A first look at the function space for six-particle scattering amplitudes in QCD

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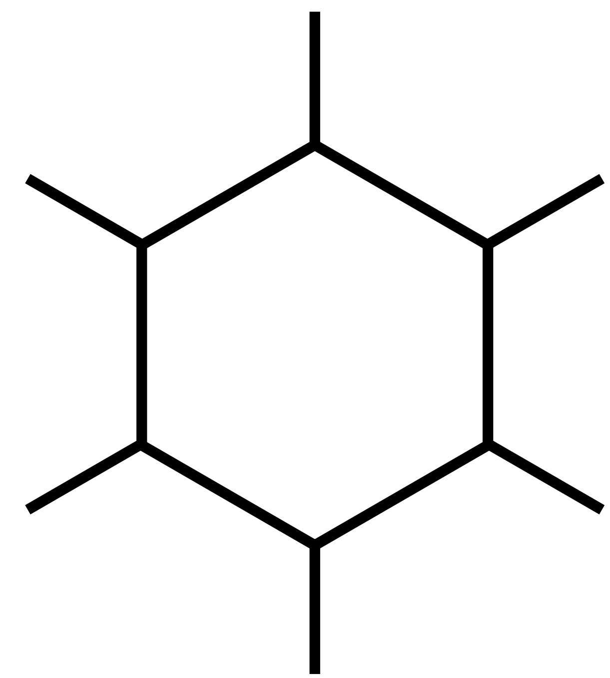
Motivation

High demand for NNLO integrals and amplitudes with many scales. State of the art is five external particles. [Cf. Simone Zoia's talk.]

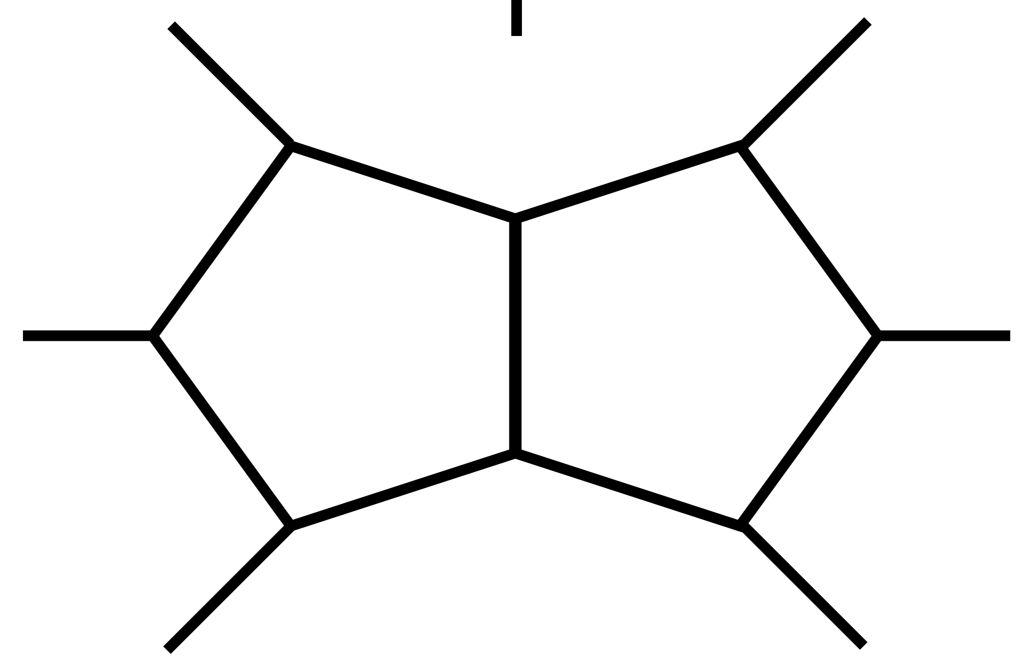
We will have a first look at 2 to 4 integrals at two loops.

A lot is known in N=4 super Yang-Mills, but QCD is much more complicated. Can we find similar structures?

Outline



1. D-dimensional hexagon integral and one-loop function space



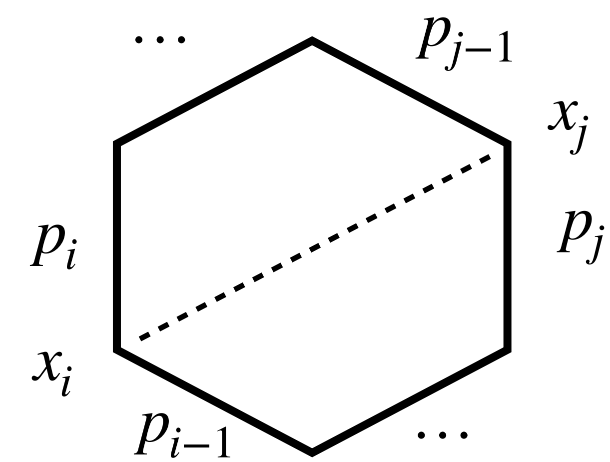
2. Planar two-loop six-point integrals on the maximal cut

3. Discussion and outlook

D-dimensional vs. 4-dimensional kinematics

At n points in D dimensions, $\frac{1}{2}n(n-3)$ kinematic variables

Independent set:



$$x_{ij}^2 := (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

For n=6, there are 9 such variables

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}\}$$

$$s_{ij} = (p_i + p_j)^2, \qquad s_{ijk} = (p_i + p_j + p_k)^2.$$

In four dimensions, only 8 of them are independent (in general $3n - 10$), because of a Gram determinant condition:

$$G := \det\left(p_i \cdot p_j\right) = 0\,, \quad i,j \in \{1,\ldots 5\}$$

For example solved by momentum twistor variables.

Canonical differential equations and function space

We derive differential equations for all master integrals in the problem. By choosing an appropriate basis we find canonical differential equations: [Henn 2013]

[cf. talks by Guilherme Pimentel, Simone Zoia]

$$d\mathbf{f}(\mathbf{v};\epsilon) = \epsilon \left[\sum_{i,j} A_i \, d\log \alpha_j(\mathbf{v}) \right] \mathbf{f}(\mathbf{v};\epsilon)$$

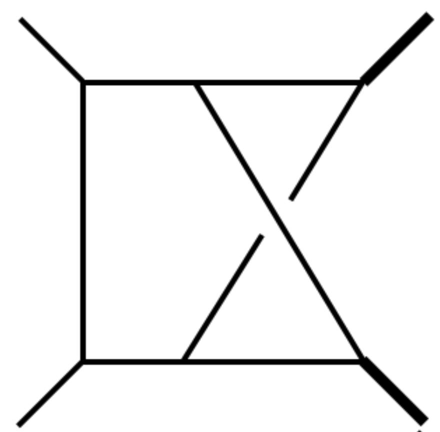
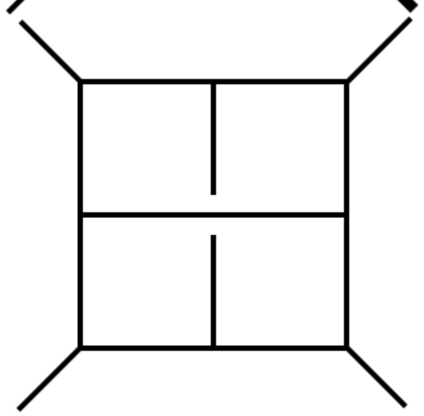
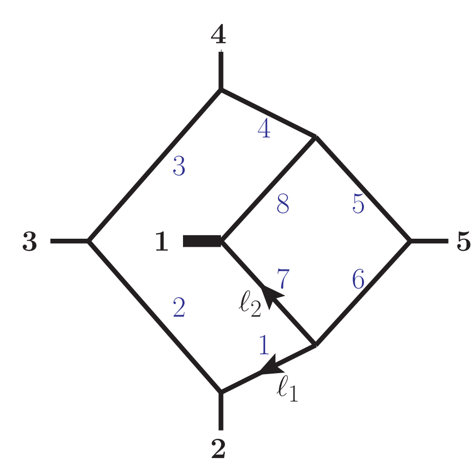
Vector of N master integrals

Constant NxN matrices

Symbol 'alphabet' determines function space [cf. talk by Song He]

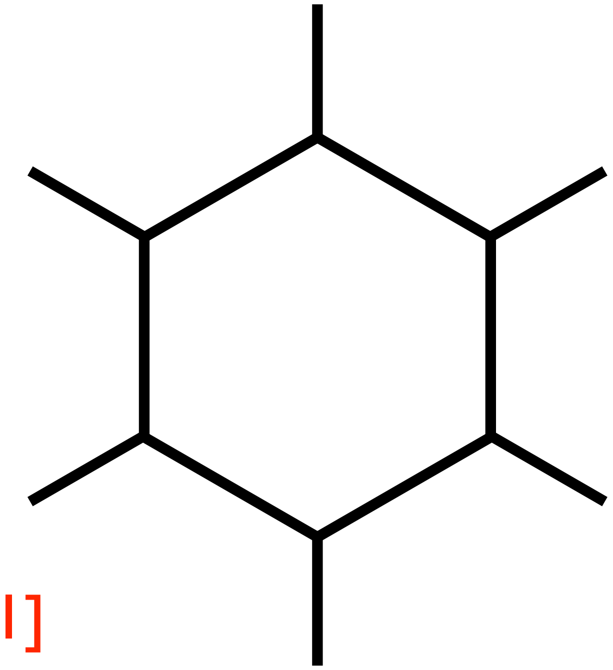
General form of RHS for elliptic case and beyond subject of intense study. [see e.g. talk by Matthias Wilhelm]

State-of-the-art examples

Type of integrals/kinematics		#MI	#vars	#letters
Integrals for two-loop V V' production [Caola, JMH, Melnikov, Smirnov, 2014]		51	3	19
Three-loop non-planar four-point integrals [JMH, Mistlberger, Wasser, Smirnov, 2020; see poster by Piotr Bargiela]		76	1	2
Five-point two-loop integrals with one off-shell leg [Abreu et al 2021; cf. talk by Simone Zoia]		135	5	63

Part I : D-dimensional hexagon integral and one-loop function space

Hexagon integral



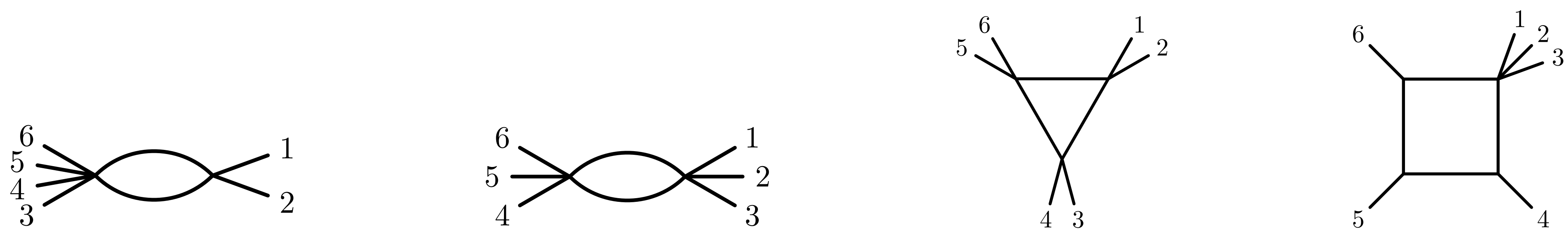
At D=6, this integral is known, it is a weight three function that has a dual conformal symmetry. [JMH, Drummond, Dixon 2011; Spradlin, Volovich 2011]

This is part of a beautiful dual conformal function space for N=4 sYM, related to the A3 cluster algebra. But this is another story… [See Lance Dixon's talk.]

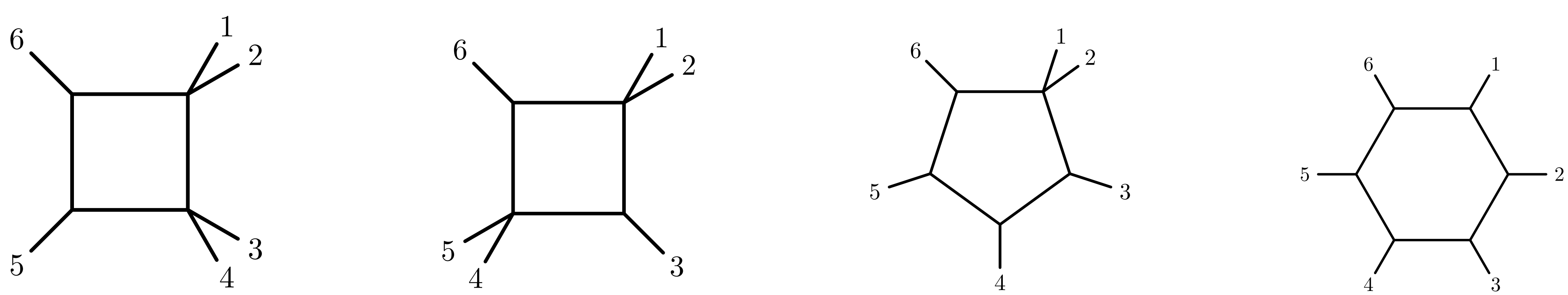
At D $\neq$ 6, it is a genuine function of 8 dimensionless variables. This is needed for computing higher-order terms in eps in conventional dimensional regularization.

System of differential equations

The integrals with up to five propagators are known, but it is easiest to recompute them directly in the notation for six-particle scattering.



(a) I_1 (b) I_7 (c) I_{10} (d) I_{13}



(e) I_{19} (f) I_{25} (g) I_{28} (h) I_{33}

Total of 33 master integrals.

Basis of uniform weight integrals

One-loop (n-1)-gons and n-gons in n dimensions are pure uniform weight functions, when normalized by their leading singularities. [shown for even n in Spradlin, Volovich 2011]

We relate all these integrals to a basis in the same dimension, via dimensional shifts. [Tarasov; Lee]

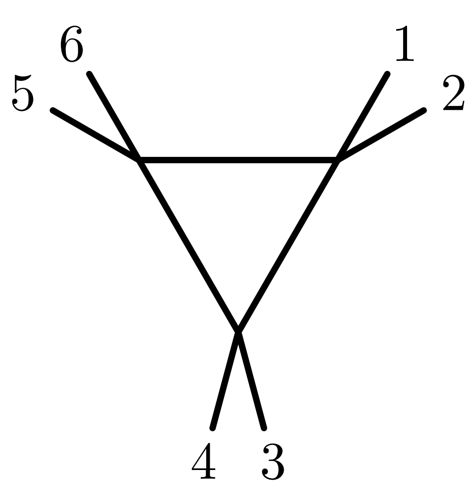
We verify that this basis leads to canonical differential equations.

One-loop hexagon alphabet

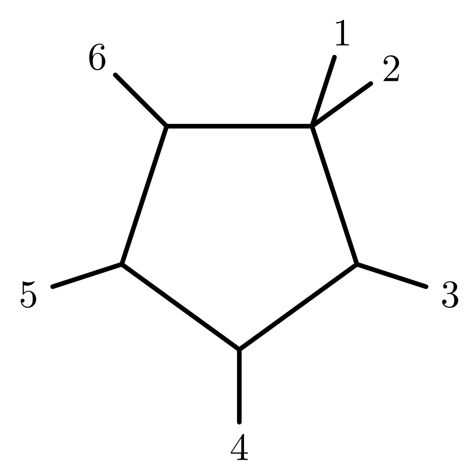
103 alphabet letters (48 even and 45 odd).

93 from one-mass pentagon kinematics (plus cyclic),
10 new letters [in agreement with Chen, Ma, Yang, 2022]

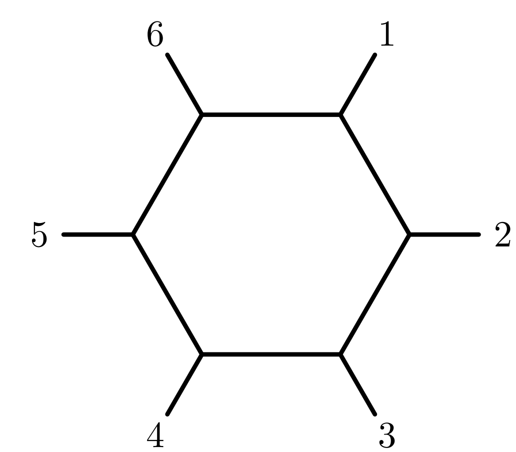
3 types of square roots:



$\rightarrow \sqrt{\Delta_3(v_1, v_3, v_5)}$

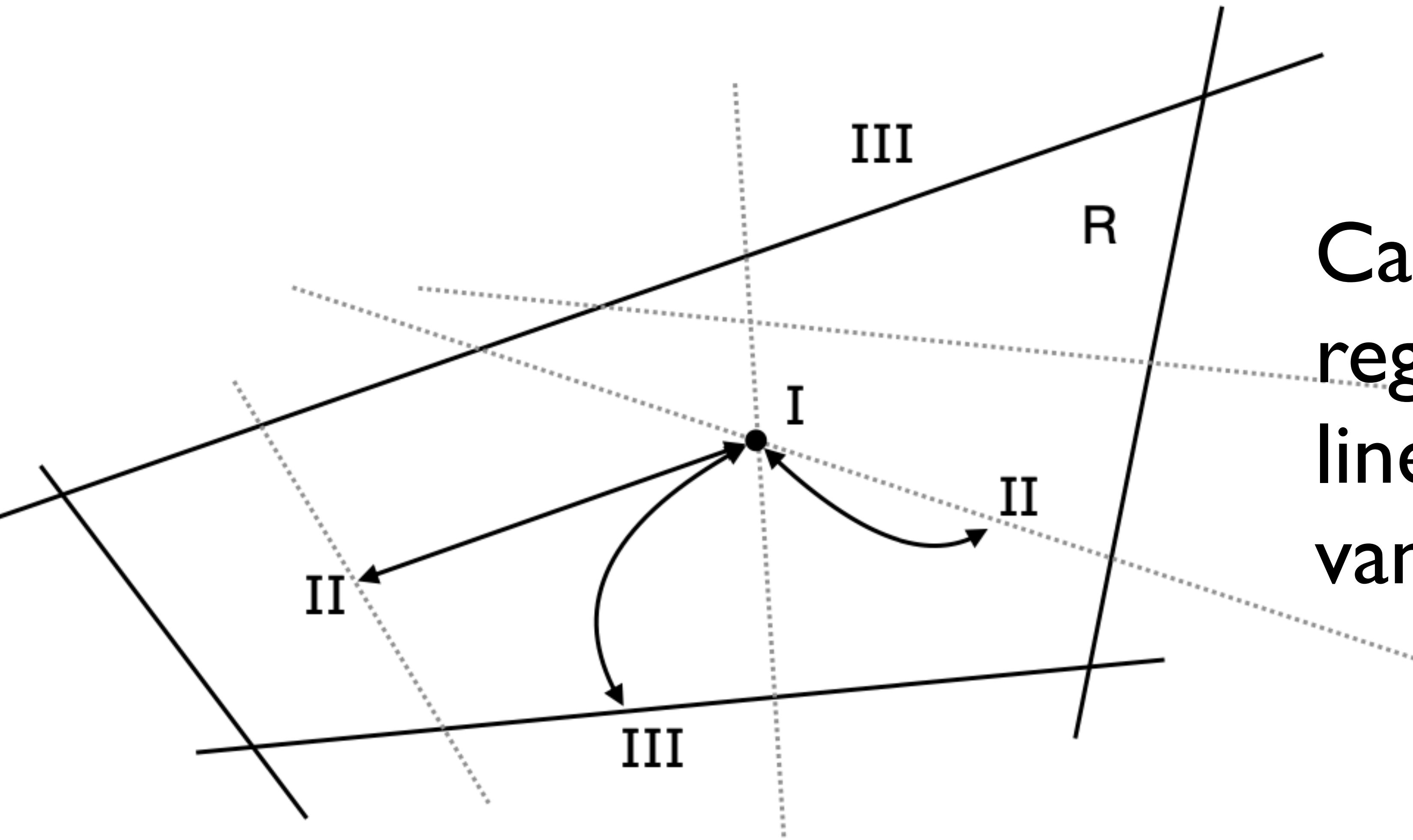


$\rightarrow \sqrt{\Delta_5(v_4, v_5, v_6, v_7, v_8, v_2)}$



$\rightarrow \sqrt{\Delta_6}$

Boundary values follow from absence of spurious singularities



Cartoon of the Euclidean region R, bounded by solid lines. Dotted lines represent vanishing alphabet letters.

Useful parametrization:

$$\vec{v} = \{-1, -(1-x)^2, -1, -(1-x)^2, -1, -(1-x)^2, -1+x, -1+x, -1+x\}$$

$$\mathbb{A} \to \mathbb{A}_{\text{line}} = \{x, 1-x, x-\rho, x-\bar{\rho}\}, \qquad \rho = \frac{1}{2}(1+i\sqrt{3})$$

Numerical evaluation of hexagon integral

kinematic point	$G(p_1,\ldots,p_5)=0$	$\tilde{I}_{33}$
$\{-1,-1,-1,-1,-1,-1,-\frac{1}{2},-\frac{5}{8},-\frac{17}{14}\}$	$\checkmark$	$1.698786104665747146 3i\epsilon^3 + 6.628732165493197145i\epsilon^4 + \mathcal{O}(\epsilon^5)$
$\{-\frac{2}{3},-\frac{7}{10},-\frac{9}{11},-\frac{15}{17},-\frac{24}{29},-\frac{30}{37},-\frac{37}{43},-\frac{47}{53},-\frac{53}{59}\}$	$\times$	$1.296647495236338202 7i\epsilon^3 + 5.241756401399539064i\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)$

Values obtained from a one-fold integral representation, following [Caron-Huot, JMH, 2014].

Validated using AMFlow [Liu, Ma 2022] and SecDec [Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke 2015], both on and off the Gram determinant constraint (for four-dimensional external states).

Take-home message

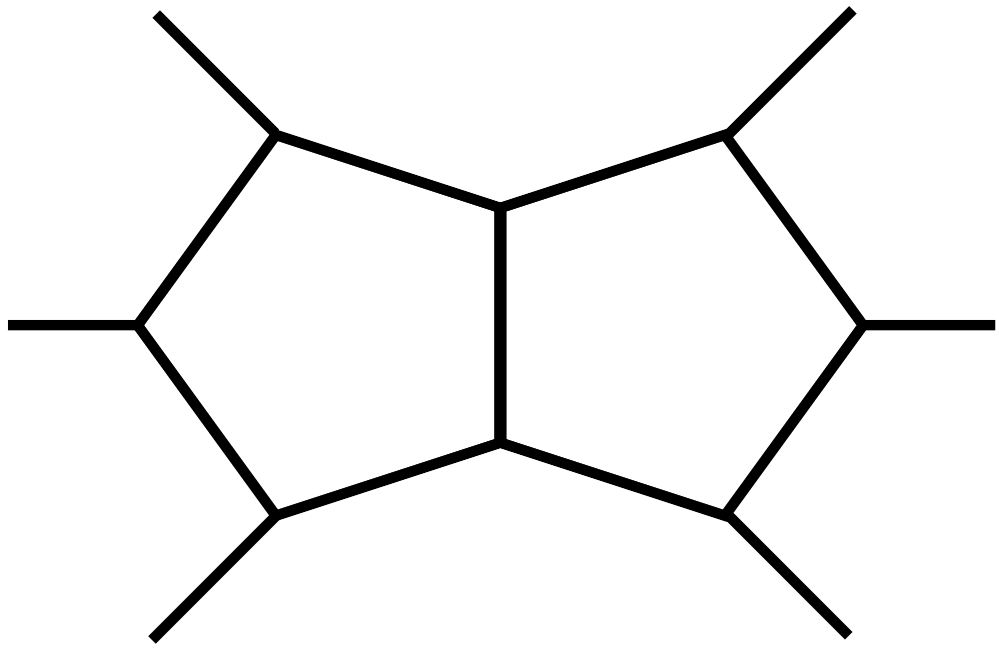
We found the full D-dimensional one-loop hexagon alphabet, needed in conventional dimensional regularization. There are 10 new letters compared to the four-dimensional helicity scheme.

We focused on the weight-four term of the hexagon, which can contribute to finite terms at two loops. We found a one-fold integral representation that is readily evaluated in the Euclidean region.

This weight four term is expected to appear in $\mathcal{O}(\epsilon^2)$ one-loop amplitudes [cf. talk by Alex Edison].

Part 2 : Planar two-loop six-point integrals on the maximal cut

Dimensional regularization scheme choice affects the number of master integrals

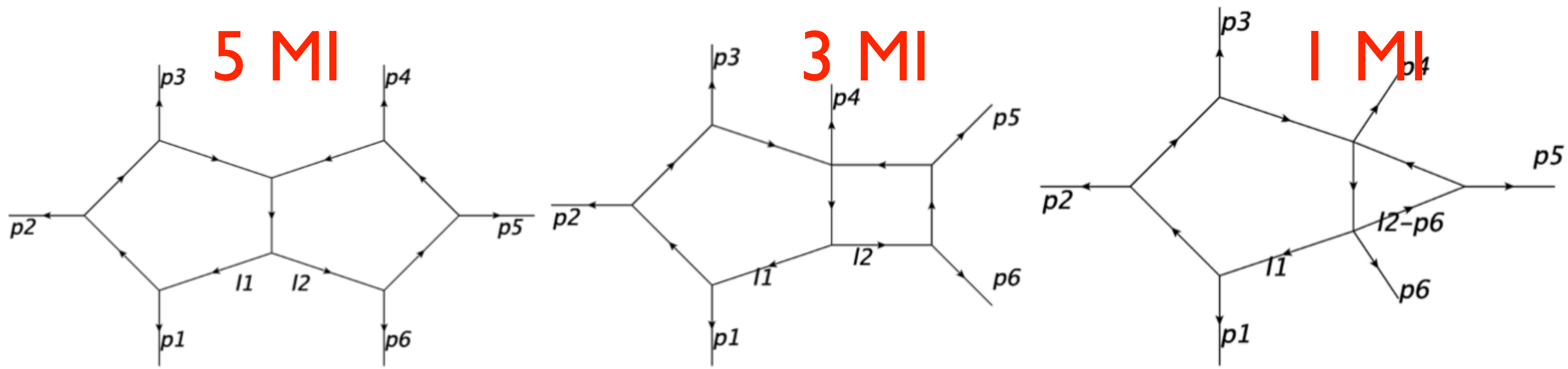


For D-dimensional external states (conventional dimensional regularization): 7 Master integrals (MI)

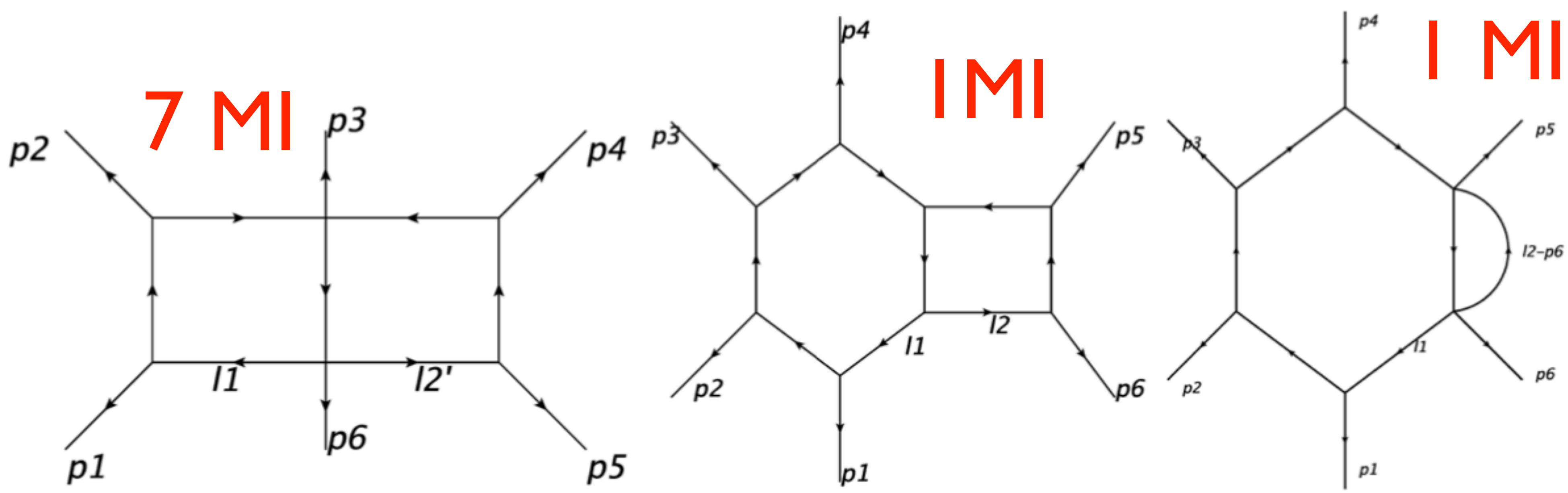
For four-dimensional external states (four-dimensional helicity scheme): 5 MI

In a first step, we perform the calculation for four-dimensional external states.

Genuine two-loop planar six-particle integrals



(a) Double-pentagon (b) Pentagon-box (c) Pentagon-triangle

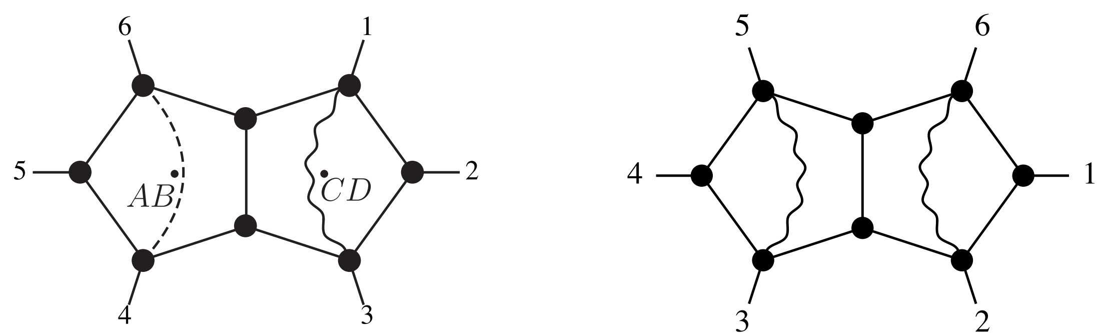


(d) Double-box (e) Hexagon-box (f) Hexagon-bubble

We consider the maximal cut of these integral sectors.

Search for uniform weight master integrals

In four dimensions, integrals with constant leading singularities are known from [Arkani-Hamed et al, 2010]:



They are finite, uniform weight-four integrals, and were computed in [Drummond, JMH, Dixon, 2011].

Issues to be solve for D-dimensional calculation:

1) How to find enough basis integrals?

2) How to extend the definition of the numerators to D dimensions, so that D-dimensional leading singularities are correctly normalized?

Uniform weight integrals from Baikov analysis

Our strategy is as follows:

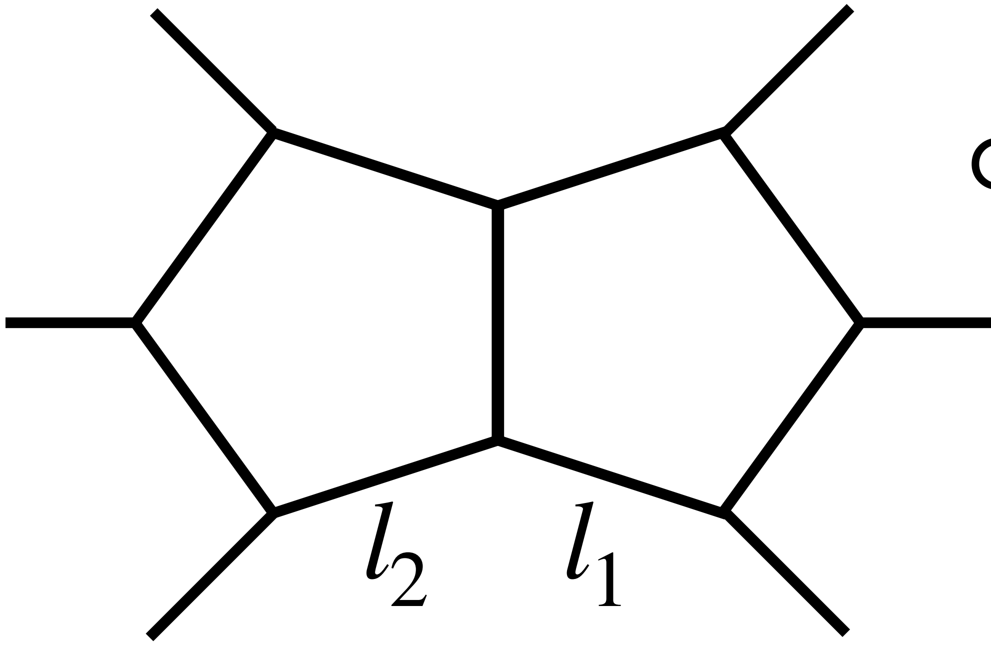
1) We include.$(6-2\epsilon)$-dimensional integrands into basis, as well as evanescent numerator terms (Gram determinants)

2) We use Baikov analysis to include information beyond four dimensions about the leading singularities

[Abreu, Dixon, Herrmann, Page, Zeng '18; Chicherin, Gehrmann, JMH, Wasser, Zhang, Zoia '18]

In this way, we find a sufficient number of MI for our basis.

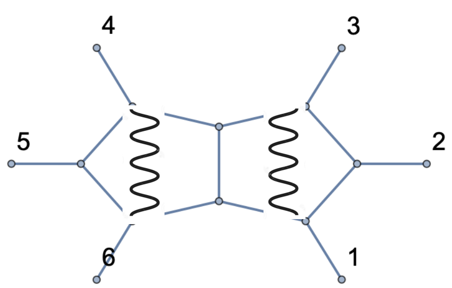
Example of uniform weight construction (1)

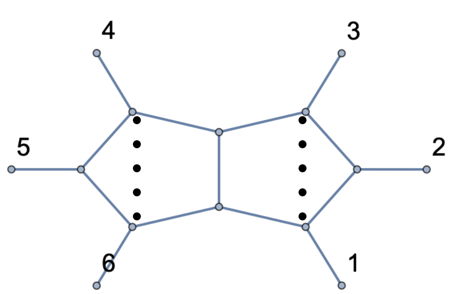


Original numerators in spinor helicity language:

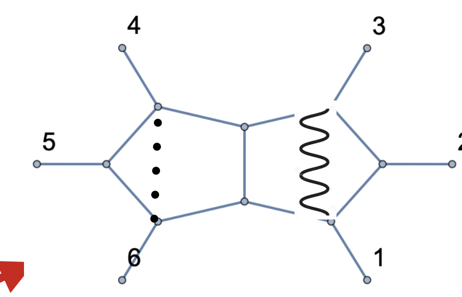
$$w_1 = p_1 + \frac{[23]}{[13]}\lambda_2\tilde{\lambda}_1, \quad w_2 = w_1^*,$$
$$v_1 = p_6 + \frac{[54]}{[64]}\lambda_5\tilde{\lambda}_6, \quad v_2 = v_1^*.$$

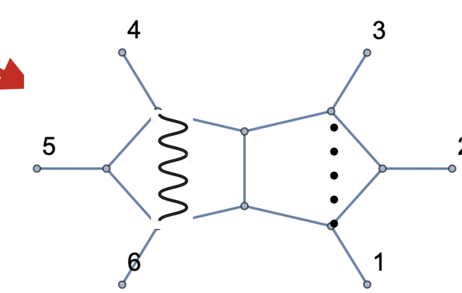
5 Master integrals



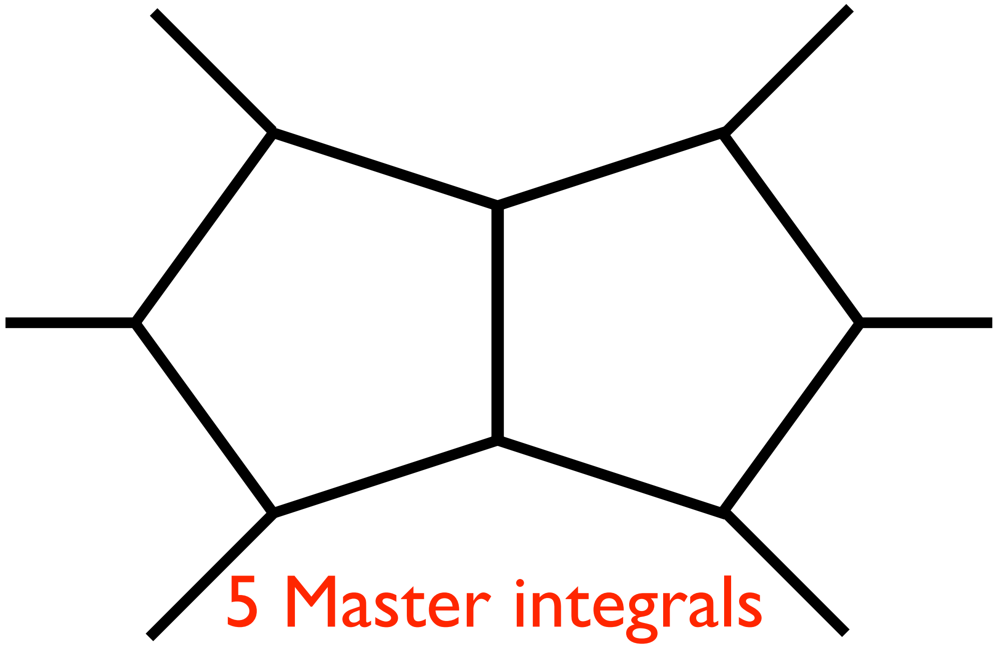


$$N_1^{\text{DP-a}} = (w_2+v_2)^2 s_{13}s_{46}(l_1-w_1)^2(l_2-v_1)^2$$
$$N_2^{\text{DP-a}} = (w_2+v_1)^2 s_{13}s_{46}(l_1-w_1)^2(l_2-v_2)^2$$
$$N_3^{\text{DP-a}} = (w_1+v_2)^2 s_{13}s_{46}(l_1-w_2)^2(l_2-v_1)^2$$
$$N_4^{\text{DP-a}} = (w_1+v_1)^2 s_{13}s_{46}(l_1-w_2)^2(l_2-v_2)^2$$





Example of uniform weight construction (2)



5 Master integrals

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

Parity odd combinations

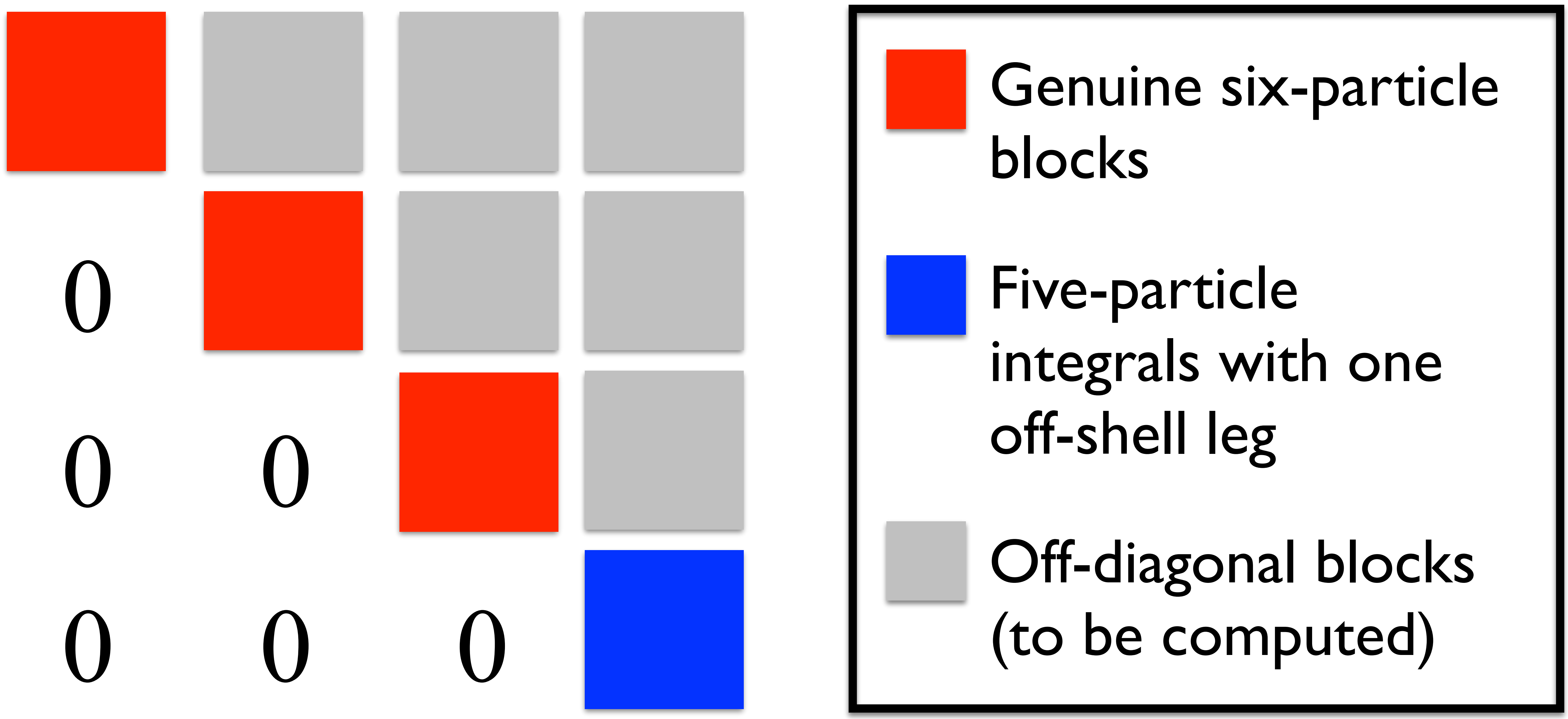
Evanescent numerator (5 × 5 Gram determinant):

$$\mu_{12} = \frac{G\begin{pmatrix} l_1 & p_1 & p_2 & p_3 & p_6 \\ l_2 & p_1 & p_2 & p_3 & p_6 \end{pmatrix}}{G(1,2,3,6)} .$$

Six-dimensional scalar integral

additional term needed in D dimensions

Differential equation blocks in canonical form



We used FIRE6 and FiniteFlow to perform the IBP reduction

Kinematics parametrised using four-dimensional momentum twistor variables.

Alphabet letters from diagonal blocks we calculated

Letters in diagonal blocks we calculated + cyclic symmetry: 205 letters

170 even:

$$W_1 = s_{12}$$
$$W_{67} = -s_{12}s_{45} + s_{34}s_{45} - s_{12}s_{46} + s_{34}s_{46} + s_{34}s_{56} \ldots$$
$$W_{145} = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]$$
$$\ldots$$

30 odd under parity:

$$W_{158} = \frac{Tr_+(1235) + Tr_+(1236)}{Tr_-(1235) + Tr_-(1236)}$$
$$\ldots$$

15 odd under flip of hexagon square root:

$$\frac{F_4 - (\text{rational factor})}{F_4 + (\text{rational factor})}$$

Part 3 : Discussion and outlook

Discussion

Progress on six-particle function space in QCD

- One-loop hexagon integral in D dimensions; can be used for calculations in conventional dimensional regularisation

- Uniform weight basis on maximal cuts of planar two-loop six particle integrals (assuming D=4 helicity formalism); identified new alphabet letters

- Full six-particle two-loop computation underway

Open questions / next steps

What is the full six-particle alphabet, once the off-diagonal matrix elements are known?

In scattering amplitudes, the functions appear together with algebraic coefficients, which are related to leading singularities. Can we predict what coefficients can appear?

What 'hexagon functions' appear in actual amplitudes, taking into account analyticity, symmetry, and interplay with leading singularities? Which ideas from N=4 sYM carry over to QCD?

Extra slides

Relation to dual conformal hexagon alphabet

The dual conformal 9-variable (A3 cluster algebra) letters from N=4 sYM can be written as:

$$\boxed{u_1,\ u_2,\ u_3,\ 1-u_1,\ 1-u_2,\ 1-u_3,\ y_1,\ y_2,\ y_3}$$

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{456}}, \quad u_3 = \frac{s_{34}s_{16}}{s_{345}s_{156}}, \quad y_i = \frac{u_i - z_+}{u_i - z_-}$$

$$z_\pm = \frac{-1+u_1+u_2+u_3 \pm \sqrt{\Delta}}{2} \qquad \Delta = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3$$

They are contained in our letters:

$$u_1 = \frac{W_1W_4}{W_{46}W_{48}}, \quad u_2 = \frac{W_2W_5}{W_{46}W_{47}}, \quad u_3 = \frac{W_3W_6}{W_{47}W_{48}}$$

$$1-u_1 = -\frac{W_{81}}{W_{46}W_{48}}, \quad 1-u_2 = -\frac{W_{79}}{W_{46}W_{47}}, \quad 1-u_3 = -\frac{W_{80}}{W_{47}W_{48}},$$

$$y_1 = W_{146}W_{153}, \quad y_2 = \frac{1}{W_{147}W_{154}}, \quad y_3 = \frac{1}{W_{151}W_{152}}.$$