Topological Methods in QFT: A Review of Recent Progress

Zohar Komargodski (SCGP Simons Center for Geometry and Physics)

A Few Reviews:

Generalized Symmetries in Condensed Matter

John McGreevy (UC, San Diego) (Apr 6, 2022)

e-Print: 2204.03045 [cond-mat.str-el]

Symmetry Protected Topological phases of Quantum Matter

T. Senthil (May 15, 2014)

Published in: *Ann.Rev.Condensed Matter Phys.* 6 (2015) 299 • e-Print: 1405.4015 [cond-mat.str-el]

Notes on generalized global symmetries in QFT

Eric Sharpe (Virginia Tech.) (Aug 19, 2015)

Published in: *Fortsch.Phys.* 63 (2015) 659-682 • e-Print: 1508.04770 [hep-th]

Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond

#4

Clay Cordova (Chicago U., EFI and Chicago U.), Thomas T. Dumitrescu (UCLA), Kenneth Intriligator (YITP, Stony Brook and SUNY, Stony Brook), Shu-Heng Shao (YITP, Stony Brook) (May 19, 2022)

Contribution to: 2022 Snowmass Summer Study • e-Print: 2205.09545 [hep-th]

Symmetries have played a central role in physics for centuries.
Quantum Mechanics elevated the importance of symmetries even further.

GROUP THEORY

AND ITS APPLICATION TO THE
QUANTUM MECHANICS OF ATOMIC SPECTRA

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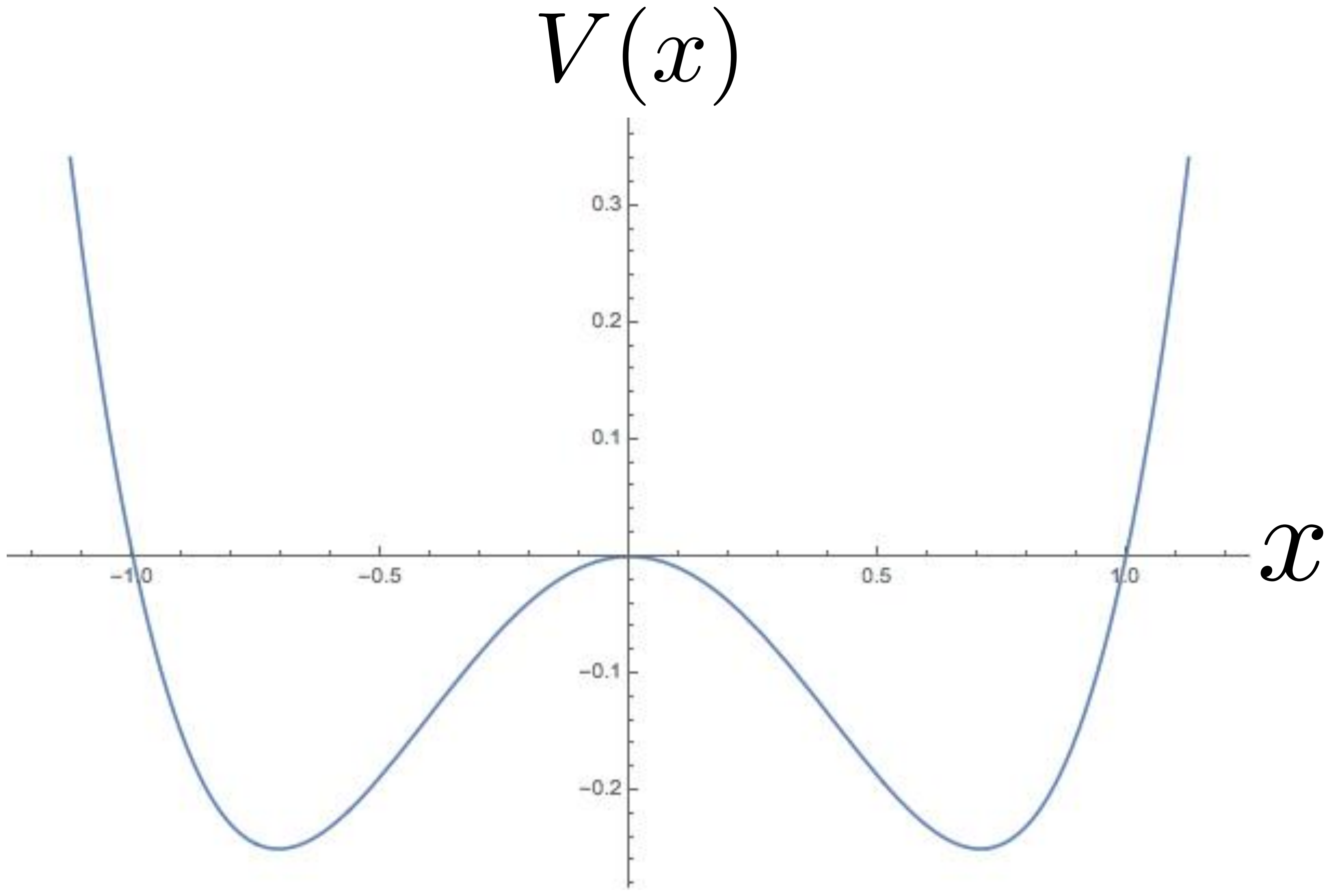
Outline:

- An elementary example
- Review the ``textbook” notion of a symmetry
- Discuss recent generalizations
- Define anomalies and generalized anomalies
- Discuss applications for the dynamics of some strongly coupled theories!

To motivate the discussion I will start from an example in QM. It is sufficiently elementary for any graduate course, and yet, its complete understanding requires somewhat unfamiliar ideas.

$$H = \frac{1}{2}\hat{p}^2 + V(x)$$

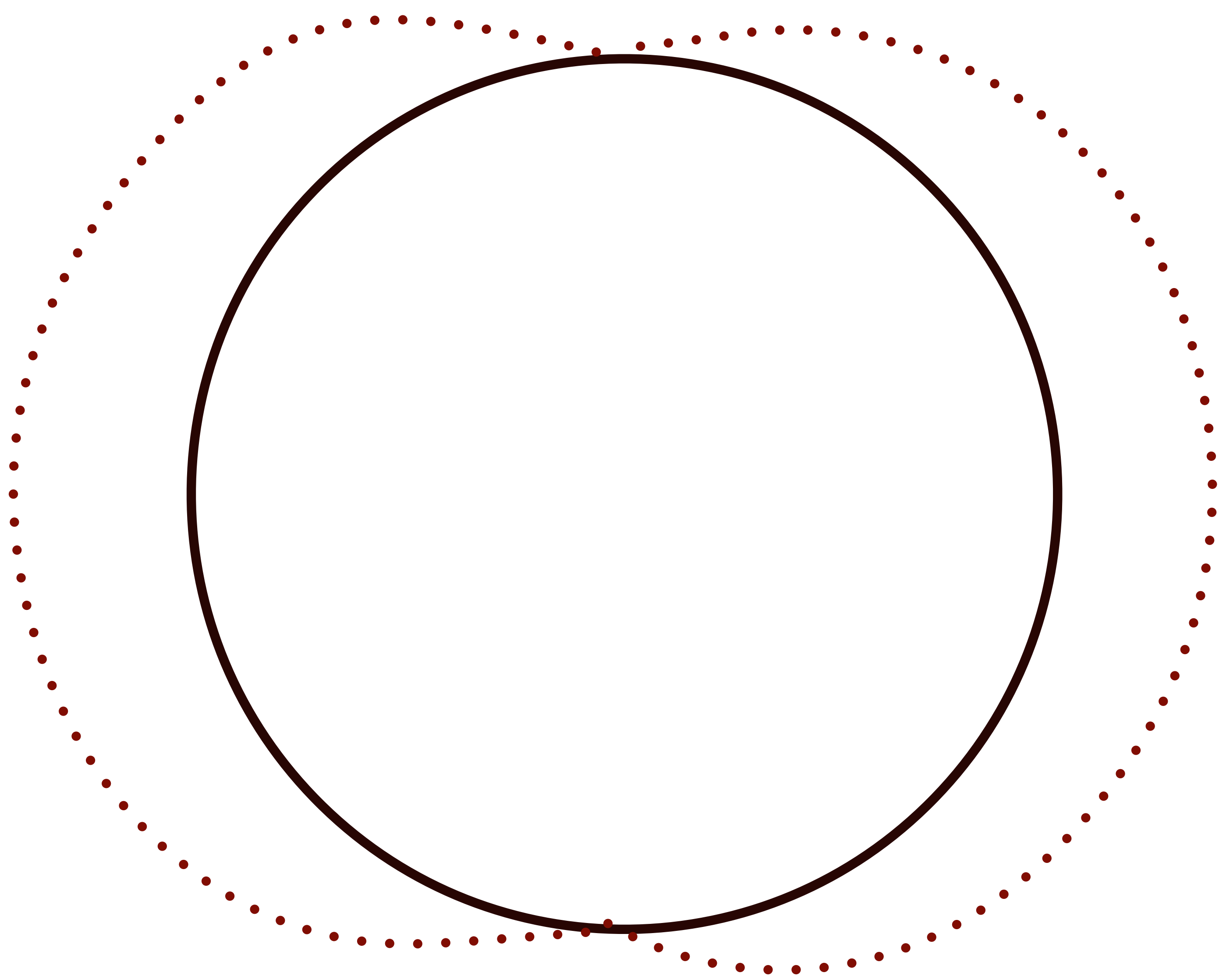
Instantons lift the classical two-fold degeneracy and the ground state is gapped



Sometimes there is a destructive interference between instantons that happens to all orders and leads to exact degeneracy. Consider an electron on a ring with two minima — at the top and bottom of the ring

$$H = \frac{1}{2}\hat{p}^2 - g\cos(2\phi)$$

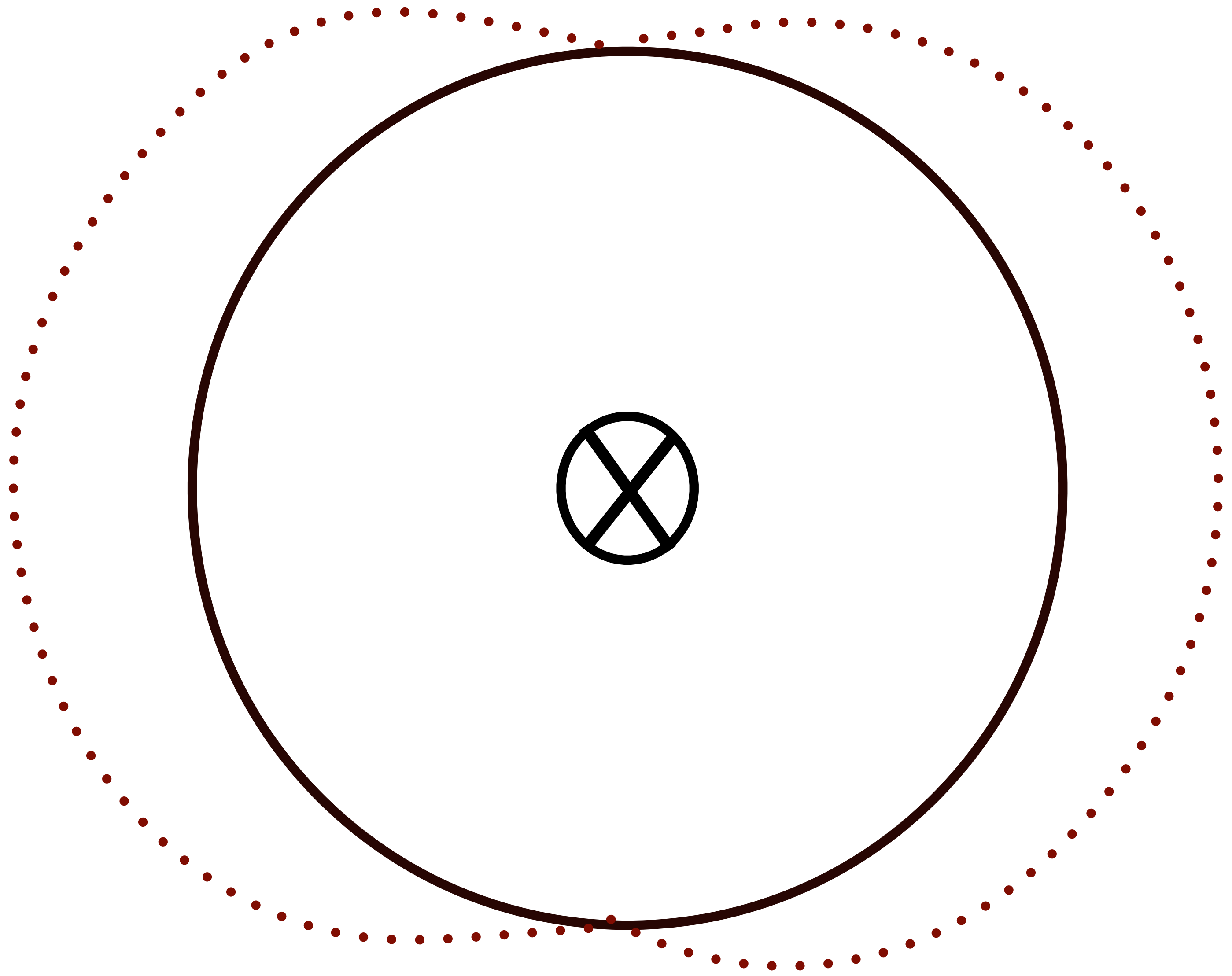
$$\hat{p} = -i\frac{d}{d\phi}$$



Quantum tunneling (instantons) lift the classical two-fold degeneracy. The ground state is symmetric under $\phi \to \phi + \pi$ and $\phi \to -\phi$.

The last step is to introduce a half-integer flux unit in the core of the ring (an AB solenoid)

$$H = \frac{1}{2}\left(\hat{p} - \frac{1}{2}\right)^2 - g\cos(2\phi)$$



This is an interacting quantum model for which for all g there are exactly two ground states. At g=0 this is obvious since the spectrum is

$$E_n = \frac{1}{2}\left(n - \frac{1}{2}\right)^2$$

And clearly n=0 and n=1 give the same ground state energy. Why this degeneracy survives for all g is what we intend to clarify.

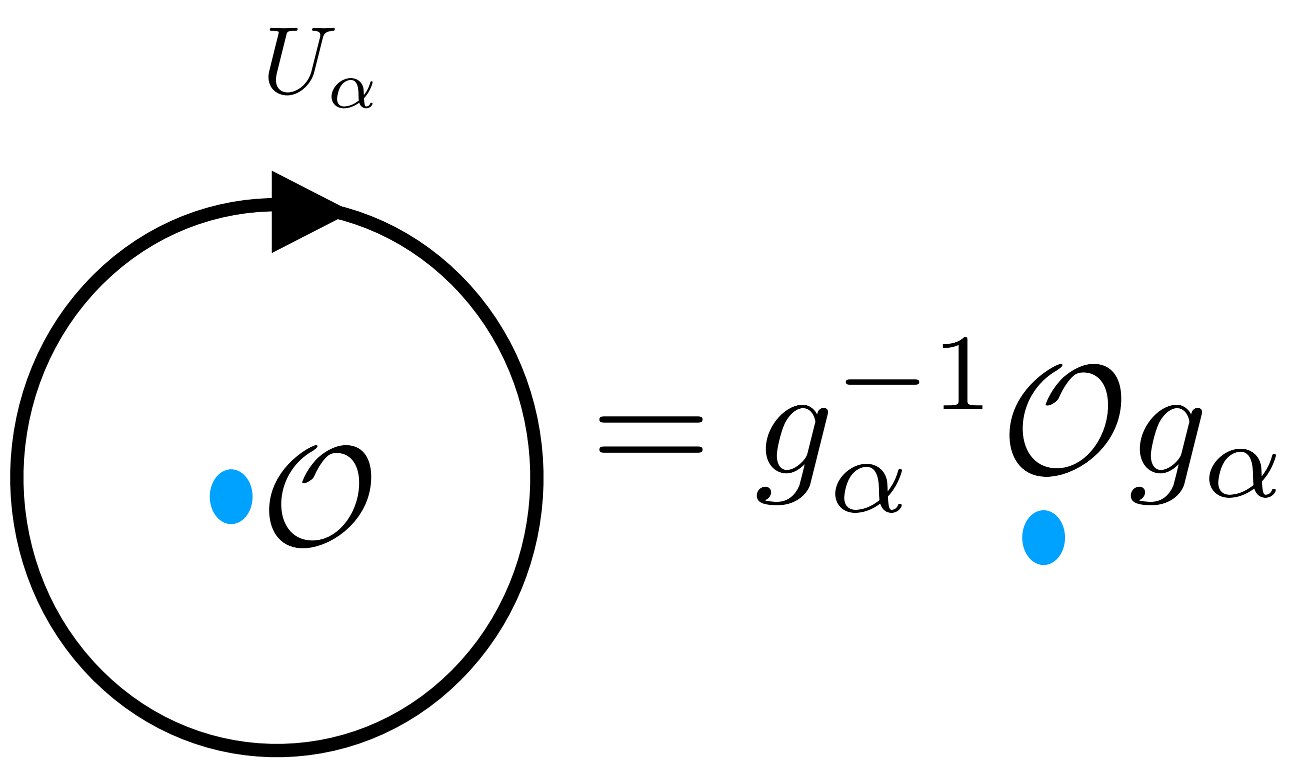
What is a symmetry ?

Ordinary continuous symmetries are associated to a conserved current $\partial^\mu j_\mu = 0$. One can then define the charge operators

$$U_\alpha = \exp(i\alpha \int_\Sigma d^{d-1}x \star j)$$

with Σ a d-1 dimensional (co-dimension 1) surface. Conservation of the current guarantees that U_α is independent of small deformations of Σ .

When U_α wraps local operators we get the usual action of the symmetry



Using this property and the invariance under small deformations, all the Ward identities follow.

Recent Generalizations

The idea is to associate topological operators to generalized symmetries. This generalization works in several directions.

- Ordinary discrete symmetries: while there is no continuous current, there is still a topological co-dimension 1 operator. This is a somewhat trivial (and not really new) remark.

- There could be topological operators of co-dimension 2 etc. A p-form symmetry is associated to a topological operator of co-dimension p+1. Ordinary symmetries correspond to p=0.

Free Maxwell theory has a 1-form electric symmetry and a d-3 - form magnetic symmetry.

$$\mathcal{L} = \frac{1}{4e^2} F^2$$

$$U_\alpha^E = \exp(i\alpha \int d^{d-2}x \star F)$$

$$j_{\mu_1 \cdots \mu_{d-2}} = \epsilon_{\mu_1 \cdots \mu_d} F^{\mu_{d-1}\mu_{d-2}} \quad ; \quad U_\alpha^M = \exp(i\alpha \int d^2x \star j)$$

Classically these conservation laws describe the conservation of electric and magnetic charges inside any region of space.

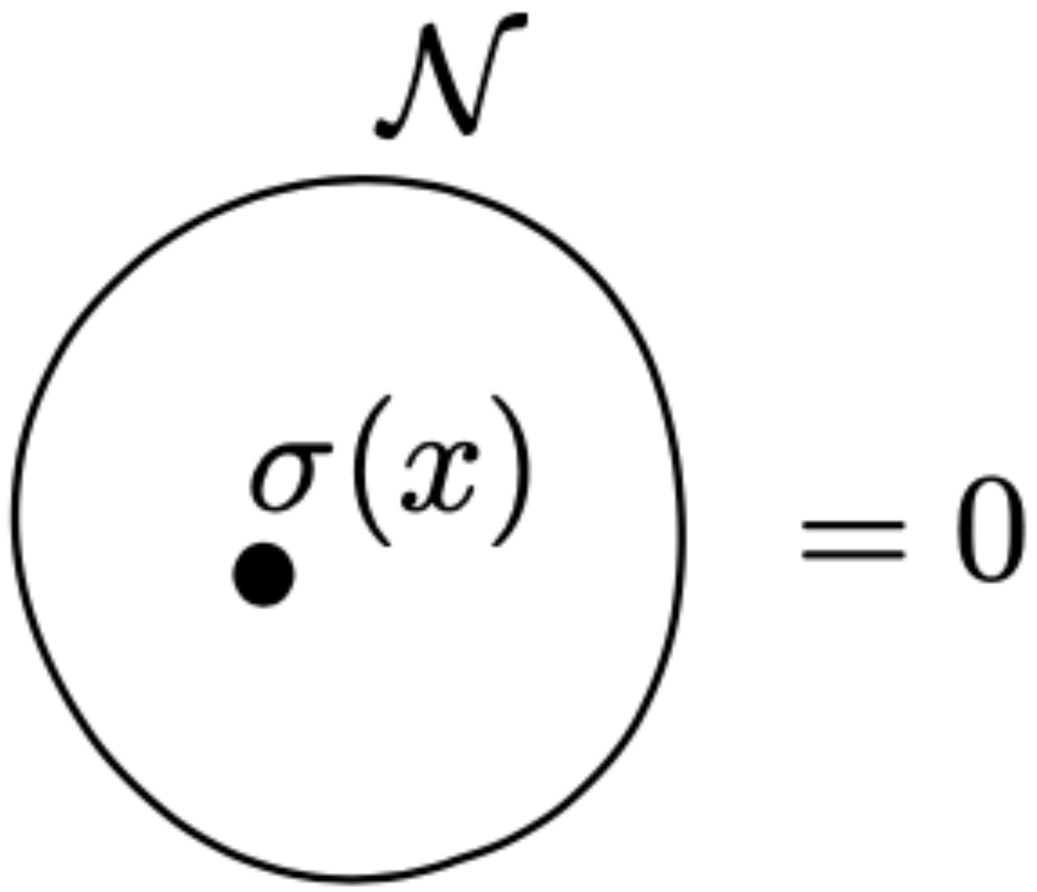
In d=4, both the 1-form electric symmetry and the 1-form magnetic symmetry are spontaneously broken — the Goldstone boson is the photon!

New look at QED in four-dimensions: The Photon as a Goldstone boson and the topological interpretation of electric charge

A. Kovner (Los Alamos), B. Rosenstein (Taiwan, Inst. Phys.) (Nov 3, 1992)

Published in: *Phys.Rev.D* 49 (1994) 5571-5581 • e-Print: hep-th/9210154 [hep-th]

- Finally, there could be topological surfaces not associated to a group element at all, namely, they could be non-invertible. A non-invertible symmetry could annihilate some local operators.



Topological Defect Lines and Renormalization Group Flows in Two Dimensions

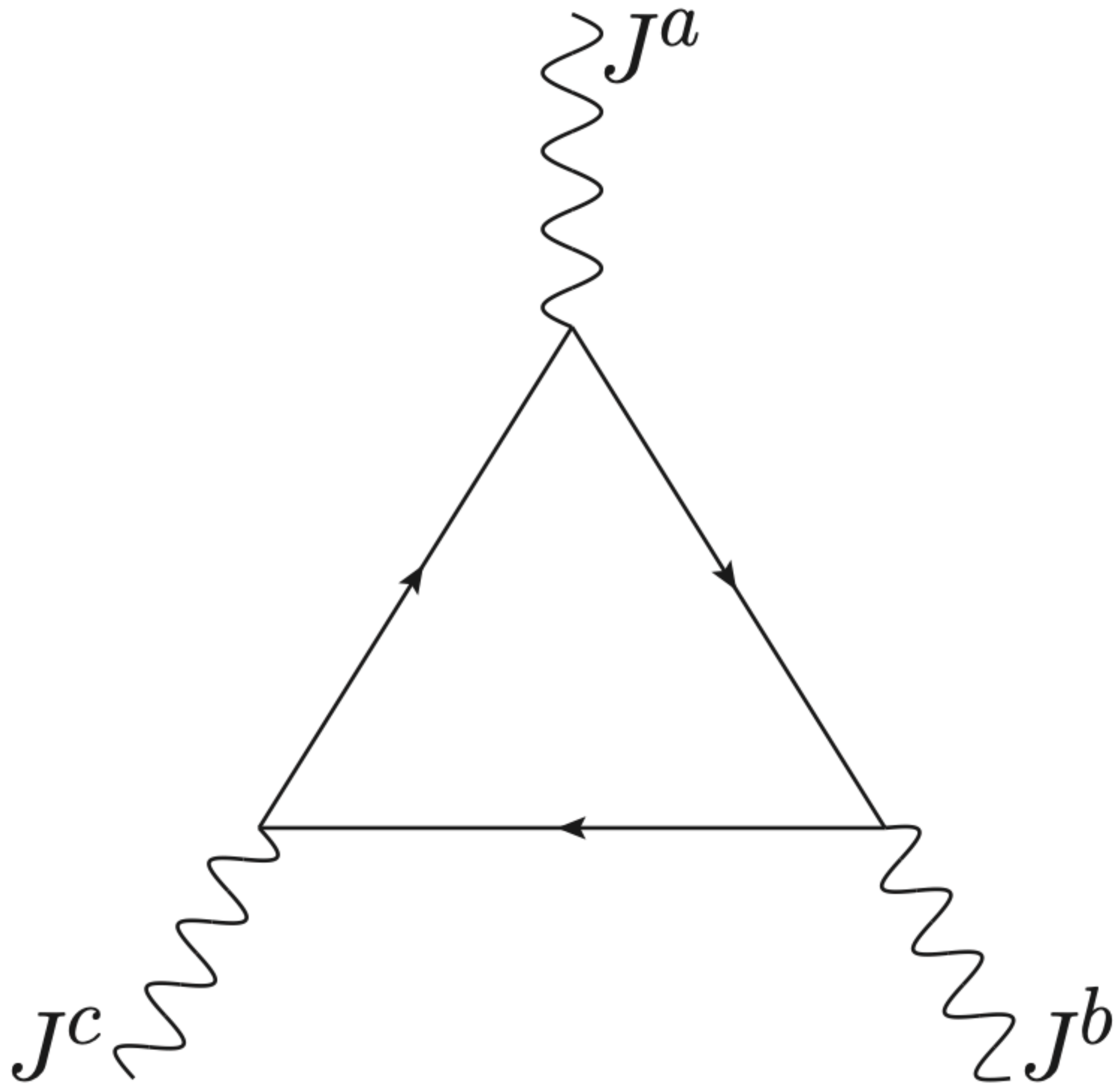
Chi-Ming Chang (UC, Davis, QMAP), Ying-Hsuan Lin (Caltech), Shu-Heng Shao (Princeton, Inst. Advanced Study), Yifan Wang (Princeton U.), Xi Yin (Harvard U., Phys. Dept.) (Feb 12, 2018)

Published in: *JHEP* 01 (2019) 026 • e-Print: 1802.04445 [hep-th]

Anomalies and Generalized Anomalies

The notion of ’t Hooft anomaly is very familiar for ordinary, continuous 0-form symmetries. It captures the lack of charge conservation in the presence of classical background sources (not to be confused with ABJ anomaly — which simply means that there is no symmetry at all).

$$\partial_\mu j^\mu \sim F \wedge F$$



Being that ’t Hooft anomalies are quantized c-number effects (hence cannot be renormalized) they have been an indispensable tool for several decades.

The most striking applications are of course for the chiral SU(3)xSU(3) symmetry in QCD, where the ’t Hooft anomaly leads to predictions for the neutral pion decay and the violation of G-parity in meson scattering.

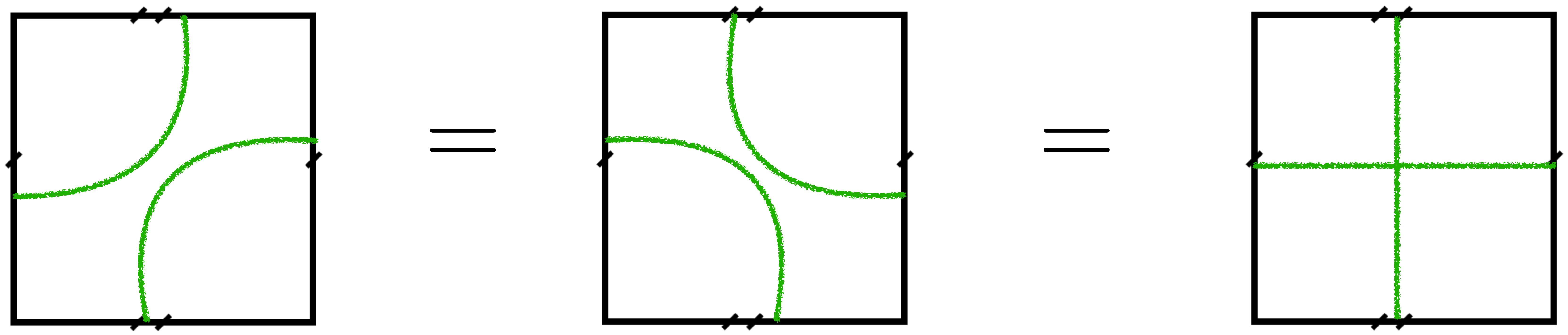
TASI 2003 lectures on anomalies

Jeffrey A. Harvey (Chicago U., EFI) (Sep, 2005)

e-Print: hep-th/0509097 [hep-th]

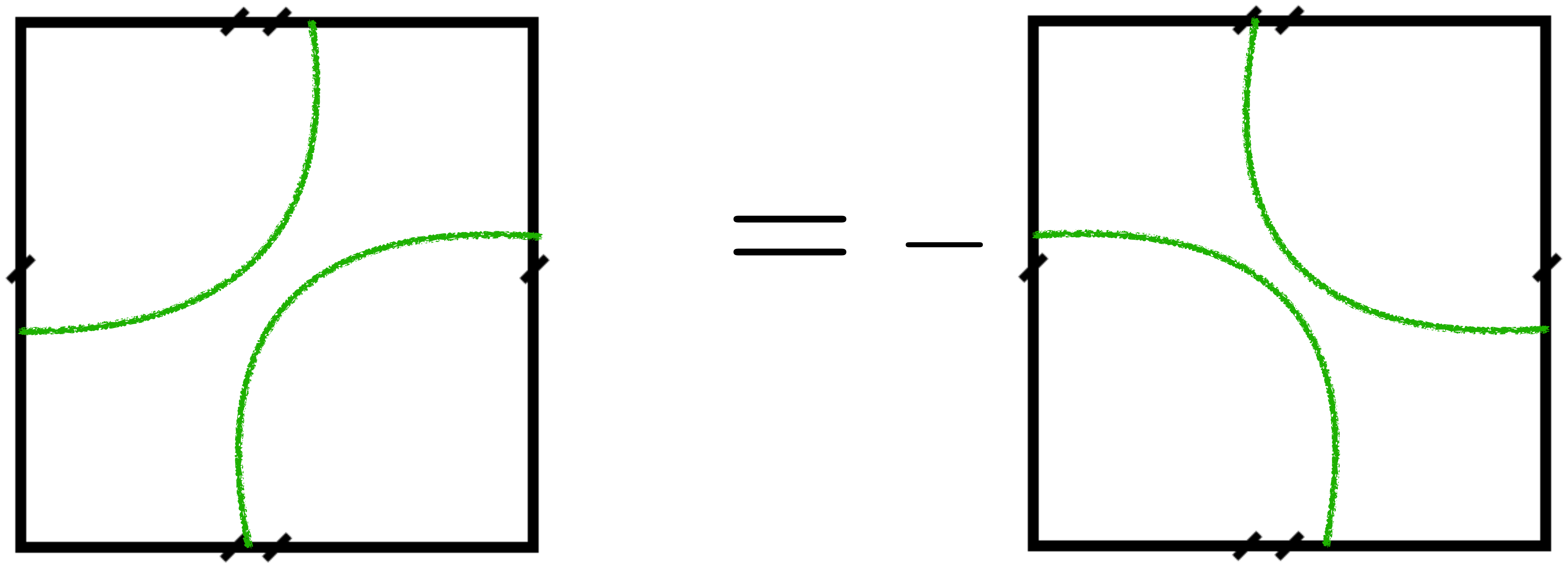
The notion of ’t Hooft anomaly can be generalized to discrete 0-form symmetries, to p-form symmetries, and even to non-invertible symmetries and also further, to more complicated structures that I won’t mention.

The idea is to consider networks of the topological charges and require that these networks can consistency re-connect.



If the networks can consistently re-connect we say that there is no anomaly, and otherwise there is a 't Hooft anomaly.

$\mathbb{Z}_2$ anomaly in 1+1 dimensions (bosonic)

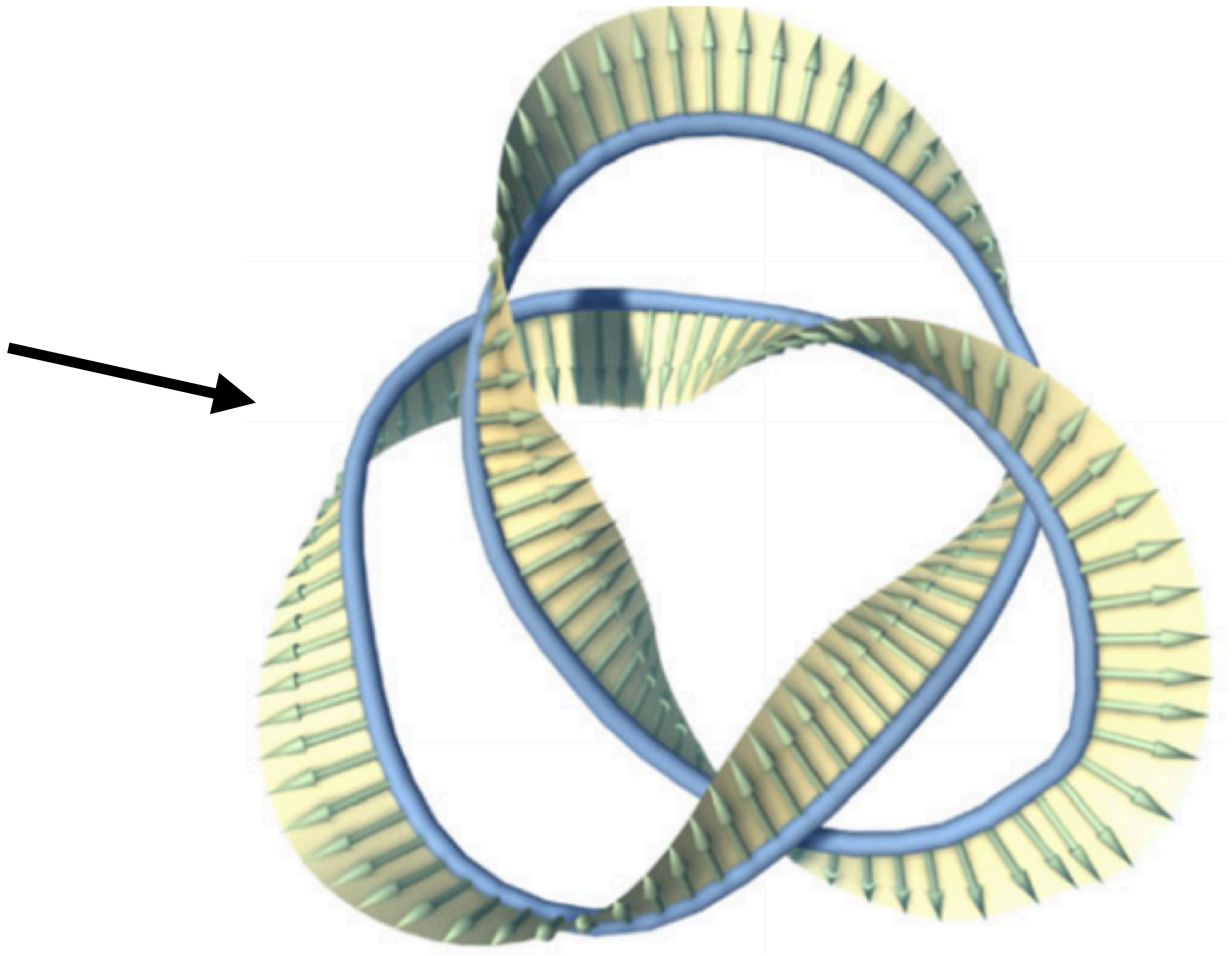


This minus sign might be hard to compute!! Not a triangle diagram...

Continuous anomalies imply that the ground state must be gapless either due to symmetry breaking or massless fermions.

Discrete anomalies are a little less constraining. They can be satisfied by:

- Symmetry breaking
- Massless modes
- A topological field theory



For some discrete anomalies it is possible to rule out the last option. The consequences of such anomalies are pretty much identical to the continuous case.

Anomaly Obstructions to Symmetry Preserving Gapped Phases

Clay Córdova (Chicago U. and Chicago U., EFI and Princeton, Inst. Advanced Study), Kantaro Ohmori (Princeton, Inst. Advanced Study) (Oct 10, 2019)

e-Print: 1910.04962 [hep-th]

We are finally ready to revisit our elementary QM example

$$H = \frac{1}{2}\left(\hat{p} - \frac{1}{2}\right)^2 - g\cos(2\phi)$$

This theory has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 0-form symmetry corresponding to the transformations $\phi \to \phi + \pi$ and $\phi \to -\phi$.

However, there is a phase in the algebra of those symmetries

$$U_{\phi\to-\phi}U_{\phi\to\phi+\pi} = -U_{\phi\to\phi+\pi}U_{\phi\to-\phi}$$

This phase precludes us from unambiguously laying out a network of topological operators and hence it is a ’t Hooft anomaly. It forces the system to have at least two degenerate ground states!

This minus sign can be calculated in the g=0 theory and, since the sign cannot change continuously, it holds for any g.

For QM models with just $\mathbb{Z}_2$ symmetry such anomalies cannot arise, one needs “at least” $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) = \mathbb{Z}_2$$

Applications

We will go over examples in 2d, 3d, and 4d where anomalies and generalized symmetries give us an insight into the low-energy physics of strongly interacting models.

2d

$$\mathcal{L} = \frac{-1}{4g^2}\mathrm{Tr}F^2 + i\Psi^T\gamma^0 D_\mu\gamma^\mu\Psi$$

Ψ is a non-chiral Majorana fermion in the adj representation of SU(N).

This theory has a $\mathbb{Z}_N$ 1-form symmetry along with a $\mathbb{Z}_2$ chiral symmetry but also exponentially many $\sim 2^{2N}$ non-invertible symmetries.

Analyzing the consequences of this huge number of symmetries one finds some concrete predictions:

- Deconfinement (even though no fundamental quark is present)
- An exponential vacuum degeneracy (!)
- The string tension in the theory with a small quark mass:

$$T_k \sim |m| \sin\left(\frac{\pi k}{N}\right)$$

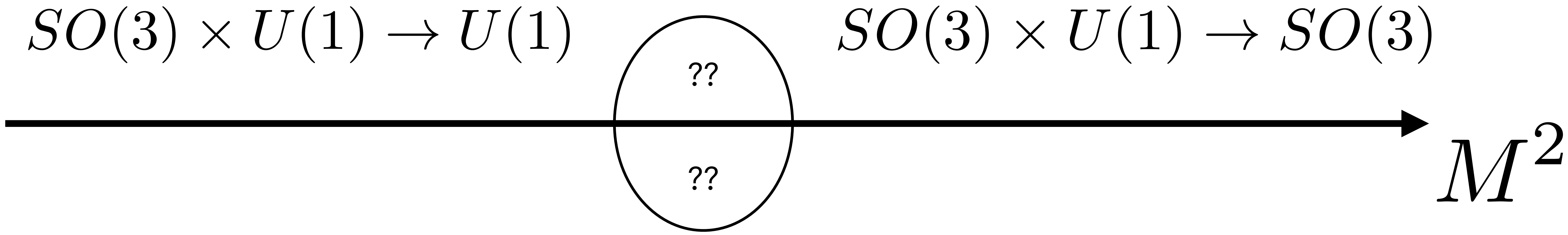
3d

Consider U(1) gauge theory with two charged bosons:

$$\frac{-1}{4e^2}F^2 + |D\Phi^i|^2 + M^2|\Phi^i|^2 + \lambda|(\Phi^i|^2)^2$$

$j_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}F^{\nu\rho}$ is a 0-form U(1) symmetry (called the topological charge). In addition we have the SO(3) rotations acting on Φ^i.

There is a mixed anomaly between SO(3) and U(1) and this precludes a trivial phase. Landau-Ginzburg theories never have such phase diagrams, which is why this type of dynamics is called ``beyond Landau-Ginzburg.”



Time reversal symmetry has interesting anomalies in 3d, given by a phase which is a 16th root of unity.

This has implications for nonAbelian gauge theories with fermions.

Let us consider SU(2) gauge theory with an adjoint Majorana fermion. The symmetries and anomalies imply that it must be deconfined and nontrivial in the infrared.

$$SU(2) + \text{adjoint } \lambda \longrightarrow \Psi + U(1)_2$$

$$\Psi \sim \text{Tr}\lambda^3$$

4d

There again many interesting examples. I will mention only one example before we close.

$$S = \int d^4x \; \mathrm{Tr}\left(\frac{-1}{4g^2} F \wedge \star F + \frac{i\theta}{8\pi^2} F \wedge F \right)$$

This is just the famous Yang-Mills theory. At $\theta = 0$
we believe it is gapped and confined but at $\theta = \pi$
there is a new anomaly and the theory must either break time reversal symmetry or be deconfined!

- Generalized symmetries and discrete anomalies in all dimensions, with and without fermions.
- Stringent constraints on the infrared physics. Many new proposals for various gauge theories.
- A parallel, analogous story for condensed matter systems.
- Intriguing connections to mathematics.
- Many topics we did not cover: two-groups, fractionalization classes, symmetry preserving boundary conditions etc.

Thank you!