,a_n,t), \qquad G(;x)=1$$

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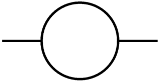

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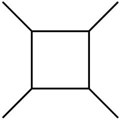

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Symbol [Goncharov, Spradlin, Vergu, Volovich (2010)]

$$\mathrm{d}G(\vec{a};z) = \sum_i F_i \,\mathrm{dlog}(\phi_i) \quad \Rightarrow \quad \mathcal{S}(G(\vec{a};z)) = \sum_i \mathcal{S}(F_i) \otimes \log(\phi_i)$$

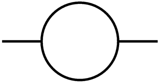
- Tensor product
- $\log(ab) = \log(a) + \log(b)$

$\Rightarrow$ Easy to manipulate and well understood

$\longrightarrow$ talks by Song, Simone, Guilherme, Lance, Johannes, Matt, Andrew & Hofie

Goncharov polylogarithms and the symbol

At one loop: Polylogarithms



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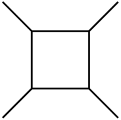

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Idea: Decompose complicated functions to simple functions!

$\rightarrow$

$= \log(\phi_i)$

Exploiting the power of the symbol

Dramatic simplification

- Two-loop six-gluon remainder function [Duhr, Del Duca, Smirnov (2010)]
  `18 pages` $\rightarrow$ `2 lines` [Goncharov, Spradlin, Vergu, Volovich (2010)]

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Symbol alphabet

- Related to cluster algebras for six- and seven-gluon remainder functions in planar $\mathcal{N} = 4$ SYM theory [Golden, Goncharov, Spradlin, Vergu, Volovich (2013)],... $\longrightarrow$ Song's talk

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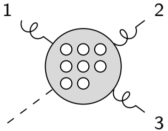
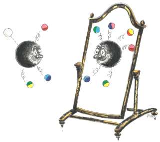
Symbol bootstrap

- Six- and seven-gluon remainder functions in planar $\mathcal{N}=4$ SYM theory up to seven- and four-loop order [Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdoğan, von Hippel, McLeod, Liu, Papathanasiou, ...]
- Solutions to differential equations [Henn, Herrmann, Parra-Martinez (2018)], ...
- Maximally transcendental part of Higgs $\rightarrow$ 3 gluons in planar Yang-Mills theory up to 8-loop order!

$$u=\frac{s_{12}}{s_{123}},\quad v=\frac{s_{23}}{s_{123}},\quad w=\frac{s_{13}}{s_{123}}$$

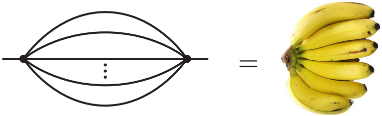
$$\phi_i\in\{u,v,w,1-u,1-v,1-w\}$$





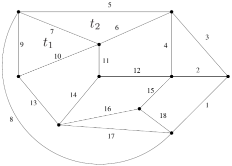
[Dixon, McLeod, **MW** (2020)], [Dixon, Gürdoğan, McLeod, **MW** (2021,2)] $\rightarrow$ Lance's talk

Beyond polylogarithms: Elliptics & Calabi-Yau manifolds



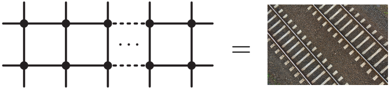
Calabi-Yau $(L-1)$-fold

[Broadhurst, Fleischer, Tarasov (1993)], ..., [Bönisch, Duhr, Fischbach, Klemm, Nega (2021)]



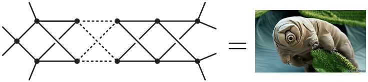
K_3 = Calabi-Yau 2-fold

[Brown, Schnetz (2010)]



Calabi-Yau $(L-1)$-fold

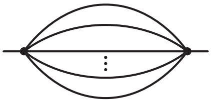
[Bourjaily, He, McLeod, von Hippel, **MW** (2018)], [Vergu, Volk (2020)]



Calabi-Yau $(2L-2)$-fold

[Bourjaily, McLeod, von Hippel, **MW** (2018)]

Beyond polylogarithms: Elliptics & Calabi-Yau manifolds

 = 

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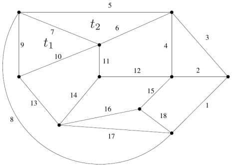

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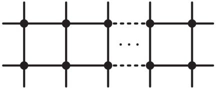 = 

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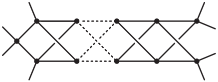
 = 

Calabi-Yau $(2L-2)$-fold

[Bourjaily, McLeod, von Hippel, **MW** (2018)]

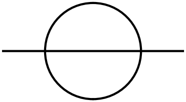
Question: Similar understanding, structures & techniques?

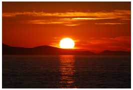
This talk: Symbology for Elliptics

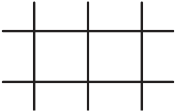
Plan for this talk:

- Review of elliptic generalizations of polylogarithms [Brown, Levin (2011)], ..., [Brödel, Duhr, Dulat, Tancredi (2017)]
- Symbology for elliptic polylogarithms
  Analog of $\log(a) + \log(b) = \log(ab)$? [**MW**, Zhang (2022)]
- Hidden simplicity → Bootstrap
  [Morales, Spiering, **MW**, Yang, Zhang (in progress)]

Examples:







[Broadhurst, Fleischer, Tarasov (1993)], ..., [Adams, Bogner, Weinzierl (2013)], ...

[Caron-Huot, Larsen (2012)], [Bourjaily, McLeod, Spradlin, von Hippel, **MW** (2017)], [Kristensson, **MW**, Zhang (2021)]

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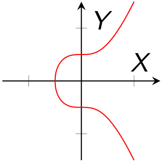
Elliptic polylogarithms I

Elliptic curve

$$Y^2 = 4X^3 - g_2 X - g_3$$



$$y = \sqrt{P(x)}, \qquad \deg_x P(x) = 3, 4$$

Elliptic polylogarithms [Brödel, Duhr, Dulat, Tancredi (2017)]

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x dx' \, \psi_{n_1}(c_1, x') \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x'\right), \qquad \mathrm{E}_4(; x) = 1$$

Integration kernels

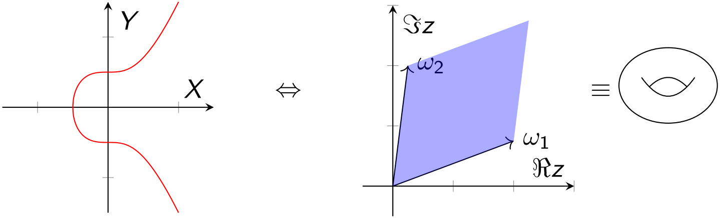
$$\psi_1(c, x) = \frac{1}{x - c}, \qquad \psi_0(0, x) = \frac{1}{y},$$

$$\psi_{-1}(c, x) = \frac{y|_{x=c}}{y(x - c)}, \qquad \psi_{\pm(n>1)}(c, x) = \dots$$

Not pure $\rightarrow$ No symbol

Elliptic curve and torus

$Y^2 = 4X^3 - g_2 X - g_3$ $\qquad\qquad$ $\mathbb{C}/\{\omega_1\mathbb{Z} + \omega_2\mathbb{Z}\}$



- Elliptic curve to torus: Abel map

$$z_c^+ = \int_{-\infty}^{c} \frac{dx}{y}$$

- Torus to elliptic curve: Weierstraß function

$$(X, Y) = (\wp(z), \wp'(z))$$

- Cave: each kinematic point c corresponds to two points $(c, \pm y|_{x=c})$ on the curve and has two images $z_c^{\pm}$ on the torus!

Elliptic polylogarithms II

Elliptic polylogarithms on the normalized torus $(\tau = \omega_2/\omega_1, w = z/\omega_1)$
[Brödel, Duhr, Dulat, Tancredi (2017)] [Brödel, Duhr, Dulat, Penante, Tancredi (2018)]

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{smallmatrix}; w\right) = \int_0^w dw'\, g^{(n_1)}(w'-w_1)\widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ w_2 & \dots & w_k \end{smallmatrix}; w'\right), \qquad \tilde{\Gamma}(;w) = 1$$

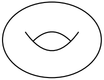
Integration kernels generated by the Eisenstein-Kronecker series:

$$\frac{(\partial_{w'}\theta_1(w')|_{w'=0})\theta_1(w+\alpha)}{\theta_1(w)\theta_1(\alpha)} = \sum_{n\geq 0} \alpha^{n-1} g^{(n)}(w)\,,$$

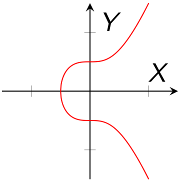
where $\theta_1(w) \equiv \theta_1(w|\tau) = \sum_{n=-\infty}^{+\infty} (-1)^{n-\frac{1}{2}} e^{\pi i \tau (n+\frac{1}{2})^2} e^{(2n+1)iw}$

Properties

- Length $k \neq$ weight $\sum_k n_k$
- Pure $\rightarrow$ Symbol [Brödel, Duhr, Dulat, Penante, Tancredi (2018)]

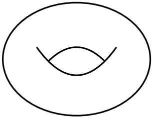


Translation between elliptic polylogarithms



$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right)$

$\Leftrightarrow$



$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{smallmatrix}; w\right)$

Relations between integration kernels

$$\psi_1(c,x)dx = \frac{1}{x-c}dx = \sum_{\sigma=\pm}[g^{(1)}(w-w_c^\sigma) - g^{(1)}(w-w_\infty^\sigma)]dw$$

$$\psi_{-1}(c,x)dx = \frac{y_c}{y(x-c)}dx = \sum_{\sigma=\pm}[\sigma g^{(1)}(w-w_c^\sigma) + \sigma g^{(1)}(w_c^\sigma)]dw$$

$$\psi_0(0,x)dx = \frac{dx}{y} = \omega_1 \underbrace{g^{(0)}(w)}_{1} dw$$

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Symbol for elliptic polylogarithms

Symbol for Goncharov polylogarithms [Goncharov, Spradlin, Vergu, Volovich (2010)]

$$\mathrm{d}G(\vec{a};z)=\sum_i F_i\,\mathrm{dlog}(\phi_i) \quad \Rightarrow \quad \mathcal{S}(G(\vec{a};z))=\sum_i \mathcal{S}(F_i)\otimes\log(\phi_i)$$

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Elliptic generalization: $\underline{\widetilde{\Gamma}}_k^{(n)} = (2\pi i)^{k-n} \widetilde{\Gamma}_k^{(n)}$ of weight n and length k
[Brödel, Duhr, Dulat, Penante, Tancredi (2018)]

$$\mathrm{d}\underline{\widetilde{\Gamma}}_k^{(n)} = \sum_i \underline{\widetilde{\Gamma}}_{k-1}^{(n-j_i)} \,\mathrm{d}\Omega^{(j_i)}(w_i) \quad \Rightarrow \quad \mathcal{S}(\underline{\widetilde{\Gamma}}_k^{(n)}) = \sum_i \mathcal{S}(\underline{\widetilde{\Gamma}}_{k-1}^{(n-j_i)}) \otimes \Omega^{(j_i)}(w_i)$$

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Remarks:

- Symbol letters $\Omega^{(n)}$: $\underline{\widetilde{\Gamma}}(\substack{n \\ 0}; w) = \Omega^{(n)}(w) - \Omega^{(n)}(0)$
- Factors of $(2\pi i) \Rightarrow$ Degenerate limit $\tau \to i\infty$: $\Omega^{(n)} \to \log$

Elliptic Symbology

Elliptic Symbology Analog of $\log(a) + \log(b) = \log(ab)$?

- Parity $\Omega^{(j)}(-w) = (-1)^{j+1}\Omega^{(j)}(w)$
- Quasi-periodicity
  $\Omega^{(n)}(w+\tau) = \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}\Omega^{(n-j)}(w)$

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- Relation between kernels
  Recall $\psi_1(c,x)dx = \frac{dx}{x-c} = \sum_{\sigma\in\pm}[g^{(1)}(w-w_c^\sigma) - g^{(1)}(w-w_\infty^\sigma)]dw$

  Trivial identity

$$\int_a^b \psi_1(c,x)dx = \log\frac{b-c}{a-c}$$
$$= \sum_{\sigma\in\pm} \Omega^{(1)}(w_b^+ - w_c^\sigma) - \Omega^{(1)}(w_b^+ - w_\infty^\sigma) - (b\to a)$$

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- Further identities?

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  $\Omega^{(n)}(w+\tau) = \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}\Omega^{(n-j)}(w)$
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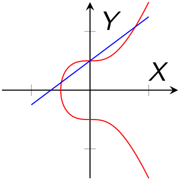
  Recall $\psi_1(c,x)dx = \frac{dx}{x-c} = \sum_{\sigma\in\pm}[g^{(1)}(w-w_c^\sigma) - g^{(1)}(w-w_\infty^\sigma)]dw$

  Trivial identity

$$\int_a^b \psi_1(c,x)dx = \log\frac{b-c}{a-c}$$
$$= \sum_{\sigma\in\pm} \Omega^{(1)}(w_b^+ - w_c^\sigma) - \Omega^{(1)}(w_b^+ - w_\infty^\sigma) - (b\to a)$$

- Further identities?
  - Abel's addition theorem for $\Omega^{(0,1)}$
  - Symbol prime for $\Omega^{(2)}$

Abel's addition theorem for elliptic integrals



$$\mathcal{C}: \quad y^2 = P_4(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
$$\mathcal{C}': \quad y = -x^2 + b_1 x + b_2$$

Intersect in points (x_1, y_1), (x_2, y_2), $(x_3, -y_3)$
with $y_i = \sqrt{P_4(x_i)}$

Abel's addition theorem [Abel]

$I = \sum_{(x_i,y_i)\in\mathcal{C}\cap\mathcal{C}'} \int^{(x_i,y_i)} \text{rational}(x,y)$ contains at most logarithms of b_i

[**MW**, Zhang (2022)]

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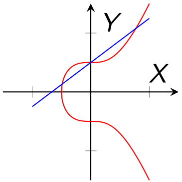

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- Torus images: $w_c^+ = \dfrac{1}{\omega_1}\displaystyle\int_{-\infty}^{c} \dfrac{dx}{y}$

  Abel's theorem: $w_{x_1}^+ + w_{x_2}^+ = w_{x_3}^+ \Rightarrow$ Identity for $\Omega^{(0)}(w) = 2\pi i w$

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Abel's addition theorem for elliptic integrals

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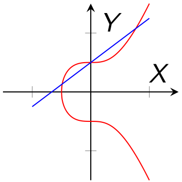

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- Recall $\psi_{-1}(c, x) = \frac{y_c}{y(x-c)} = \sum_{\sigma=\pm}[\sigma g^{(1)}(w - w_c^\sigma) + \sigma g^{(1)}(w_c^\sigma)]dw$

  Abel's theorem $\Rightarrow$ Identity for $\Omega^{(1)}(w)$

[**MW**, Zhang (2022)]

Identities for $\Omega^{(2)}$?

Identities for $\Omega^{(2)}$? $\Rightarrow$ elliptic Bloch relations

[Bloch (2000)], [Zagier, Gangl (2000)], [Brödel, Kaderli (2019)], [Bolbachan (2019)]

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Five-term identity for the Bloch-Wigner dilogarithm
$D = \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$:

$$D(t) + D(s) + D\left(\tfrac{1-t}{1-ts}\right) + D(1-ts) + D\left(\tfrac{1-s}{1-ts}\right) = 0$$

Generalize to the elliptic Bloch-Wigner function

$$D^E(t,\tau) = \sum_{l\in\mathbb{Z}} D(e^{2\pi i\tau l}t)$$

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Problem We do not use the five-term identity for D, we use the symbol!

$\Omega^{(2)} \sim D^E$ is already a symbol letter $\Rightarrow$ Define symbol prime

Symbol prime

Consider

$$\mathcal{S}(\widetilde{\underline{\Gamma}}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; w)) = \Omega^{(0)}(w) \otimes \Omega^{(1)}(w) - \Omega^{(2)}(w) \otimes (2\pi i \tau)$$

Define symbol prime

$$\mathcal{S}'(\underbrace{\Omega^{(2)}(w)}) = \underbrace{\Omega^{(0)}(w)} \otimes' \underbrace{\Omega^{(1)}(w)}$$



Properties

- Reduces complicated identities of $\Omega^{(2)}$ to identities of $\Omega^{(0)}, \Omega^{(1)}$
  $\rightarrow$ Simplifications
- ...

[**MW**, Zhang (2022)]

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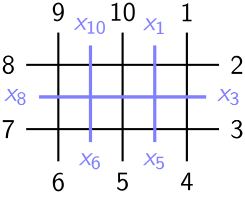
Example: 10-point elliptic double box

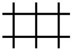


$$= \int d^4x_l d^4x_k \frac{(x_1 - x_5)^2}{(x_1 - x_l)^2(x_3 - x_l)^2(x_5 - x_l)^2} \times \frac{(x_3 - x_8)^2(x_6 - x_{10})^2}{(x_l - x_k)^2(x_6 - x_k)^2(x_8 - x_k)^2(x_{10} - x_k)^2}$$

- Dual momenta $x_a - x_{a+1} = p_a$
- (Seven independent) cross-ratios $x_{a,b;c,d} = \frac{(x_a - x_b)^2(x_c - x_d)^2}{(x_a - x_c)^2(x_b - x_d)^2}$
- First elliptic contribution to planar $\mathcal{N} = 4$ SYM theory
  Only contribution to a particular component of the planar 10-point N^3MHV amplitude [Caron-Huot, Larsen (2012)]

Structure of the symbol ($\tau = \omega_2/\omega_1$):

$$\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$$



$$\left.\vphantom{\frac{2\pi i}{\omega_1}}\right)$$

$$= \sum_{ikl} c^{ikl} \log(\phi_k) \otimes \log(\phi_l) \otimes \left[\sum_j \log(\phi_{ij}) \otimes \left(\frac{2\pi i}{\omega_1}\int_{-\infty}^{c_j} \frac{dx}{y}\right) + \boldsymbol{\Omega}_i \otimes (2\pi i \tau)\right]$$

with $\boldsymbol{\Omega}_i = \sum \Omega^{(2,1,0)}$ and

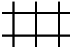
$$\mathcal{S}'(\boldsymbol{\Omega}_i) = \sum_j \left(\frac{2\pi i}{\omega_1}\int_{-\infty}^{c_j} \frac{dx}{y}\right) \otimes' \log(\phi_{ij})$$

$$= \quad \{\underbrace{\log(\phi_i)}_{53}, \underbrace{\left(\frac{2\pi i}{\omega_1}\int_{-\infty}^{c_j} \frac{dx}{y}\right), 2\pi i \tau}_{6}\}$$

[**MW**, Zhang (2022)]

Symbol of the 10-point elliptic double box

Structure of the symbol ($\tau = \omega_2/\omega_1$):

$\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$  $\left.\right)$

$$= \sum_{ikl} c^{ikl} \log(\phi_k) \otimes \log(\phi_l) \otimes \left[\sum_j \log(\phi_{ij}) \otimes \left(\frac{2\pi i}{\omega_1} \int_{-\infty}^{c_j} \frac{dx}{y} \right) + \boldsymbol{\Omega}_i \otimes (2\pi i \tau) \right]$$

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Structure manifested by the symbol prime

- Double periodicity
- Integrability w.r.t. τ

$\Rightarrow$ Symbol prime explains $\boldsymbol{\Omega}_i$!

[**MW**, Zhang (2022)]

Elliptic bootstrap?

Warm up: $\mathcal{S}\left(\frac{2\pi i}{\omega_1}\ \#\right)$

Disassemble and reassemble?

 $\rightarrow$  $\rightarrow$?

[Morales, Spiering, **MW**, Yang, Zhang (in progress)]

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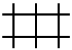
Warm up: $\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$  $\left.\right)$

Bootstrap:

$$\underbrace{(7\text{ first})\times(26\text{ second})}_{\xrightarrow{\text{integrability}}43\xrightarrow{\text{Steinmann}}28}\times(53\text{ third})\times(6\text{ last}\neq(2\pi i)\tau)$$

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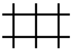
Warm up: $\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$  $\left.\right)$

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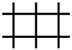
Warm up: $\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$  $\left.\right)$

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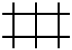
Warm up: $\mathcal{S}\left(\frac{2\pi i}{\omega_1}\right.$  $\left.\right)$

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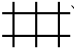
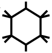
Differential equation [Paulos, Spradlin, Volovich (2012)], [Nandan, Paulos, Spradlin, Volovich (2013)]

$\partial_{u_5}\Big($  $\Big) = \frac{1}{\sqrt{\Delta_6}}\Big($  $_{D=6}\Big)$

Δ_6: six-point Gram determinant $\Rightarrow$ Fix overall constant!

[Morales, Spiering, **MW**, Yang, Zhang (in progress)]

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Conclusion and outlook

Conclusion

- Symbology for elliptic Feynman integrals & scattering amplitudes $\equiv$ Analog of $\log(a) + \log(b) = \log(ab)$ for elliptic letters $\Omega^{(n)}$
  - $\Omega^{(0,1)}$: Abel's addition theorem
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  $\rightarrow$ Reveals hidden simplicity!
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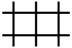
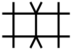
Outlook

- Simplified symbol $\to$ simplified function
- Bootstrap elliptic Feynman integrals and scattering amplitudes $+$ Elliptic letters via Schubert problem

  Works for  $\to$ Next target: 

  [Morales, Spiering, **MW**, Yang, Zhang (in progress)]
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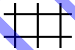
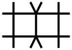
  Works for  $\rightarrow$ Next target: 

  [Morales, Spiering, **MW**, Yang, Zhang (in progress)]
- Generalize symbology to Calabi-Yau integrals

Thank you!

Bonus slide: Symbol letters

- $\Omega^{(-1)} = -2\pi i\tau$, $\Omega^{(0)}(w) = 2\pi i w$, $\Omega^{(1)}(w) = \log(\theta_1(w,\tau)/\eta(\tau))$
- Sum representation

$$\Omega^{(\text{odd } j)}(w) = \Big(\log(1-\mathrm{e}^{2\pi i w}) - \pi i w\Big)\delta_{j,1} - \frac{2j\zeta_{j+1}\tau}{(2\pi i)^j} + \frac{1}{(j-1)!}\sum_{n=1}^{\infty} n^{j-1}\log((1-\mathrm{e}^{2\pi i(n\tau-w)})(1-\mathrm{e}^{2\pi i(n\tau+w)})),$$

$$\Omega^{(\text{even } j)}(w) = -\frac{2\zeta_j w}{(2\pi i)^{j-1}} + \frac{1}{(j-1)!}\sum_{n=1}^{\infty} n^{j-1}\log\frac{1-\mathrm{e}^{2\pi i(n\tau+w)}}{1-\mathrm{e}^{2\pi i(n\tau-w)}}$$

- Degenerate limit: $y = x^2 + 2ax + b$, $\tau \to i\infty$

$$w_c^+ = \frac{1}{\omega_1}\int_{-\infty}^{c}\frac{dx}{y} \to \frac{1}{2\pi i}\log\frac{c+a-\sqrt{a^2-b}}{c+a+\sqrt{a^2-b}}$$

$$\Omega^{(\text{even } j)}(w_c^+) \to -\frac{2\zeta_j}{(2\pi i)^j}\log\frac{c+a-\sqrt{a^2-b}}{c+a+\sqrt{a^2-b}}$$

Bonus slide: Symbol alphabet

- First entry: $\{\log(u_i), \log(v_j)\}_{i=1,\ldots,5;j=1,2} \rightarrow$ First-entry condition
- Second + third entry: $\{\log(a_i), \log(\frac{a_i - z_i}{a_i - \bar{z}_i}), \log(\frac{(z_i - \bar{z}_j)(\bar{z}_i - z_j)}{(z_i - z_j)(\bar{z}_i - \bar{z}_j)})\}$ with a_i rational in momentum twistors and

$$z_{abcd}\bar{z}_{abcd} = x_{a,b;c,d}, \qquad (1 - z_{abcd})(1 - \bar{z}_{abcd}) = x_{d,a;b,c}$$

from

- Last entry: $\left\{\left(\frac{2\pi i}{\omega_1}\int_{-\infty}^{c_j}\frac{dx}{y}\right), 2\pi i\tau\right\}$ with $c_j \in \{a_*, z_1 - 1, \bar{z}_1 - 1, -z_2, -\bar{z}_2, -i\infty\}$

[Kristensson, **MW**, Zhang (2021)],[**MW**, Zhang (2022)]