Some exact results of $\mathcal{N} = 4$ SYM and S-duality

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Introduction

- 4d $\mathcal{N}=4$ supersymmetric Yang-Mills theory has many remarkable properties: hidden symmetries, geometric formulations, extremely high loop results, integrability, ...
- I will discuss a much older story of $\mathcal{N}=4$ SYM: the S-duality [Montonen–Olive, 77'], [Goddard-Nuyts-Olive, 77'].
- Concretely, we like to understand how correlators (or the dual string amplitudes) behave under modular transformation.
- The duality relates weak-strong coupling, and requires to compute correlators at finite coupling

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g_{\mathrm{YM}}^2} = \tau_1 + i\tau_2\,.$$

Goddard-Nuyts-Olive duality

S-duality of $\mathcal{N}=4$ SYM with gauge group G_N:

$$T\ :\ (G_N,\tau)\rightarrow(G_N,\tau+1)\,,\qquad \hat{S}\ :\ (G_N,\tau)\rightarrow({}^L G_N,-\frac{1}{r\,\tau})\,,$$

$r=\text{long/short root}^2$ of G_N, and ${}^L G_N$ is the Langlands dual group.

- $r=1$ for simply laced groups $SU(N), SO(2N)$ and $r=2$ for non-simply laced groups $SO(2N{+}1), USp(2N)$.
- the GNO duality

$\mathfrak{g}_N$	${}^L\mathfrak{g}_N$
$su(N)$	$su(N)$
$so(2N)$	$so(2N)$
$usp(2N)$	$so(2N{+}1)$
$so(2N{+}1)$	$usp(2N)$

only algebra is relevant for correlators of local operators

A four-point correlator

- We consider the correlators of superconformal primary operators of $\mathcal{N} = 4$ SYM,

$$\mathcal{O}_2(x, Y) = \mathrm{tr}(\phi_{I_1}(x)\phi_{I_2}(x))Y^{I_1}Y^{I_2}\,,$$

where $I_p = 1, 2, \cdots, 6$ and $Y \cdot Y = 0$. The correlators of $\mathcal{O}_2$ are well studied. [e.g. Agnese Bissi's talk]

- Two- and three-point correlators are protected.
- Supersymmetry and superconformal symmetries imply [Eden, Petkou, Schubert, Sokatchev][Nirschl, Osborn]

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle = \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle_{\mathrm{free}} + \mathcal{I}_4(x_i, Y_i)\,\mathcal{T}_{G_N}(U, V; \tau, \bar{\tau})\,,$$

where $\mathcal{I}_4$ is fixed by the symmetries and we focus on $\mathcal{T}_{G_N}$.
U, V are cross ratios & $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{\mathrm{YM}}}$.

Integrated correlators in $\mathcal{N} = 4$ SYM

- We are interested in $SL(2,\mathbb{Z})$ modular properties and the correlator at finite coupling τ.
- In general this's very difficult (impossible!); we will consider a simpler yet highly non-trivial object: integrated correlators,

$$\mathcal{C}_{G_N}(\tau,\bar{\tau}) = \int dU dV\, M(U,V)\, \mathcal{T}_{G_N}(U,V;\tau,\bar{\tau})\,.$$

  With suitable choices of the measure to preserve supersymmetry, $\mathcal{C}_{G_N}(\tau,\bar{\tau})$ may be computed exactly.
- One can reconstruct the un-integrated correlator at finite coupling, at least for first few orders in large-N expansion.

Integrated correlators in $\mathcal{N} = 4$ SYM

Two integrated correlators have been studied.

- Integrated correlator one: [Binder, Chester, Pufu, Wang] [Chester, Pufu]

$$\mathcal{C}_{G_N,1}(\tau,\bar\tau) = -\frac{8}{\pi}\int_0^\infty dr \int_0^\pi d\theta \frac{r\sin^2(\theta)}{U}\mathcal{T}_{G_N}(U,V;\tau,\bar\tau)\,,$$

with $U = 1 + r^2 - 2r\cos(\theta)$, $V = r^2$.

- Integrated correlator two: [Chester, Pufu]

$$\mathcal{C}_{G_N,2}(\tau,\bar\tau) = -\frac{32}{\pi}\int_0^\infty dr \int_0^\pi d\theta \frac{r\sin^2(\theta)}{U}(1+U+V)\bar{D}_{1111}(U,V)\mathcal{T}_{G_N}(U,V;\tau,\bar\tau)\,,$$

where $\bar{D}_{1111}(U,V)$ is the $4d$ 1-loop box integral.

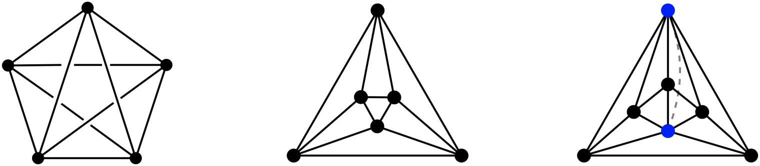
Integrated correlators and periods of Feynman integrals

Perturbation theory in the expansion of $a_{G_N} = \lambda_{G_N}/(4\pi^2)$

[Eden, Heslop, Korchemsky, Sokatchev][Bourjaily, Heslop, Tran] ... [Shun-Qing Zhang's Gong show & poster]

$$\mathcal{T}_{G_N}(U,V) = 2c_{G_N}\frac{U}{V}\sum_{\ell=1}^{\infty} a_{G_N}^{\ell}\, x_{13}^2 x_{24}^2 \int d^4x_5 \dots d^4x_{\ell+4}\, f^{(\ell)}(x_i)\,,$$

where $f^{(\ell)}$ can be represented by permutation symmetric graphs with $\ell+4$ vertices with weight-4:



$f^{(1)}$ $f^{(2)}$ $f^{(3)}$

Integrated correlators and periods of Feynman integrals

- The first integrated correlator is given by periods of $f^{(\ell)}(x_i)$

$$\mathcal{P}^{(1)}_{f^{(\ell)}} = \int \frac{d^4 x_i}{\text{vol (conf)}} f^{(\ell)}(x_i)\,.$$

Examples: [Broadhurst][Brown][Panzer][Schnetz][...]

$$\mathcal{P}^{(1)}_{f^{(1)}} = 6\zeta(3)\,, \quad \mathcal{P}^{(1)}_{f^{(2)}} = 300\zeta(5)\,, \quad \mathcal{P}^{(1)}_{f^{(3)}} = 17640\zeta(7)\,, \quad \ldots$$

- The second integrated correlator is the periods of $f^{(\ell)}(x_i)$ attached with a 1-loop box integral

$$\mathcal{P}^{(2)}_{f^{(\ell)}} = \int \frac{d^4 x_i}{\text{vol (conf)}} f^{(\ell)}(x_i) box(x_1, x_2, x_3, x_4)\,,$$

$$\mathcal{P}^{(2)}_{f^{(1)}} = 60\zeta(5)\,, \quad \mathcal{P}^{(2)}_{f^{(2)}} = -\frac{3}{2}\left(36\zeta(3)^2 + 175\zeta(7)\right)\,, \quad \ldots$$

Integrated correlators from localization

Beyond perturbation:

- Integrated correlators are determined by 4 derivatives of $\mathcal{N} = 2^*$ SYM partition function on S^4, $Z_{G_N}(m, \tau, \bar{\tau})$, via [Binder, Chester, Pufu, Wang] [Chester, Pufu]

$$\mathcal{C}_{G_N,1}(\tau,\bar{\tau}) = \tau_2^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z_{G_N}(m,\tau,\bar{\tau})\Big|_{m=0},$$
$$\mathcal{C}_{G_N,2}(\tau,\bar{\tau}) = \partial_m^4 \log Z_{G_N}(m,\tau,\bar{\tau})\Big|_{m=0},$$

where $Z_{G_N}(m, \tau, \bar{\tau})$ is computed using supersymmetric localisation [Nekorasov][Pestun]...

$$Z_{SU(N)}(m,\tau,\bar{\tau}) = \int d^N a\, \delta(\sum_i a_i) \prod_{i<j} (a_i - a_j)^2\, e^{-\frac{8\pi^2}{g_{\rm YM}^2} \sum_i a_i^2}\, Z_{1-{\rm loop}}\, |Z_{\rm inst}|^2,$$

where $Z_{1-{\rm loop}}$ and $Z_{\rm inst}$ give perturbative and instanton contributions, respectively.

Integrated correlators from localization

- Four derivatives bring down four integrated operators, which lead to integrated four-point functions.
- Localisation reduces path integral to N-dimensional integral! But it's not an easy integral and many properties (e.g. $SL(2,\mathbb{Z})$) are not manifest.
- Furthermore, $Z_{\rm inst}$ is an infinite sum of instantons; the k-instanton contribution is a k-dim contour integral.

We want to do better than the localisation formula.

Exact results of an integrated correlator

- By carefully analysing the localisation formula, we proposed an exact expression for the first integrated correlator, $\mathcal{C}_{SU(N),1}(\tau,\bar{\tau})$, for arbitrary N and τ:

$$\mathcal{C}_{SU(N)}(\tau,\bar{\tau}) = \sum_{(p,q)\in\mathbb{Z}^2} \int_0^\infty \exp\Big(-t\pi\frac{|p+q\tau|^2}{\tau_2}\Big) B_{SU(N)}(t)\, dt\,,$$

where $B_{SU(N)}(t) = \frac{t\mathcal{Q}_{SU(N)}(t)}{(t+1)^{2N+1}}$ & $\mathcal{Q}_{SU(N)}(t)$ is a degree-$(2N-2)$ palindromic polynomial in terms of Jacobi polynomials. e.g.

$$\mathcal{Q}_{SU(2)}(t) = 9t^2 - 30t + 9\,,$$
$$\mathcal{Q}_{SU(3)}(t) = 18t^4 - 99t^3 + 126t^2 - 99t + 18\,.$$

- The expression is manifestly $SL(2,\mathbb{Z})$ invariant, and all the information is contained in $B_{SU(N)}(t)$.

Exact results of an integrated correlator

Expressed as an infinite sum of non-holomorphic Eisenstein series,

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = -b_{SU(N)}(0) + \sum_{s=2}^{\infty} b_{SU(N)}(s) E(s;\tau,\bar\tau)\,,$$

with known rational coefficients $b_{SU(N)}(s)$, and

$$\begin{aligned} E(s;\tau,\bar\tau) &= \sum_{(p,q)\neq(0,0)} \int_0^\infty dt \exp\Big(-t\pi \frac{|p+q\tau|^2}{\tau_2}\Big) t^s \\ &= \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\zeta(2s-1)\Gamma(s-\frac{1}{2})}{\pi^{s-\frac{1}{2}}\Gamma(s)}\tau_2^{1-s} + \sum_{k\neq 0} \mathcal{F}_k(s;\tau_2) e^{2\pi k \tau_1}\,. \end{aligned}$$

This leads to $SL(2,\mathbb{Z})$ spectral representation [Collier, Perlmutter]:

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) = -2b_{SU(N)}(0) + \frac{1}{2i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{(-1)^s}{\sin \pi s} b_{SU(N)}(s) E(s;\tau,\bar\tau) + \text{~~cusp forms~~}\,.$$

A proof of the formula

- $\mathcal{C}_{SU(2)}(\tau,\bar\tau)$ is confirmed by an explicit computation.
- We prove the perturbative part of the integrated correlator obeys a Laplace-difference equation, which can be uniquely promoted to a $SL(2,\mathbb{Z})$ invariant equation

$$4\tau_2^2\partial_\tau\partial_{\bar\tau}\mathcal{C}_{SU(N)}(\tau,\bar\tau) - (N^2{-}1)\left[\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) - 2\mathcal{C}_{SU(N)}(\tau,\bar\tau) + \mathcal{C}_{SU(N-1)}(\tau,\bar\tau)\right] \\ - (N+1)\mathcal{C}_{SU(N-1)}(\tau,\bar\tau) + (N-1)\mathcal{C}_{SU(N+1)}(\tau,\bar\tau) = 0\,.$$

- The Laplace-difference equation (with b.d. conditions from perturbation) determines $\mathcal{C}_{SU(N)}(\tau,\bar\tau)$ in terms of $\mathcal{C}_{SU(2)}(\tau,\bar\tau)$.

Generating function for the integrated correlator

We can even obtain the generating function by summing over N

$$
\begin{aligned}
G_{SU}(\tau,\bar{\tau};z) &= \sum_{N=2}^{\infty} \mathcal{C}_{SU(N)}(\tau,\bar{\tau})\, z^N \\
&= \sum_{(p,q)\in\mathbb{Z}^2} \int_0^{\infty} \exp\Big(-\, t\pi \frac{|p+q\tau|^2}{\tau_2}\Big) F_{SU}(t,z)\, dt\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{SU}(t,z) &= \sum_{N=2}^{\infty} B_{SU(N)}(t) z^N \\
&= \frac{3tz^2\left[(t-3)(t+1)^2(3t-1)-(t-1)^2(t+3)(3t+1)z\right]}{(1-z)^{3/2}\left[(t+1)^2-(t-1)^2 z\right]^{7/2}}\,.
\end{aligned}
$$

$F_{SU}(t,z)$ is closely related to the so-called (generalised) Harer-Zagier formula in the matrix-model literature.

The integrated correlator in various regions: small-$g_{\rm YM}$, large-N, ...

Weak-coupling perturbative expansion

Weak-coupling perturbative expansion (loops), using $a = \lambda/(4\pi^2)$:

$$\mathcal{C}_{SU(N)}(\tau_2) = (N^2-1)\Big[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32} + \frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024} + \cdots\Big]\,.$$

- Non-planar contributions start to enter at 4 loops – in agreement with known results.
- It gives an all-loop prediction for any N.
- Using results of periods, first four loops are shown to agree with Feynman diagram computations; predict relations and results for unknown higher-loop Feynman integrals.

Large N: small-λ expansion

Large-N expansion: $\mathcal{C}_{SU(N)}(\tau,\bar{\tau}) \sim \sum_{g=0}^{\infty} N^{2-2g}\,\mathcal{C}^{(g)}(\lambda)$.

- Small-λ expansion

$$\mathcal{C}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n\,,$$

$$\mathcal{C}^{(1)}(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n(n-5)(2n+1)\zeta(2n+1)\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{24\,\pi^{2n+1}\Gamma(n)^2}\lambda^n\,,$$

$$\vdots$$

- They are all convergent with a finite radius $|\lambda| < \pi^2$, which has been seen in $\mathcal{N}=4$ SYM, such as cusp anomalous dimension [Basso, Korchemsky, Kotanski], amplitudes [Basso, Dixon, Papathanasiou].

Large N: large-λ expansion

- Large-λ expansion:

$$\mathcal{C}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\,\lambda^{n+1/2}}\,,$$

$$\vdots$$

- They are all asymptotic & not Borel summable, require non-perturbative completions (resurgence)

$$\Delta\mathcal{C}^{(0)}(\lambda) \sim i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \cdots\right].$$

- One also finds another type of exponential term $e^{-8N\pi/\sqrt{\lambda}}$ (see also [Hatsuda,Okuyama, 2208.01891]), essentially due to $SL(2,\mathbb{Z})$.

Large N: finite YM coupling τ

$SL(2,\mathbb{Z})$ at Large N: large-N expansion with finite YM coupling τ ("very strong coupling limit"):

$$\mathcal{C}_{SU(N)}(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar\tau) + \frac{45}{2^8 N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau)$$
$$+ \frac{3}{N^{\frac{3}{2}}}\Big[\frac{1575}{2^{15}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac{3}{2};\tau,\bar\tau)\Big] + \frac{225}{N^{\frac{5}{2}}}\Big[\frac{441}{2^{18}} E(\tfrac{9}{2};\tau,\bar\tau) - \frac{5}{2^{16}} E(\tfrac{5}{2};\tau,\bar\tau)\Big]$$
$$+ \frac{63}{N^{\frac{7}{2}}}\Big[\frac{3898125}{2^{27}} E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}} E(\tfrac{7}{2};\tau,\bar\tau) + \frac{73}{2^{22}} E(\tfrac{3}{2};\tau,\bar\tau)\Big] + \cdots$$
$$+ O\Big(\sum_{(p,q)\neq(0,0)} \exp\Big(-4\sqrt{N\pi}\frac{|p+q\tau|}{\sqrt{\tau_2}}\Big)\Big).$$

Recall $E(s;\tau,\bar\tau)$ is the non-holomorphic Eisenstein series; $q=0$ gives $e^{-2\sqrt{\lambda}}$ and $p=0$ gives $e^{-8N\pi/\sqrt{\lambda}}$.

Integrated correlators & Goddard-Nuyts-Olive duality

The integrated correlator with a general classical gauge group G_N.

- For the simply laced groups $G_N = SU(N),\ SO(2N)$.

$$T : \tau \to \tau + 1; \quad S : \tau \to -1/\tau$$

The correlator is $SL(2, \mathbb{Z})$ invariant

$$\mathcal{C}_{G_N}(\tau, \bar{\tau}) = \sum_{(p,q)\in\mathbb{Z}^2} \int_0^\infty dt\, e^{-t\pi \frac{|p+q\tau|^2}{\tau_2}} B_{G_N}(t)\,.$$

As for $\mathcal{C}_{SU(N)}(\tau, \bar{\tau})$.

Integrated correlators & Goddard-Nuyts-Olive duality

For the non-simply laced $G_N = USp(2N),\ SO(2N+1)$: only invariant under congruence subgroup $\Gamma_0(2) \subset SL(2,\mathbb{Z})$: $T,\ \hat{S}T\hat{S}$.

$$T : \tau \to \tau + 1; \quad \hat{S} : \tau \to -1/(2\tau)\,.$$

GNO duality implies

$$\hat{S} : \quad \mathcal{C}_{SO(2N+1)}(\tau,\bar{\tau}) \ \leftrightarrow\ \mathcal{C}_{USp(2N)}(\tau,\bar{\tau})\,.$$

Our ansatz:

$$\mathcal{C}_{G_N}(\tau,\bar{\tau}) = \sum_{(p,q)\in\mathbb{Z}^2} \int_0^\infty dt \left[B^1_{G_N}(t) e^{-t\pi \frac{|p+q\tau|^2}{\tau_2}} + B^2_{G_N}(t) e^{-t\pi \frac{|p+2q\tau|^2}{2\tau_2}} \right]$$

$$B^1_{USp(2N)}(t) = B^2_{SO(2N+1)}(t)\,, \quad B^2_{USp(2N)}(t) = B^1_{SO(2N+1)}(t)\,.$$

Checked for all perturbative terms & some instanton sectors.

Summary and outlook

The integrated correlators can be computed exactly; tools for studying $\mathcal{N} = 4$ SYM: S-duality, resurgence, connections with periods, matching type IIB string amplitudes, exact data for bootstrap ... ...

- Second integrated correlator ($\partial_m^4 \log Z\big|_{m=0}$) at finite N?
- Applications to more general correlators? higher weights, higher points
- Integrated correlators are equivalent to integrated string amplitudes in AdS. Integrated flat-space amplitudes?

Thank you!