Scattering amplitudes from dispersive iterations

A. Zhiboedov (CERN)

Amplitudes 2022, Prague

based on works with M. Correia, A. Sever, and **P. Tourkine**

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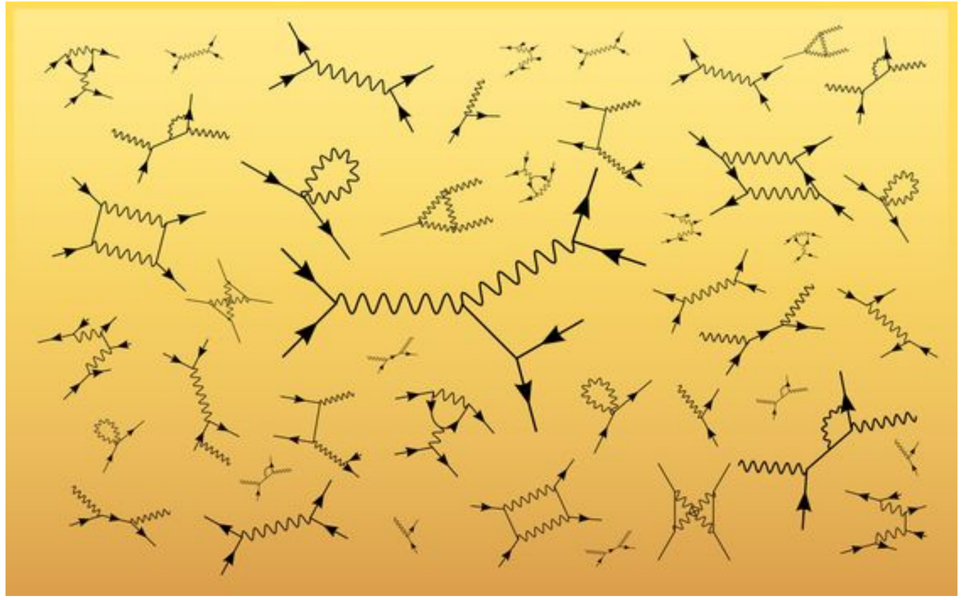

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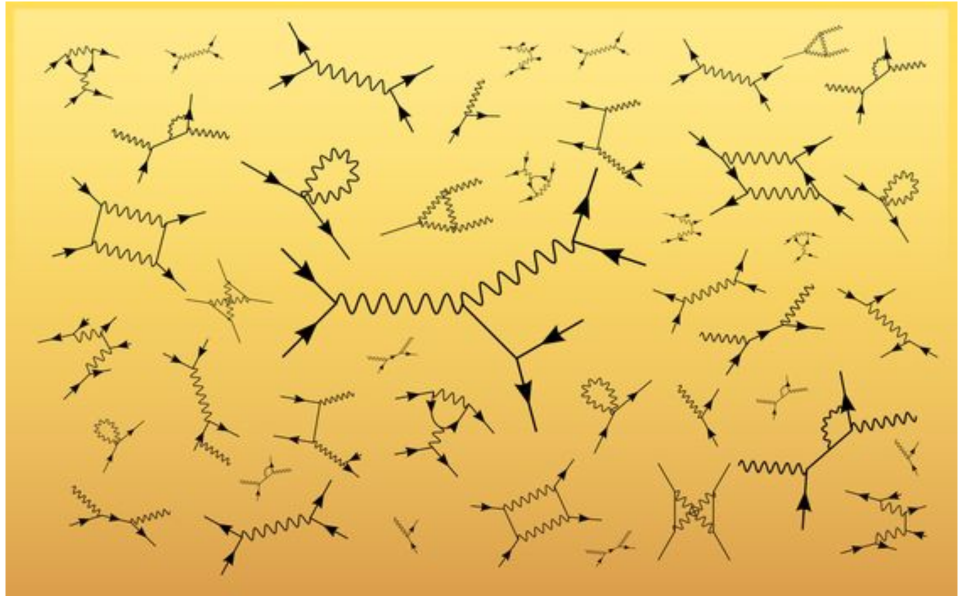

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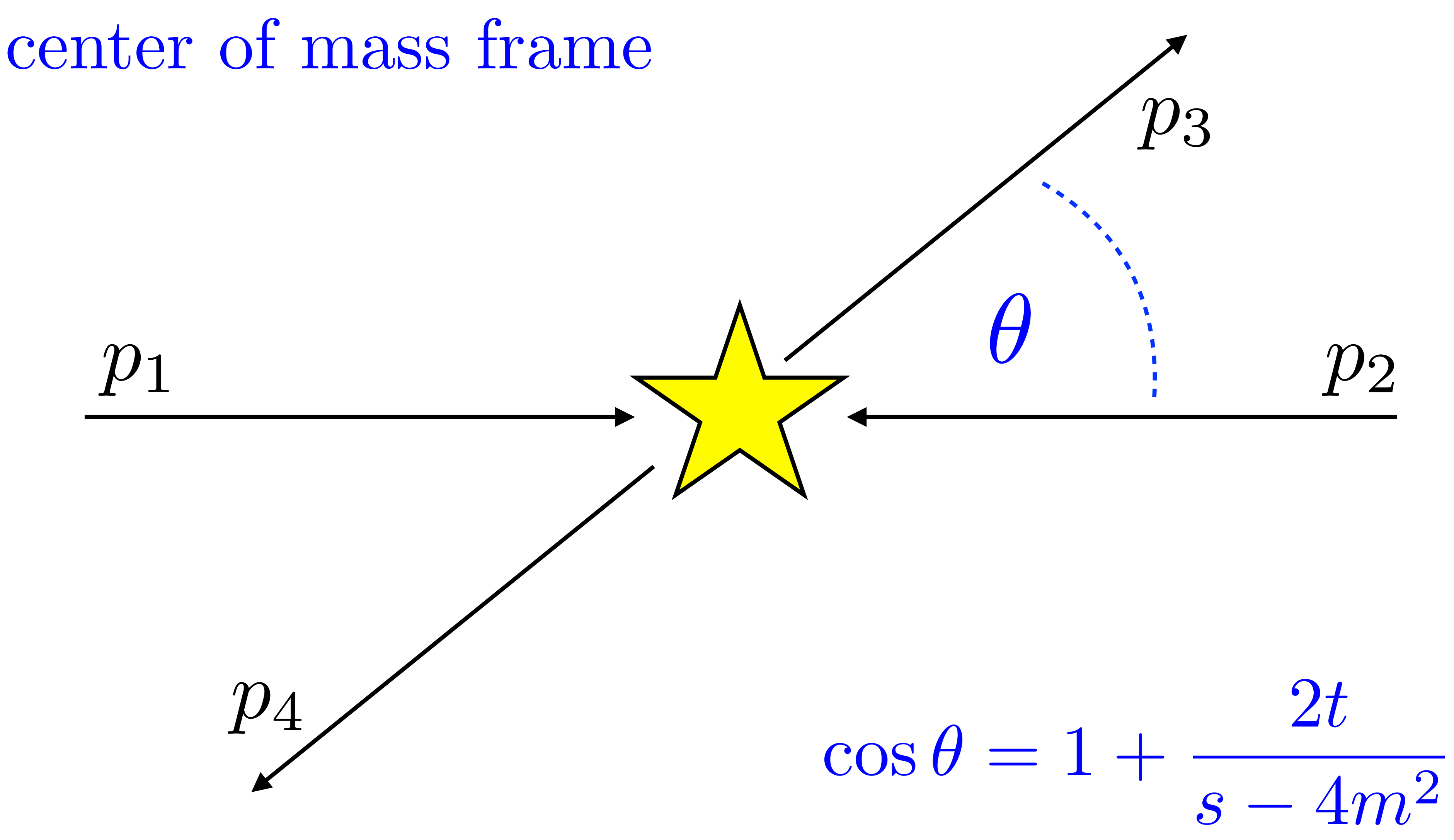
What about nonperturbative unitarity?

Let us focus on the **two-to-two scattering** amplitude $T(s,t)$ of identical particles



Nonperturbative unitarity $SS^\dagger = 1$

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* Elastic unitarity

$$T_s(s,t) = \frac{(s-4m^2)^{1/2}}{4(4\pi)^2\sqrt{s}} \int_{\sqrt{\quad} \geq 0} dz' dz'' \frac{T(s,t(z'))T^*(s,t(z''))}{(1-z^2-z'^2-z''^2+2zz'z'')^{\frac{1}{2}}} \qquad s_{\mathrm{MP}} > s \geq 4m^2$$

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* Inelastic unitarity

$$T(s,t) = 16\pi \sum_{J=0, J-\mathrm{even}}^{\infty} (2J+1) f_J(s) P_J\Big(1 + \frac{2t}{s-4m^2}\Big)$$

$$\mathrm{Im} f_J(s) \geq \frac{1}{2}\frac{(s-4m^2)^{1/2}}{\sqrt{s}} |f_J(s)|^2, \quad s \geq s_{\mathrm{MP}}$$

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* **Multi-particle unitarity**

(detailed structure of multi-particle thresholds, LC)

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Construct $T(s,t)$ in $d \geq 3$ that satisfies analyticity, crossing and nonperturbative unitarity (elastic, inelastic, MP).

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Snowmass White Paper: S-matrix Bootstrap

[Kruczenski, Penedones, van Rees '22]

Dispersive representations

What makes the S-matrix bootstrap interesting is a tension between **analyticity**, **unitarity** and **crossing**.

Dispersive representation	**Analyticity**	**Crossing**	**Unitarity**
Dispersive relations with 2 subtractions	✓	✗ *	✓
Crossing-symmetric dispersion relations	✗	✓	✓
Mandelstam representation	✓	✓	✗

[Auberson, Khuri '72][Sinha, Zahed '20]

(*) $d>2$

Current Status

Goal: Construct $T(s, t)$ which obeys the basic principles (elastic, inelastic, MP) and exhibits the correct physical structure (thresholds, LC).

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1. Algorithm (old)

- Mandelstam representation, Mandelstam equation for $\rho(s,t)$

[S. Mandelstam 58-62's]

[see also G. Chew, S. Frautschi, McCauley, K. Ter-Martirosyan, K. Wilson]

- Fixed point theorems for unitarity&crossing

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2. Implementation (new):

- Multi-particle Landau curves

[M. Correia, A. Sever, AZ '21]

- Numerical implementation in 2d and 4d

[P. Tourkine, AZ '21]

[P. Tourkine, AZ to appear]

Constructing the 4d scattering amplitude: Method

Mandelstam representation

Our starting point is the Mandelstam representation with one subtraction

$$
\begin{aligned}
T(s,t) &= \lambda + B(s,t) + B(s,u) + B(t,u), \\
B(s,t) &= \frac{s-s_0}{2}\int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\rho(s')}{(s'-s)(s'-s_0)} + \frac{t-t_0}{2}\int_{4m^2}^{\infty}\frac{dt'}{\pi}\frac{\rho(t')}{(t'-t)(t'-t_0)} \\
&\quad + (s-s_0)(t-t_0)\int_{4m^2}^{\infty}\frac{ds'dt'}{\pi^2}\frac{\rho(s',t')}{(s'-s)(t'-t)(s'-s_0)(t'-t_0)}
\end{aligned}
$$

Crossing-symmetry implies that $\rho(s,t) = \rho(t,s),$

$$s_0 = t_0 = u_0 = \frac{4m^2}{3}\ .$$

In this way (lightest particle) **maximal analyticity** & **crossing** are manifest.

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$$+ (s-s_0)(t-t_0)\int_{4m^2}^{\infty}\frac{ds'dt'}{\pi^2}\frac{\rho(s',t')}{(s'-s)(t'-t)(s'-s_0)(t'-t_0)}$$

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Unitarity with Mandelstam Rep

One can show that **dispersive elastic unitarity** takes the following form

$$\rho(s,t) = \frac{(s-4m^2)^{\frac{1}{2}}}{(4\pi)^2\sqrt{s}} \int_{z_1}^{\infty} d\eta' \int_{z_1}^{\infty} d\eta'' \frac{T_t(s+i\epsilon, t(\eta'))T_t(s-i\epsilon, t(\eta''))\theta(z-\eta_+)}{\sqrt{(z-\eta_-)(z-\eta_+)}}, \quad s_{\mathrm{MP}} > s > 4m^2,$$

$$\eta_\pm = \eta'\eta'' \pm \sqrt{\eta'^2-1}\sqrt{\eta''^2-1},$$

It relates **double spectral density** to the **square of the single discontinuity**.

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- This formula guarantees elastic unitarity for partial waves with $\mathrm{Re}[J] > 0, J \in \mathbb{C}$.
- For $J = 0$ we have separately

$$\mathrm{Im} f_0(s) = \frac{1}{2}\frac{(s-4m^2)^{1/2}}{\sqrt{s}} |f_0(s)|^2 + \theta(s-s_{\rm MP})\eta_{\rm MP}(s)$$

Landau curves

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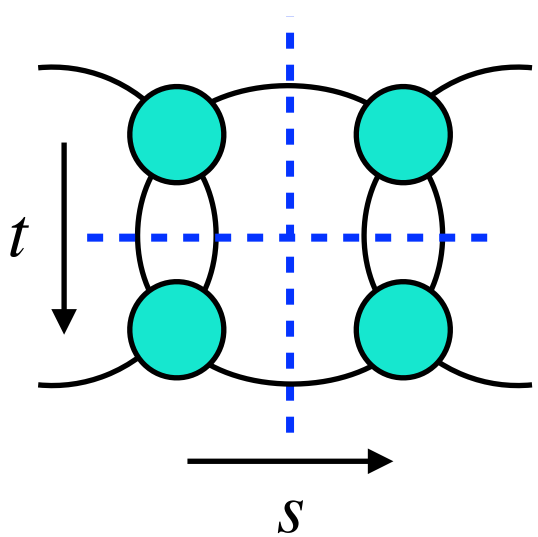
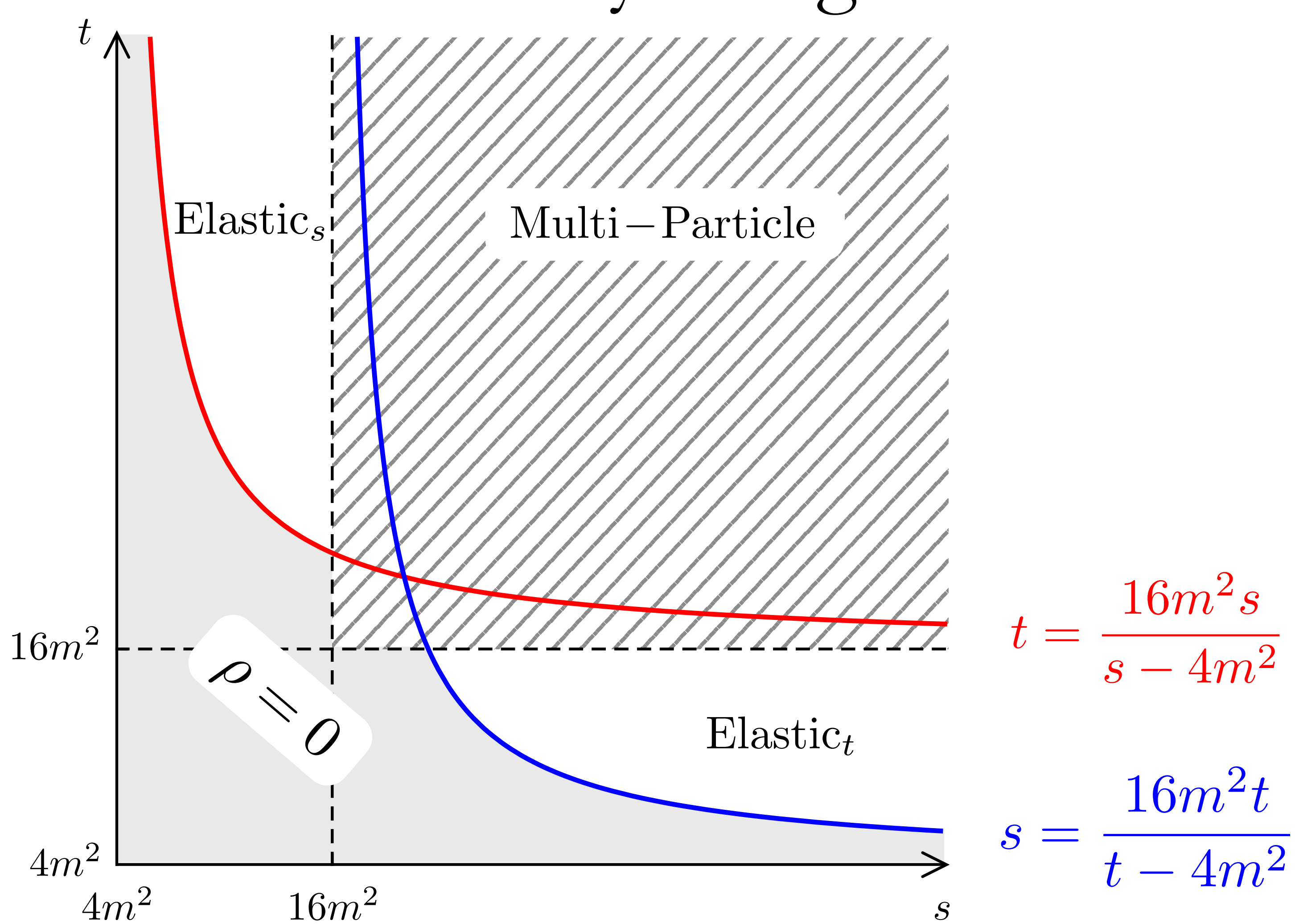
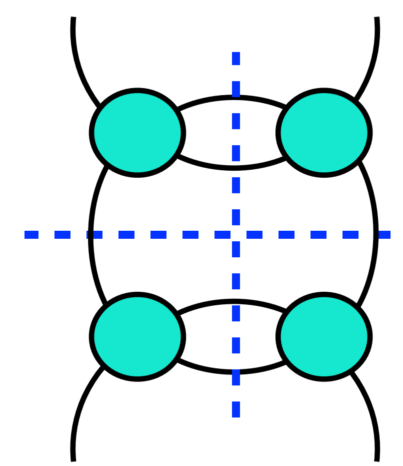
Landau curves are non-perturbative loci which follow from the analytic continuation of unitarity along which the ddisc develops.



scattering



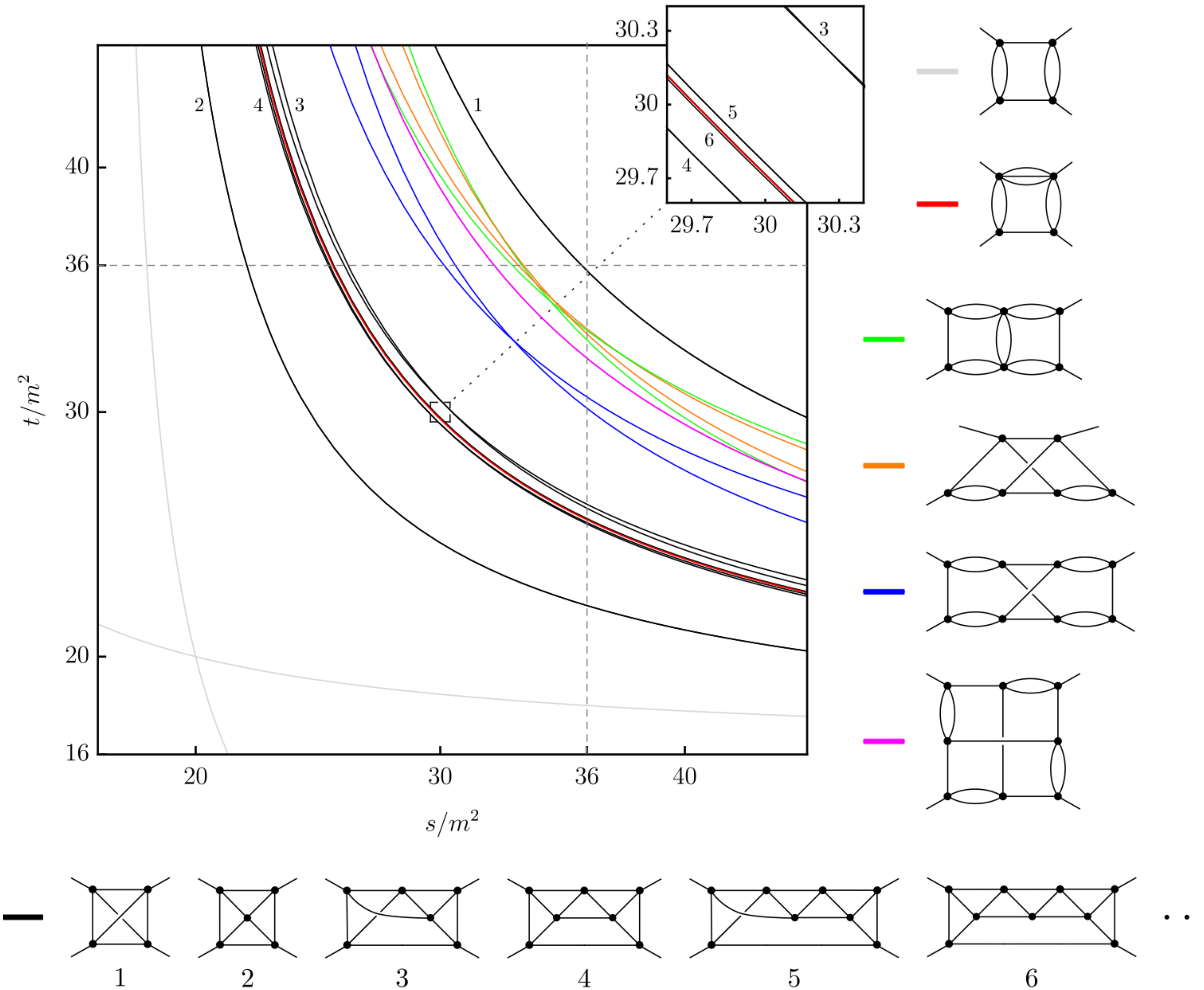


implies production

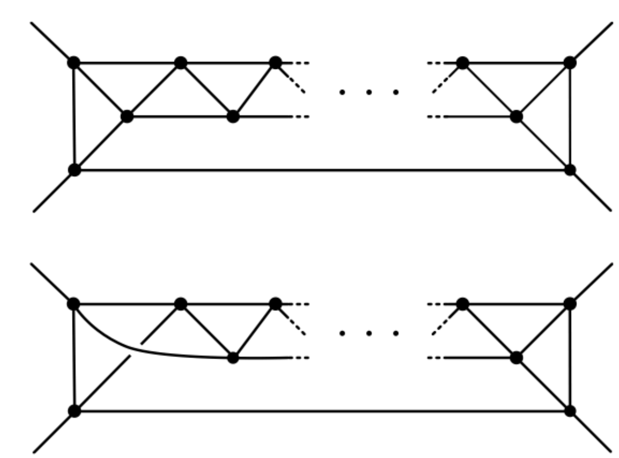
Multi-particle Landau curves

[M. Correia, A. Sever, AZ '21]

[M. Correia poster]



computer-assisted graph search



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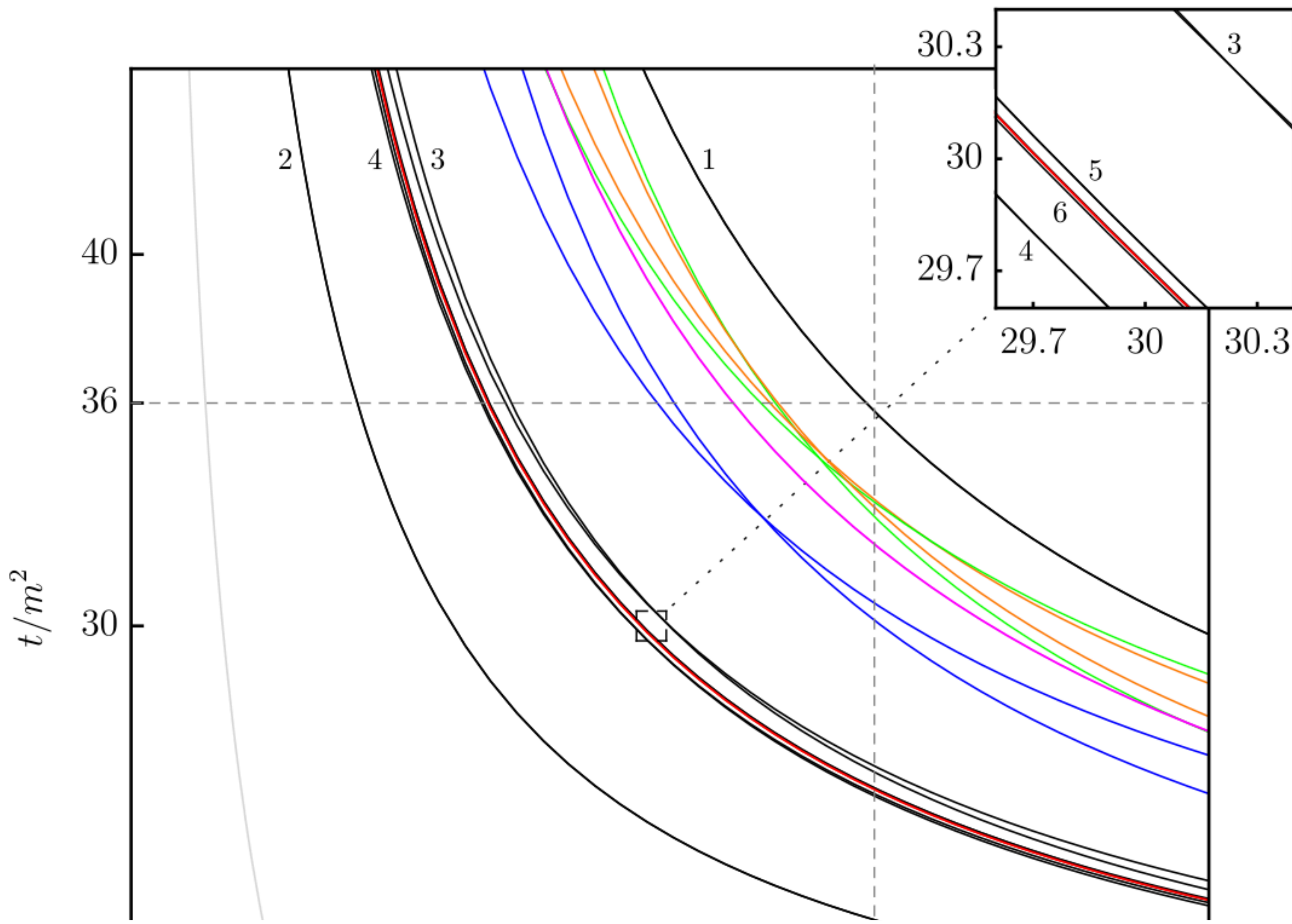
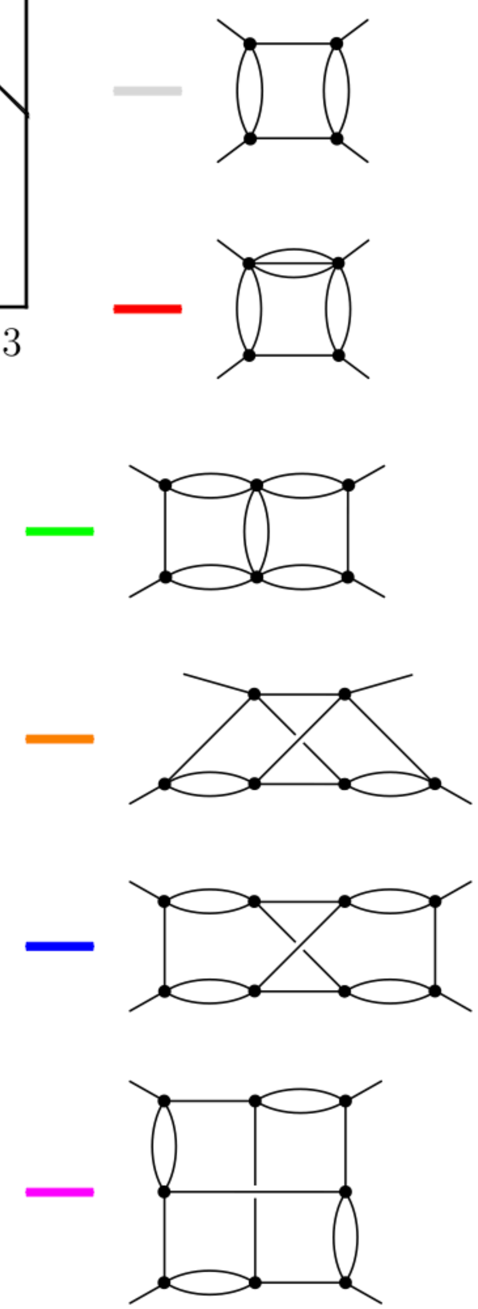
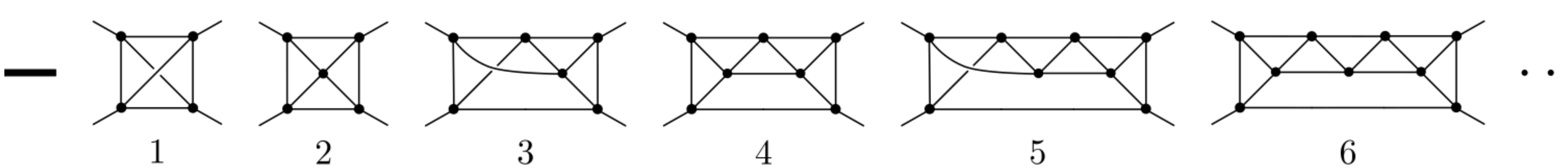
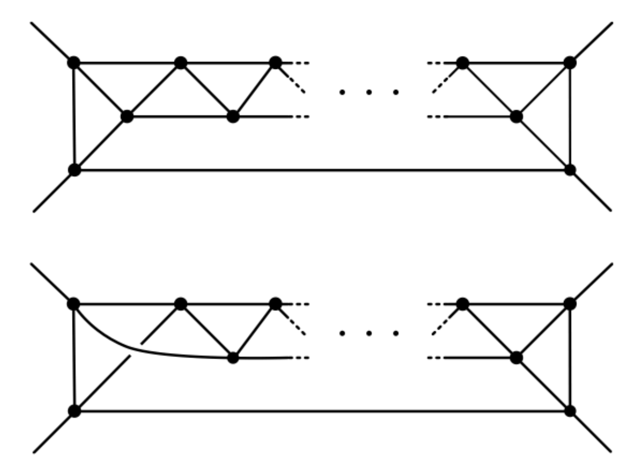




computer-assisted graph search

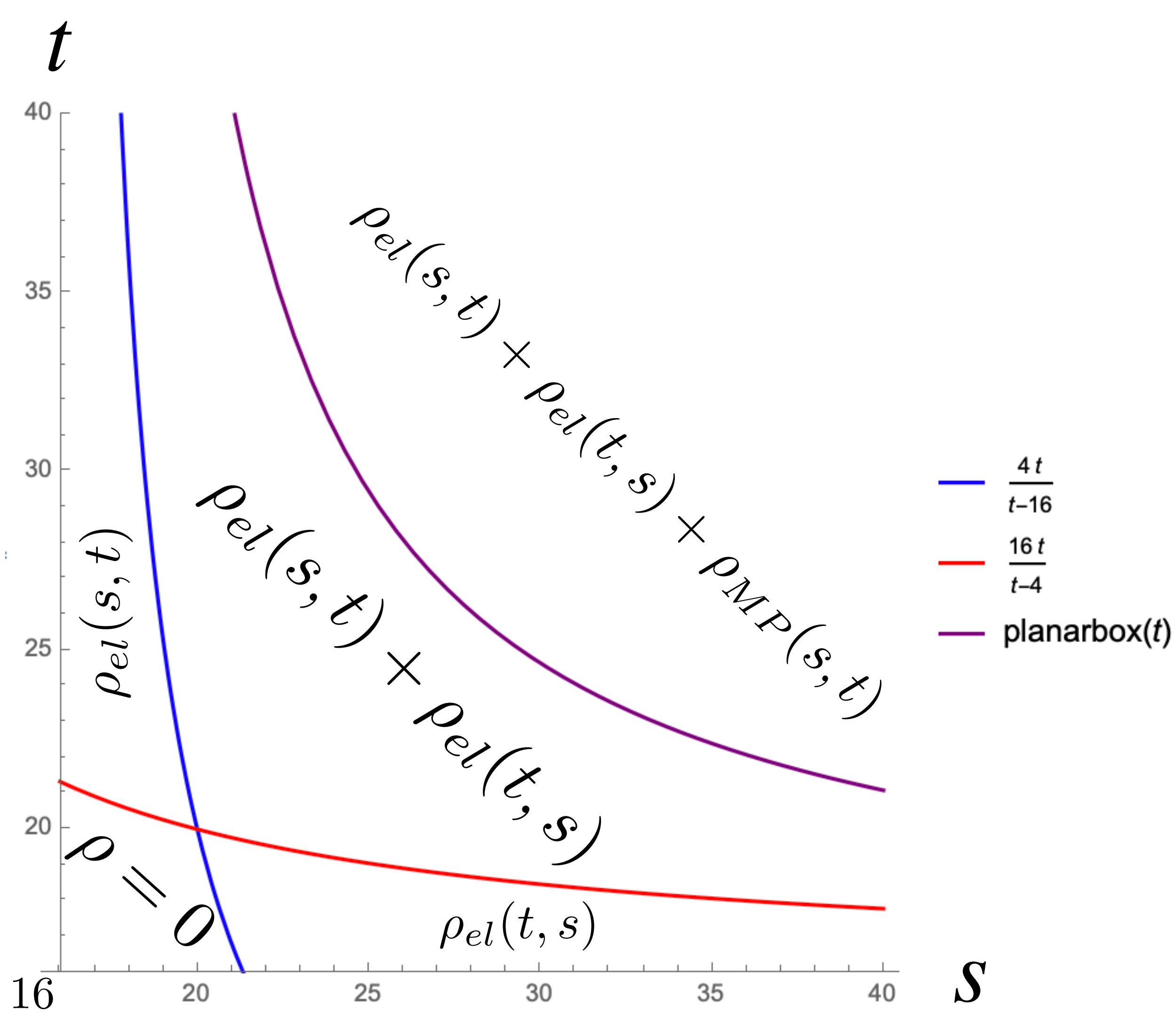
Quartic graphs ($D = 4$)									
# of vertices (V)	4	5	6	7	8	9	10	11	12
All graphs (no trivial bubbles)	2	3	23	111	788	5639	46603	410114	3587793
No trivial triangles	2	1	10	33	232	1522	12696	113034	1023415
2- or 4-particle cuts (exist)	2	1	7	25	157	955	7070	54835	429093
2- or 4-particle cuts (all legs)	2	1	4	5	12	7	10	7	9
No trivial boxes	2	1	3	4	9	4	4	3	3
α-positive Landau curves	2	1	2	1	3	1	1	1	1





Iterations

Basic idea: Construct **scattering** from given **production** and **subtraction** λ by iterating unitarity.



Given $(\lambda, \eta_{\text{MP}}(s), \rho_{\text{MP}}(s,t))$

$$T(s,t)$$

Check that inelastic unitarity is also satisfied.

Iteration scheme

single disc:

$$\rho^{(n+1)}(s) = 8\pi\left(\frac{(s-4m^2)^{1/2}}{\sqrt{s}}|f_0^{(n)}(s)|^2 + 2\theta(s-s_{\mathrm{MP}})\eta_{\mathrm{MP}}(s)\right)$$
$$-2\int_{4m^2}^{\infty}\frac{dt'}{\pi}\rho^{(n)}(s,t')[\frac{1}{t_0-t'}+\frac{\log\frac{s+t'-4m^2}{t'}}{s-4m^2}]$$

double disc:

$$T_t^{(n+1)}(s,t) = \rho^{(n)}(t) + \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{(s-s_0)\rho^{(n)}(s',t)}{(s'-s)(s'-s_0)} + \int_{4m^2}^{\infty}\frac{du'}{\pi}\frac{(u-u_0)\rho^{(n)}(u',t)}{(u'-u)(u'-u_0)}.$$

$$\rho_{\mathrm{el}}^{(n+1)}(s,t) = \mathrm{M}[T_t^{(n)}, T_t^{(n)}](s,t),$$
$$\rho^{(n+1)}(s,t) = \rho_{\mathrm{el}}^{(n+1)}(s,t) + \rho_{\mathrm{el}}^{(n+1)}(t,s) + \rho_{\mathrm{MP}}(s,t)$$

Fixed point of this mapping provides us with the solution to elastic unitarity (+some MP structure).

Atkinson theorems

One first identifies the Banach space (complete, normed) of functions (bounded, Hölder continuous) which is closed under iterations of unitarity and dispersion relations.

Atkinson theorem I: unitarity+crossing iteration is a **continuous** operator on this space.

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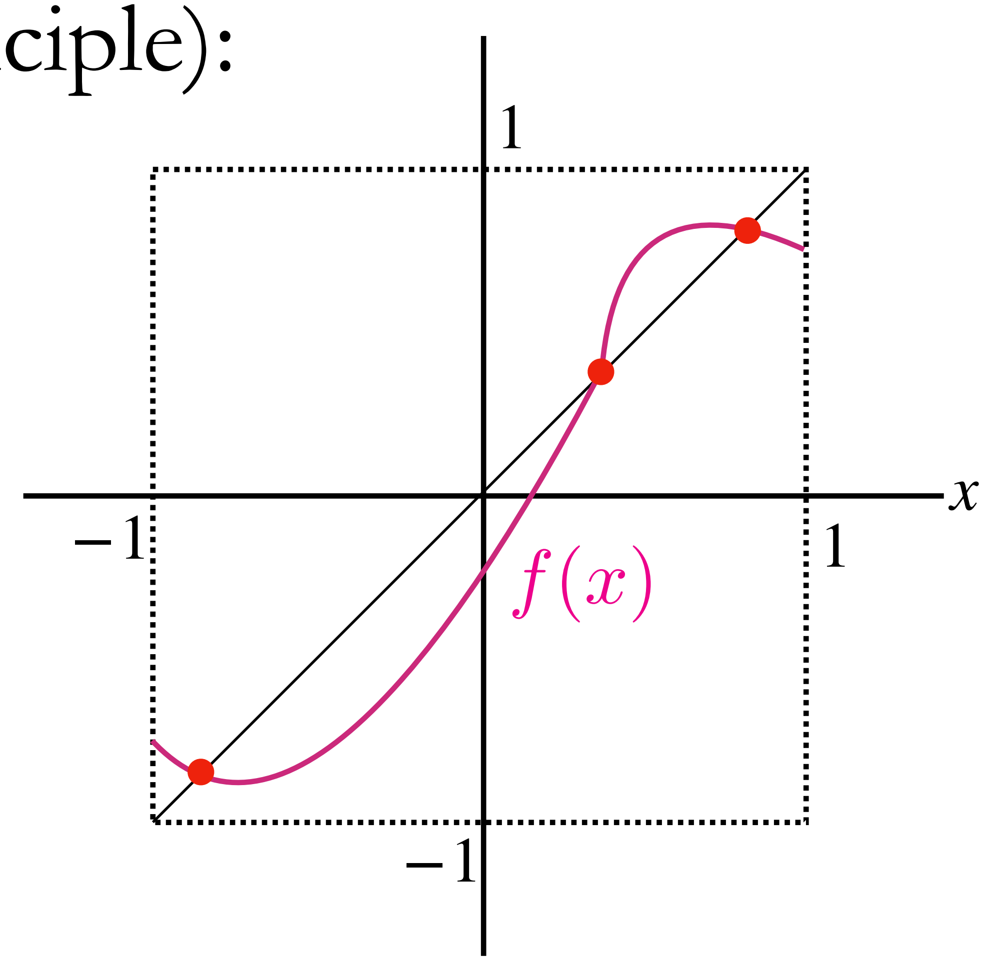
Atkinson theorem I: unitarity+crossing iteration is a **continuous** operator on this space.

After that existence of the fixed point follows (Schauder's principle):



$$x = f(x)$$

Atkinson theorems

<u>Atkinson theorem II:</u> if inelasticity is small enough, the mapping is contracting and the fixed point is unique.

$$||\mathcal{O}(\omega_1) - \mathcal{O}(\omega_2)|| \leq k||\omega_1 - \omega_2||, \qquad k < 1$$

Contraction mapping theorem: The solution is unique and can be reached by iterations.

$$||\omega - \omega_n|| \leq ||\omega_1 - \omega_0|| \frac{k^n}{1-k}.$$

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This provides us **in principle** with an infinite set of amplitude-functions parameterised by **production** that have (some of) the desired properties.

Can we construct them **in practice**?

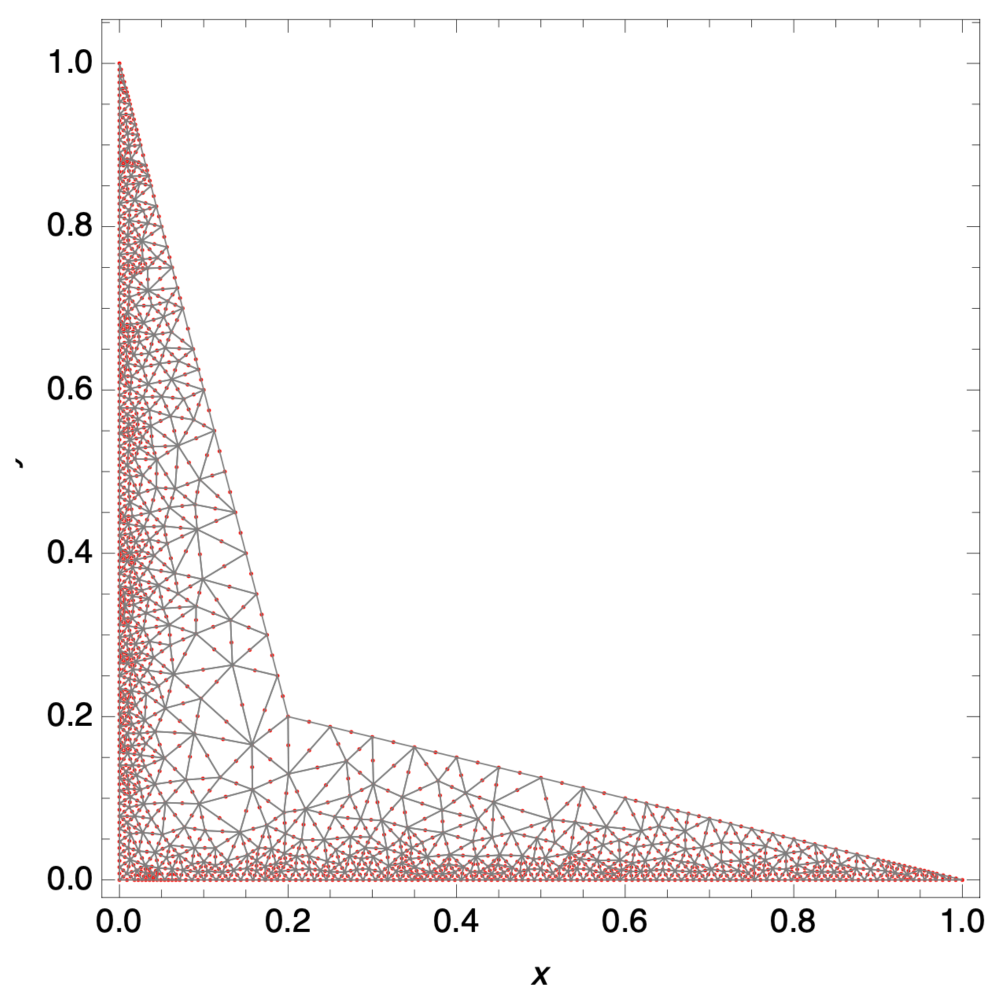
Constructing the 4d scattering amplitude: Numerical implementation

Numerical implementation

Consider the following coordinates

$$0 \leq x \equiv \frac{4m^2}{s} \geq 1 \qquad 0 \leq y \equiv \frac{4m^2}{t} \geq 1$$

Introduce a grid

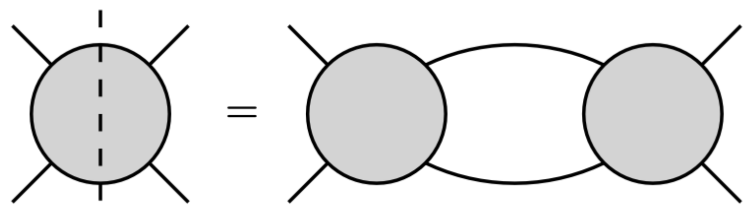


The Regge limit is treated by simply introducing cutoffs.

Relationship to Feynman diagrams

What is the relation of all this to Feynman perturbation theory?

1



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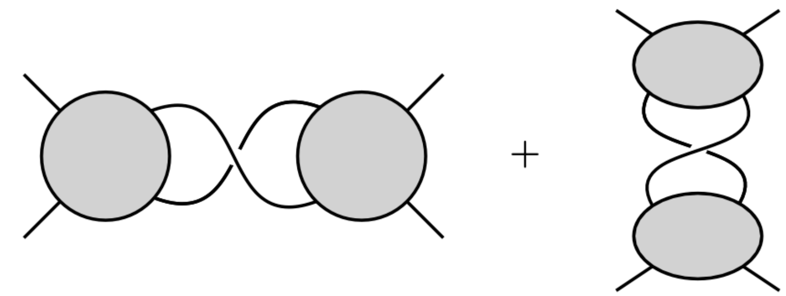

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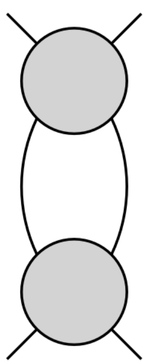

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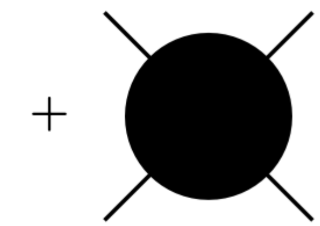


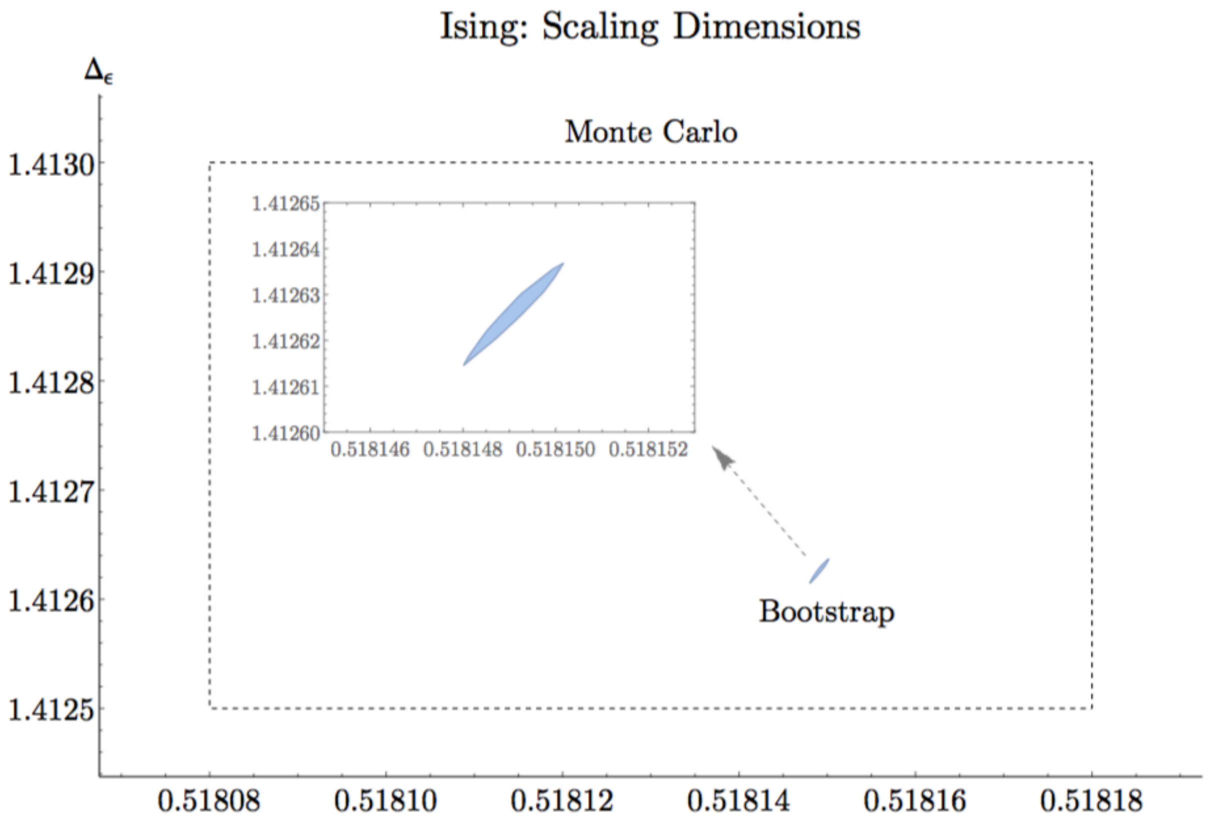
(2PI irreducible graphs)

The number of graphs and loops thus grows as follows

$$N_{n+1} = 3(N_n)^2 + 1$$
$$L = 2^n$$

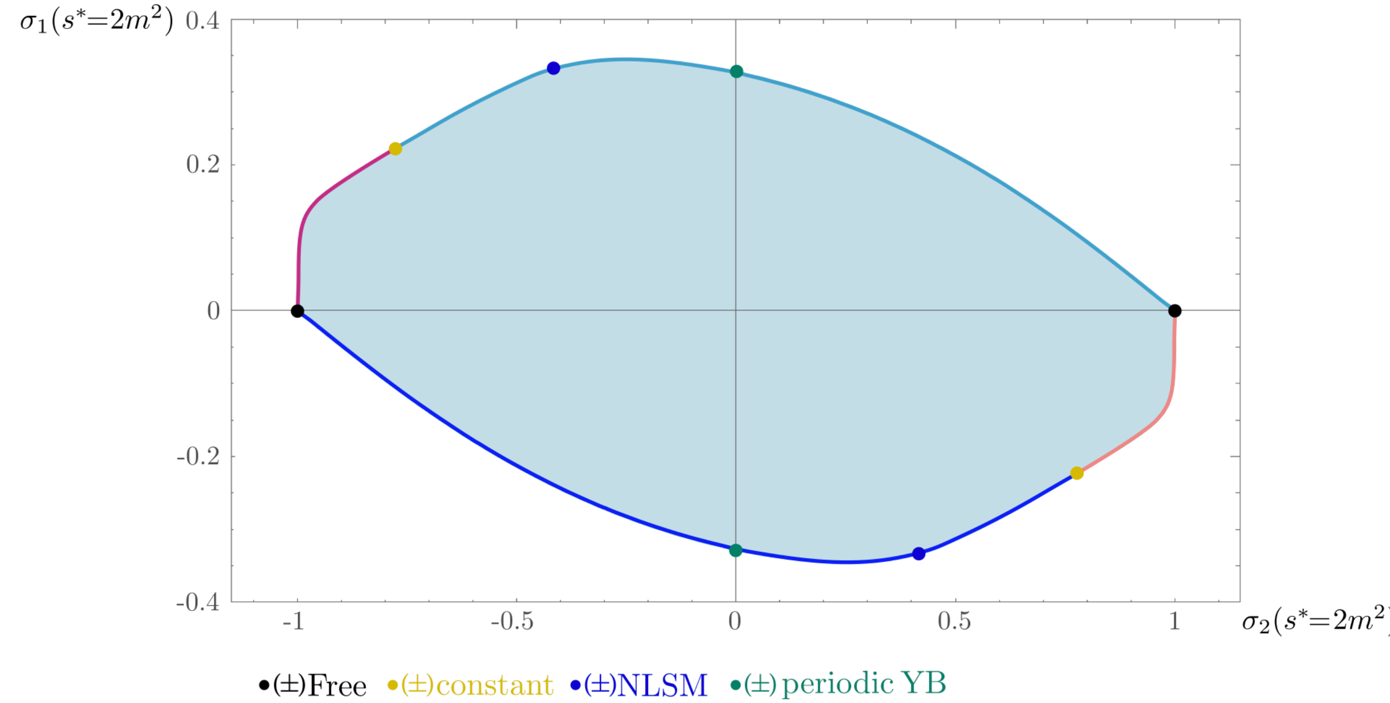
$$\Rightarrow \quad N(L) \simeq (1.86)^L$$

Using the finite grid also takes care of the Regge limit.
Spin zero partial wave is less transparent.

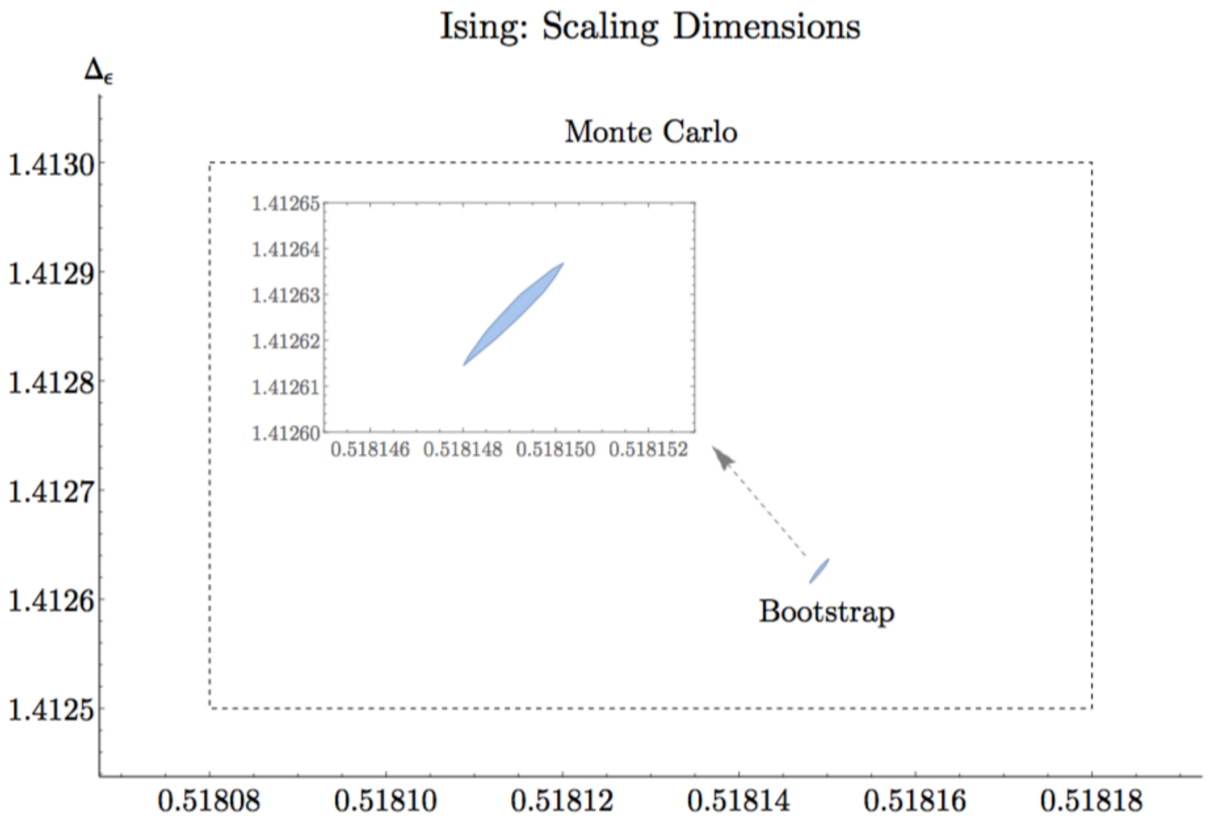


[Kos, Poland, Simmons-Duffin, Vichi '16]

d=2



[Cordova, He, Kruczenski, Vieira '19]



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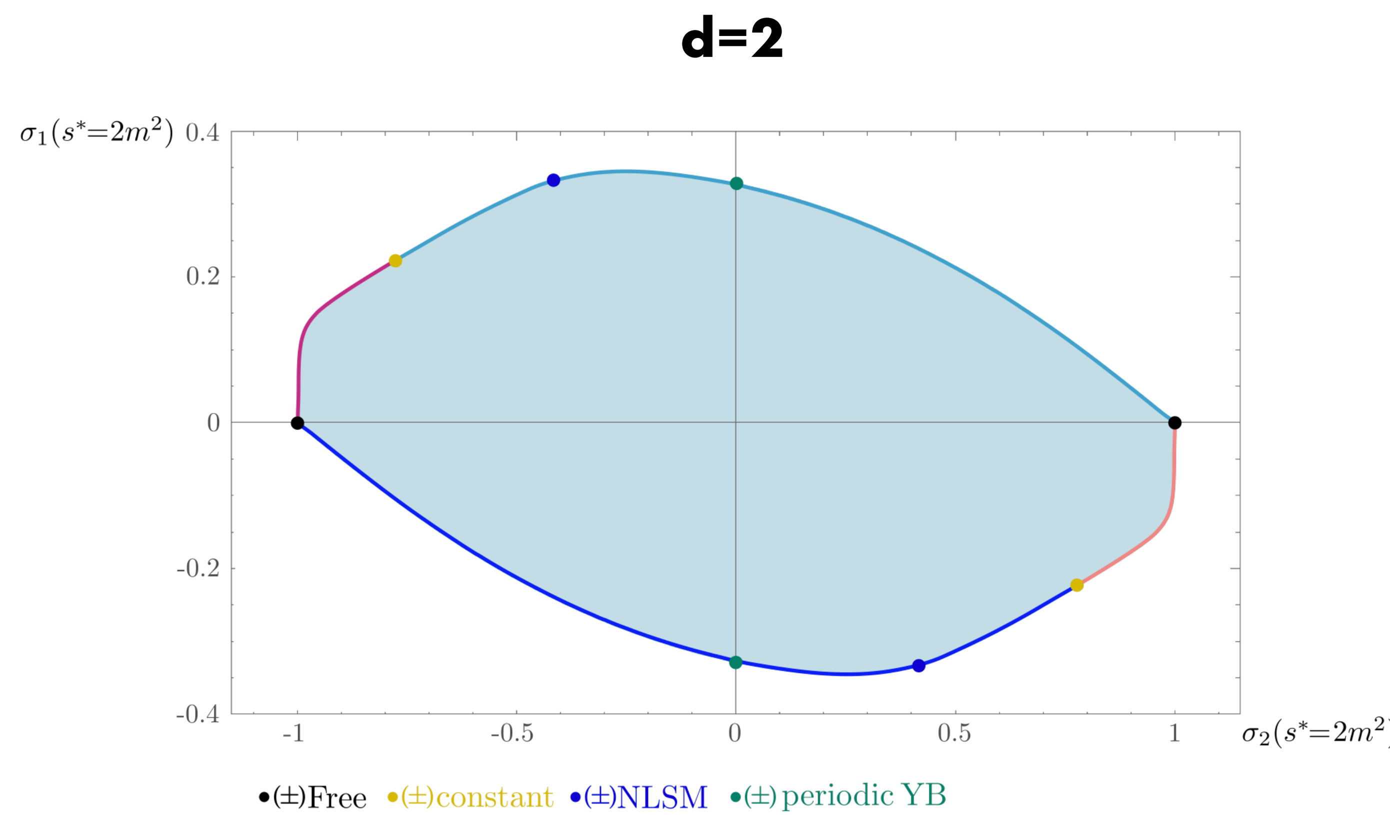

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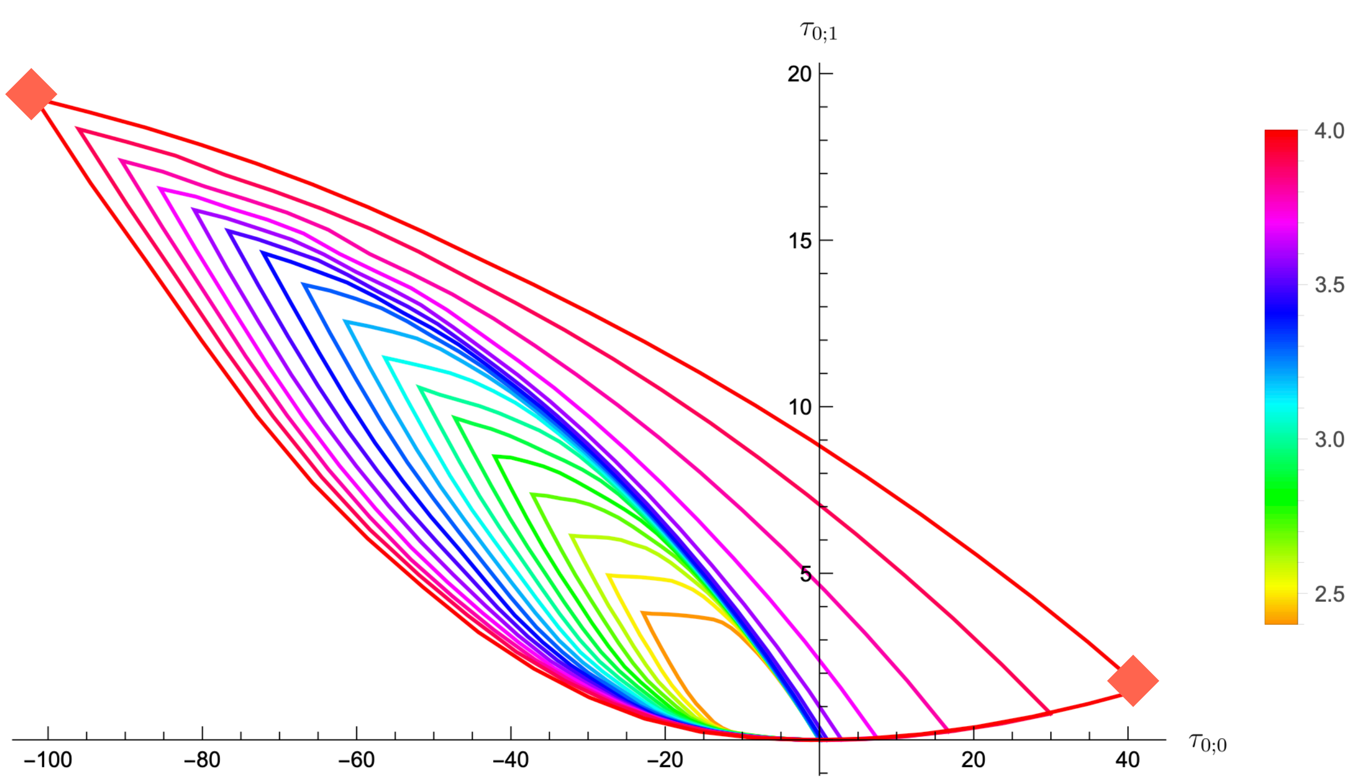
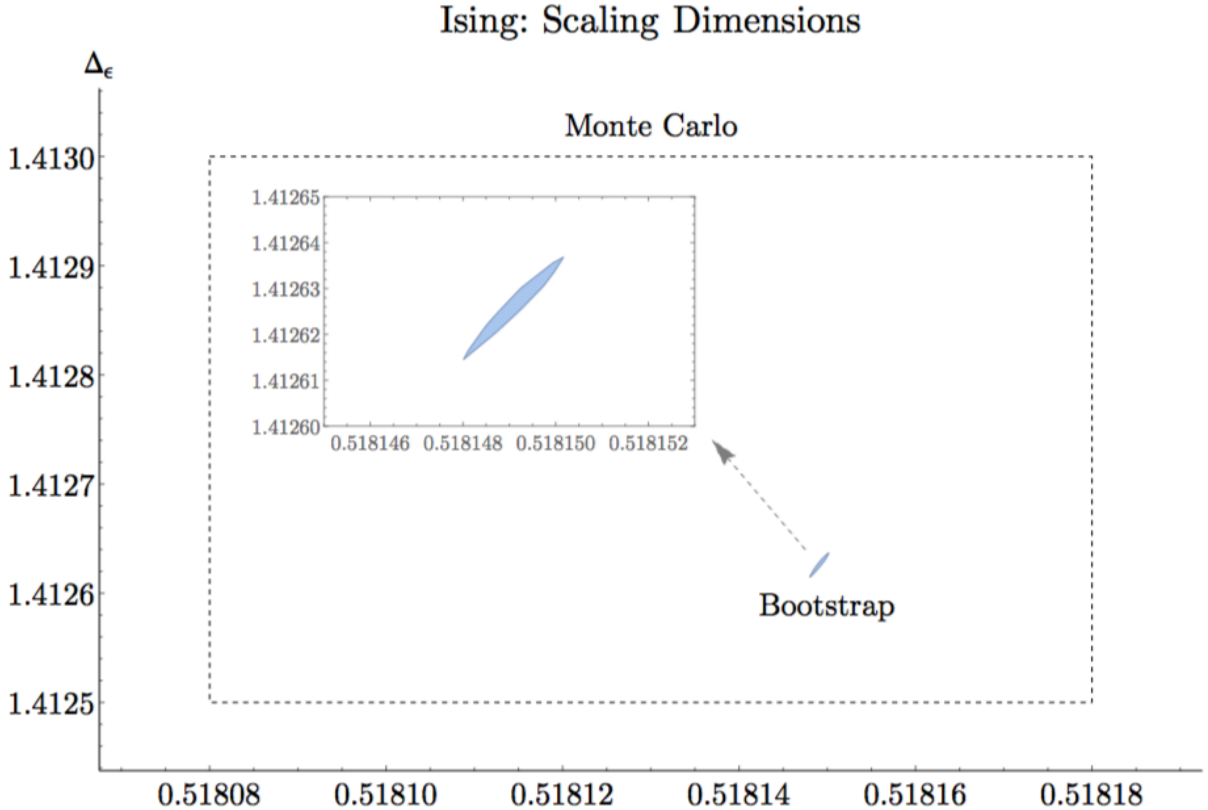


[Chen, Fitzpatrick, Karateev ’22]
[Paulos, Penedones, Toledo, van Rees, Vieira ’17]



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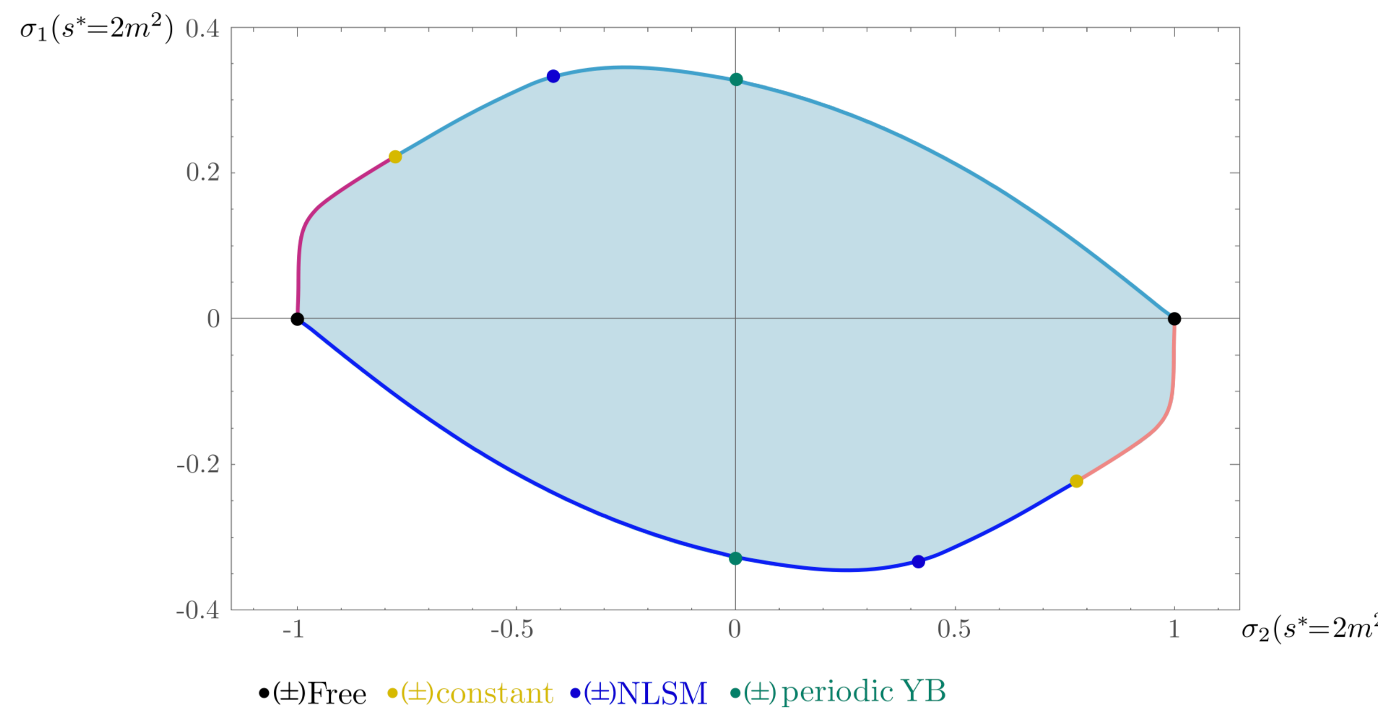

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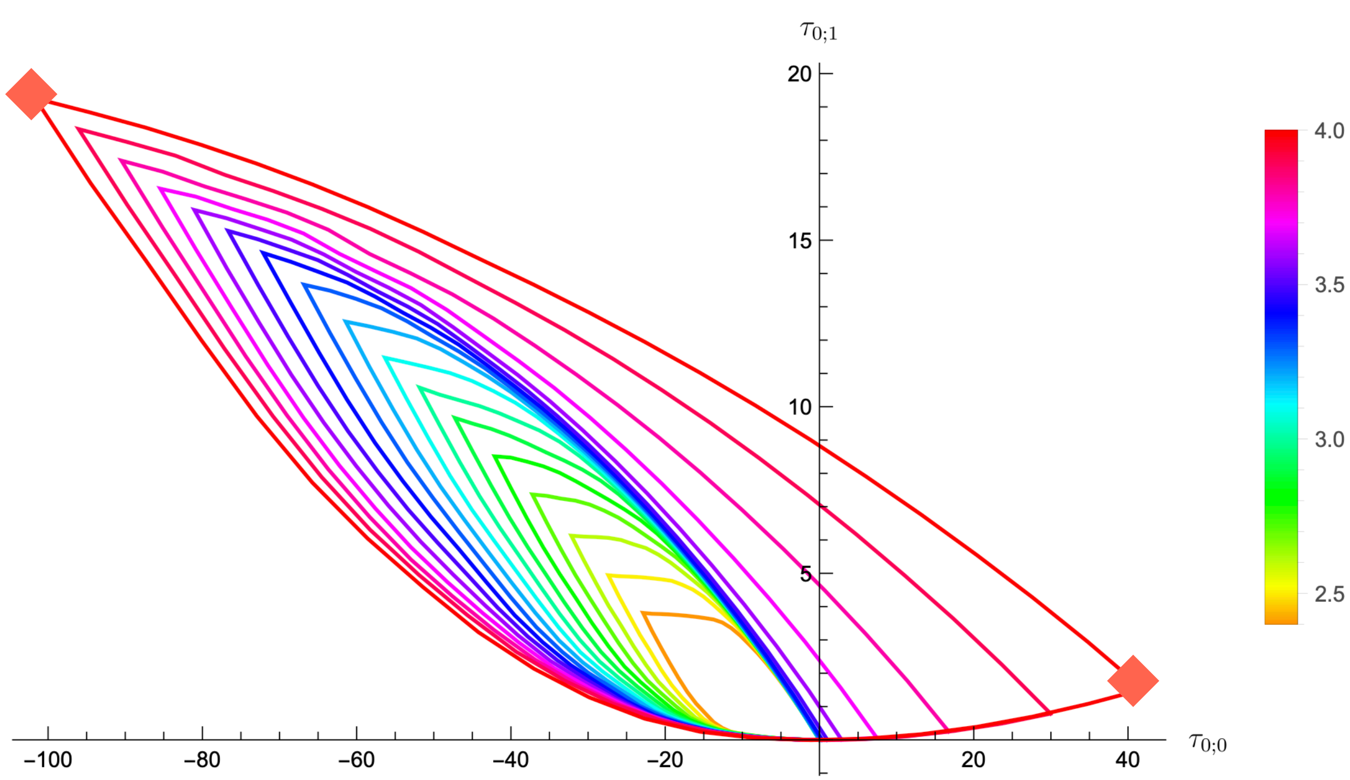

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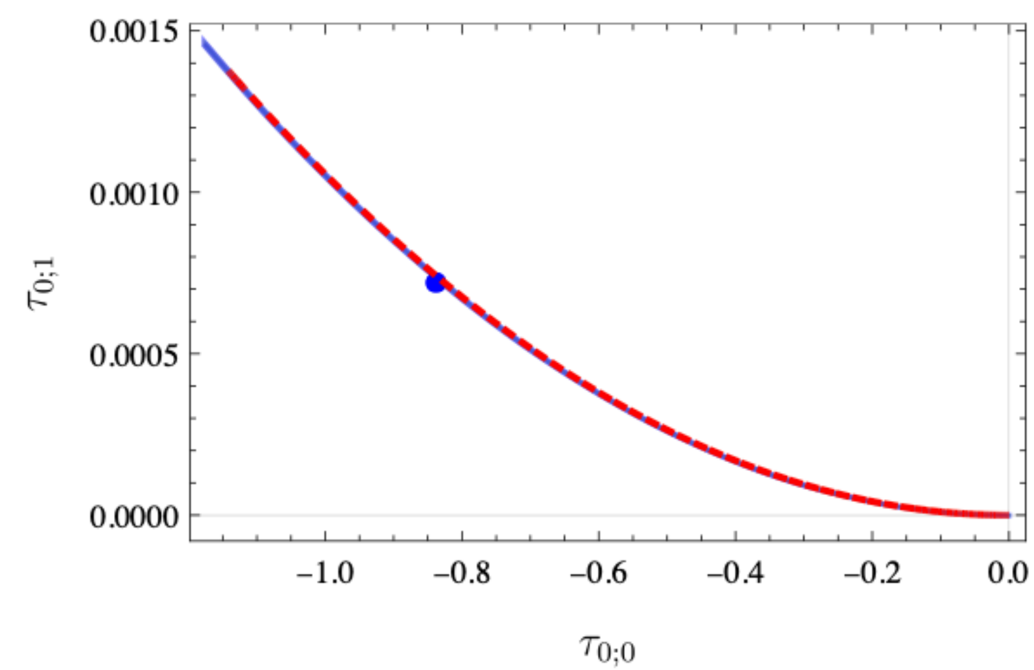

The bound is saturated by the 1-loop ϕ^4!

Strip approximation

The extremal S-matrices obtained in the literature tend to be as elastic as possible. Motivated by this we can consider in our scheme the following input

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We can also set the multi-particle double spectral density to zero (``**strip approximation**")

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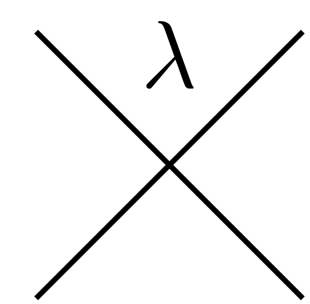
The iterations are driven by the subtraction term



Results

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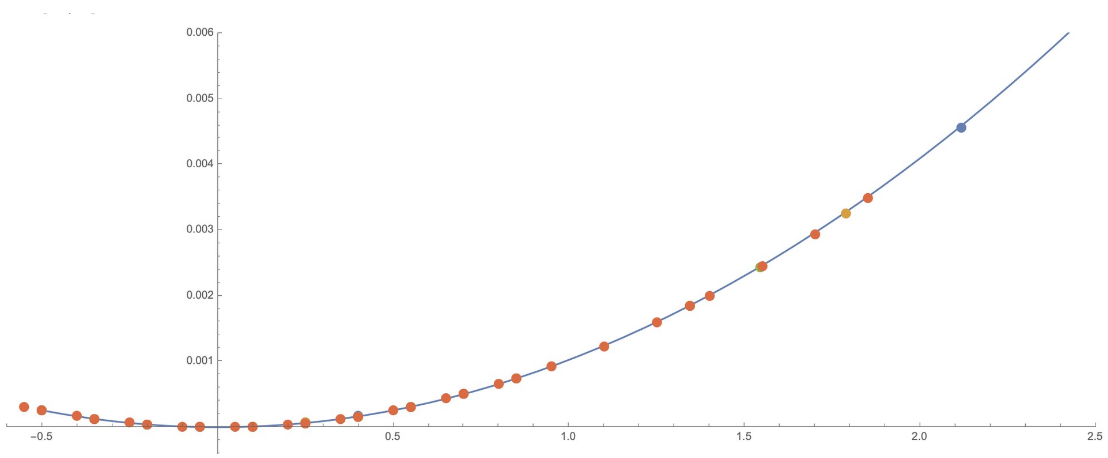
- the result (approximately) saturates the primal S-matrix bound



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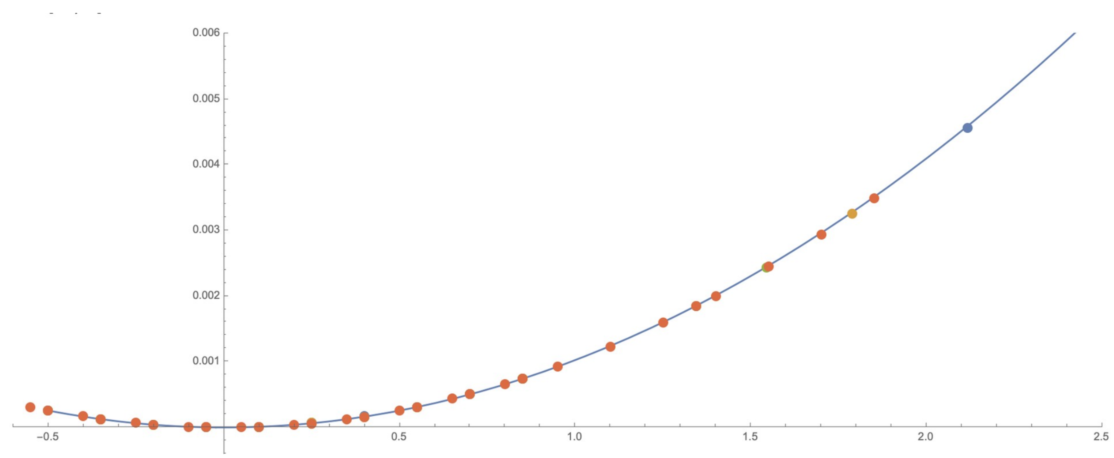
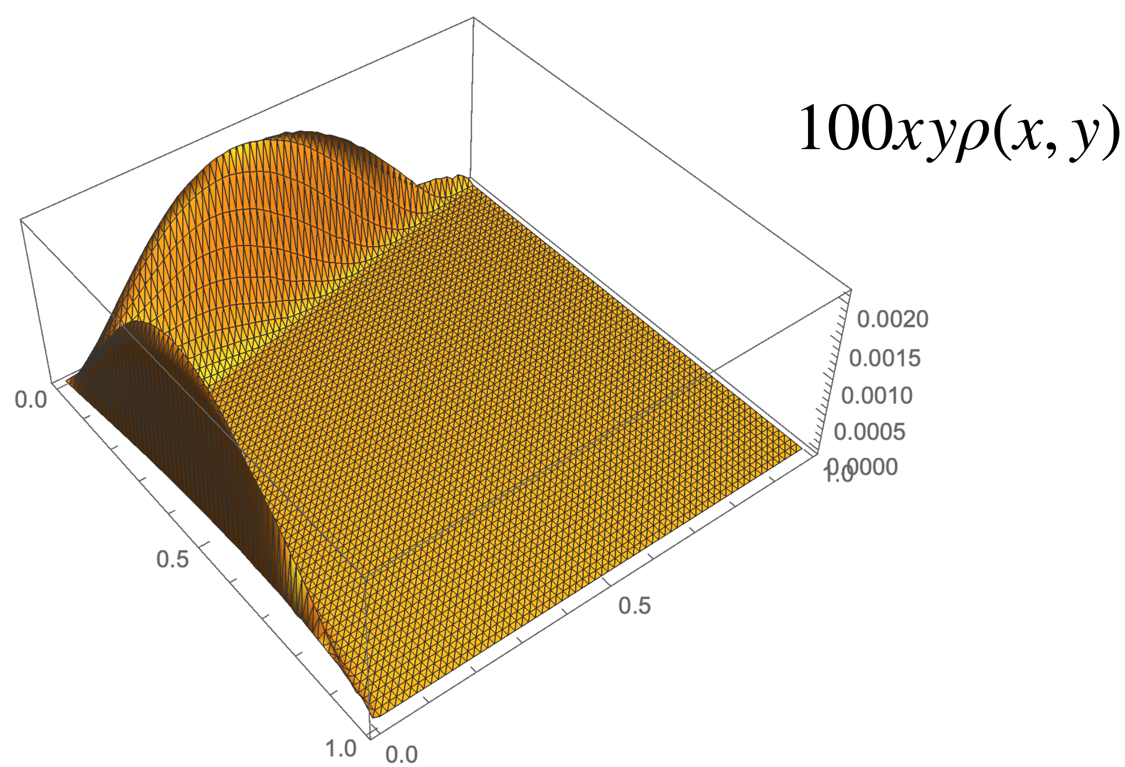
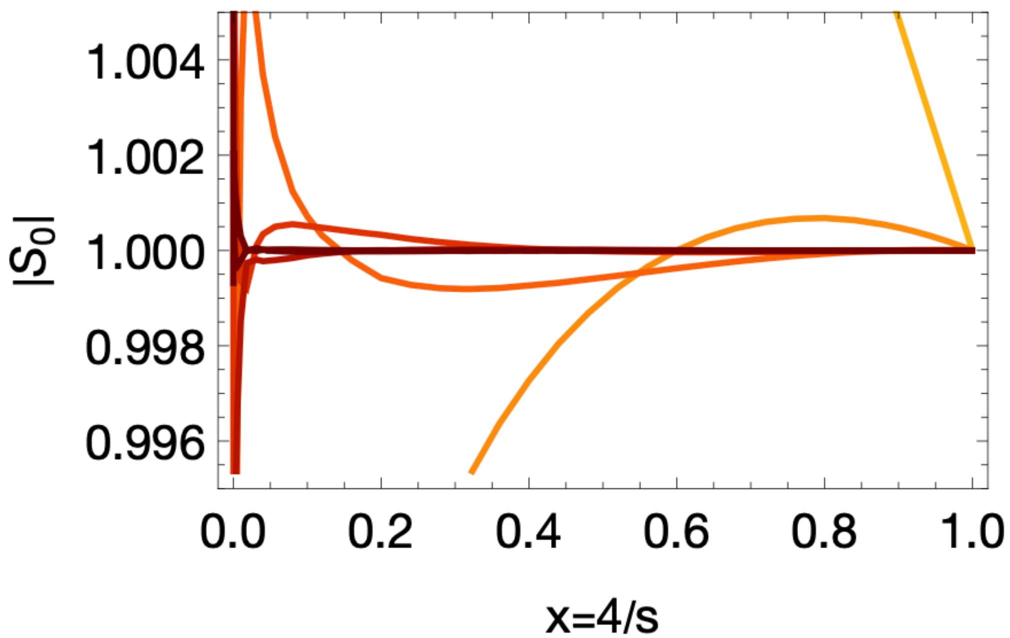


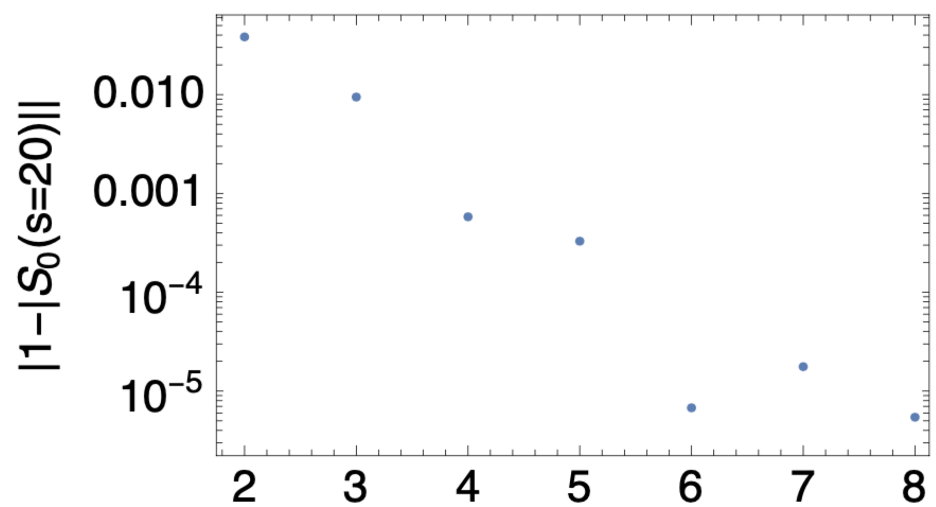
- double spectral density exhibits Landau curves



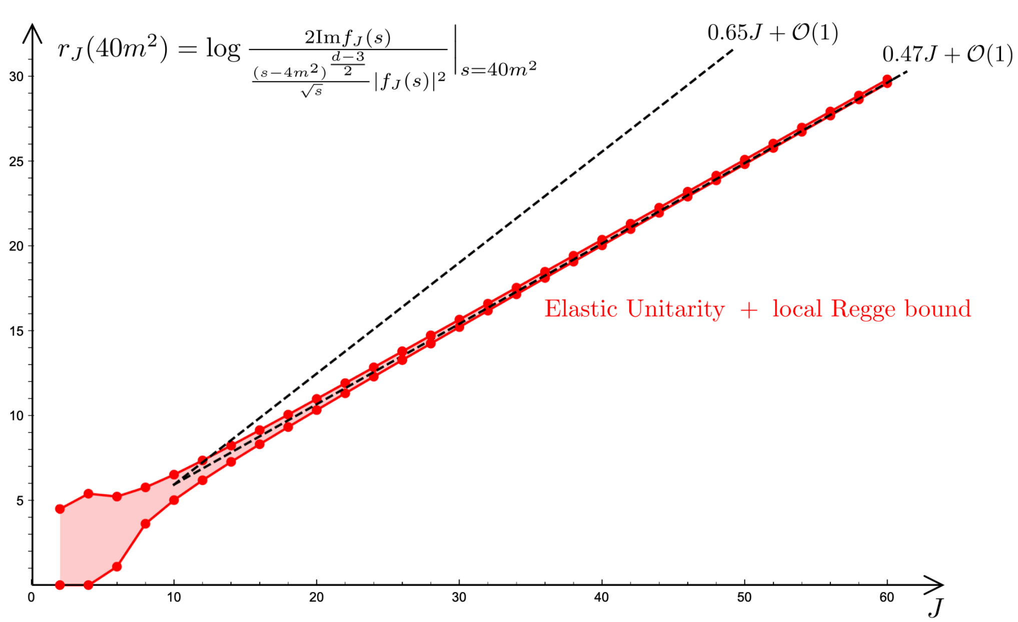
Results

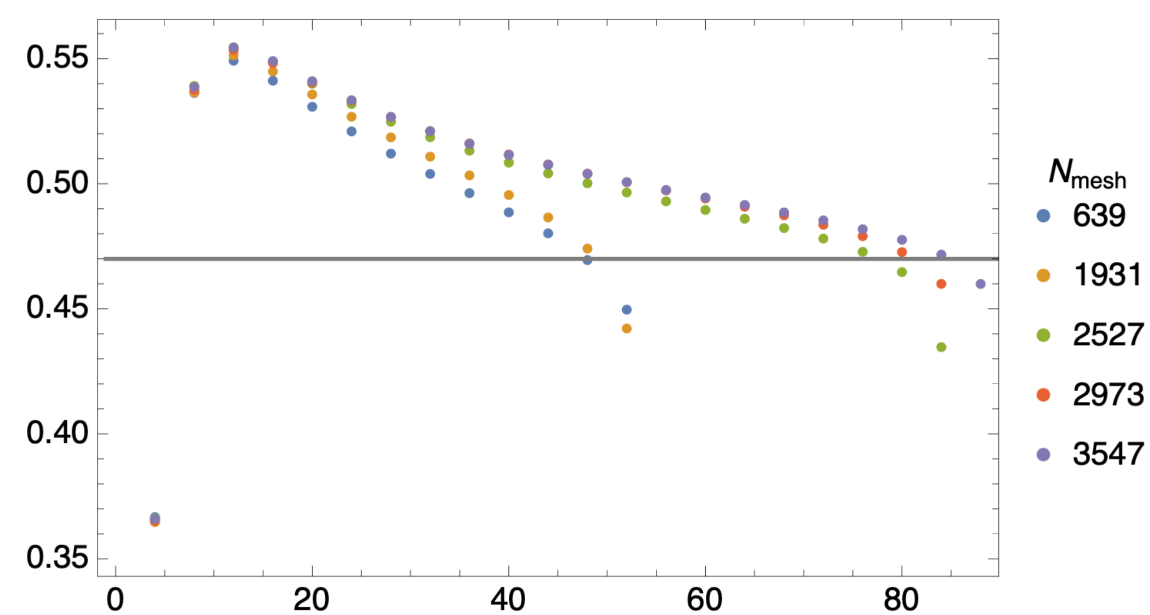
- elastic unitarity is improving quickly with the number of iterations $\lambda = 5\pi$





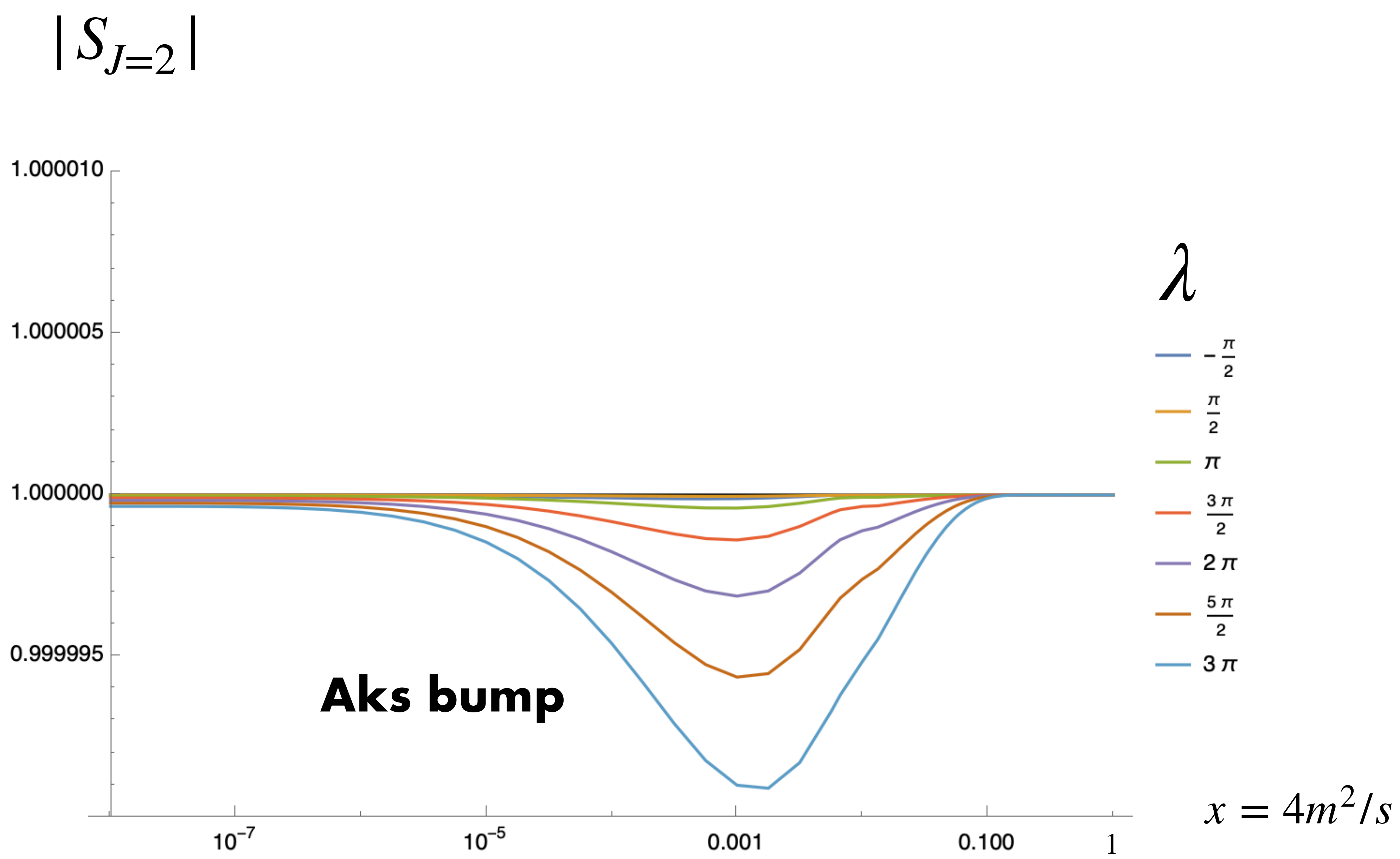
- Large spin partial waves agree with the Dragt-Martin analytic bootstrap





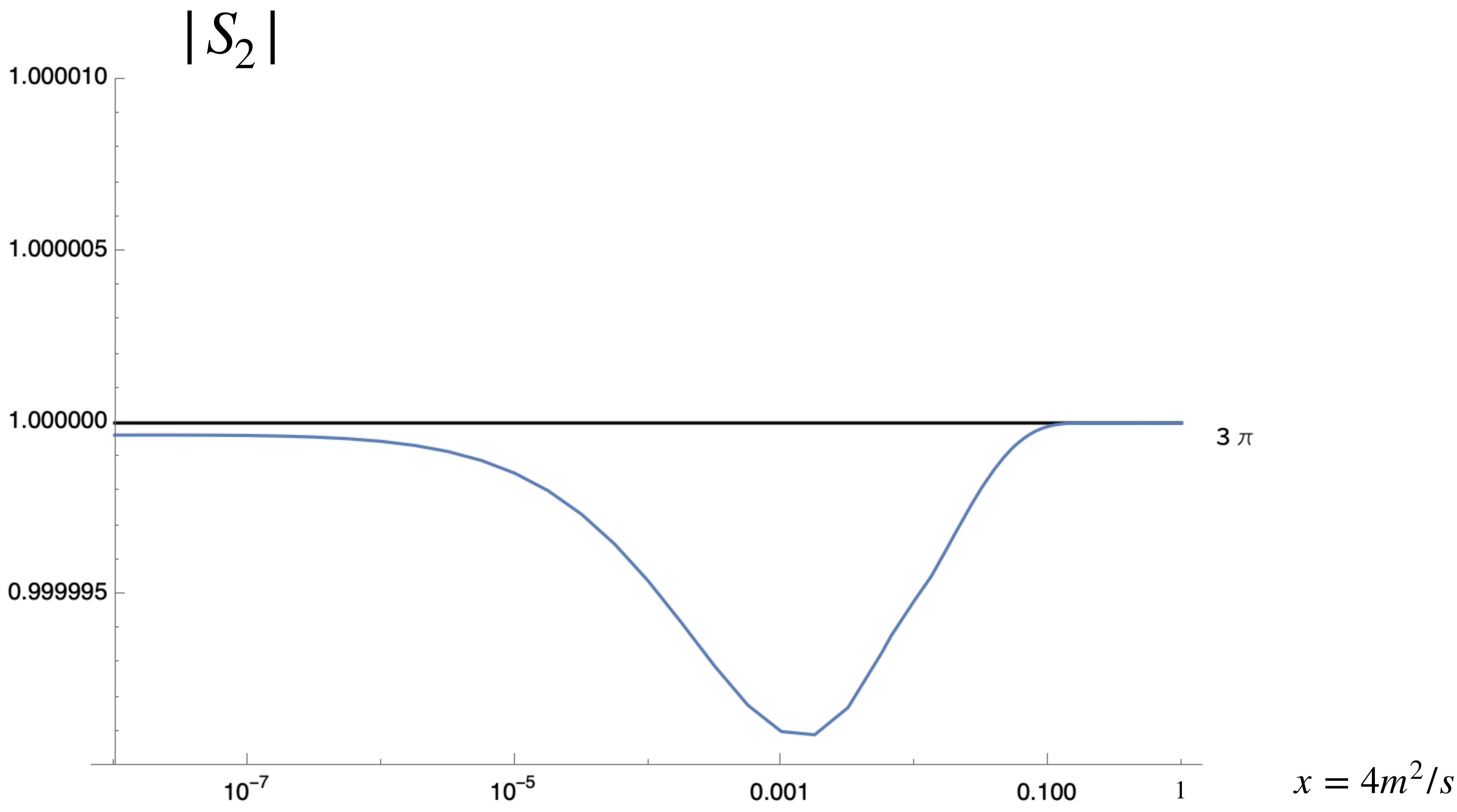
Results

- $J = 2$ partial wave exhibits inelasticity



Results

- partial waves do not go to 1 at infinite energy

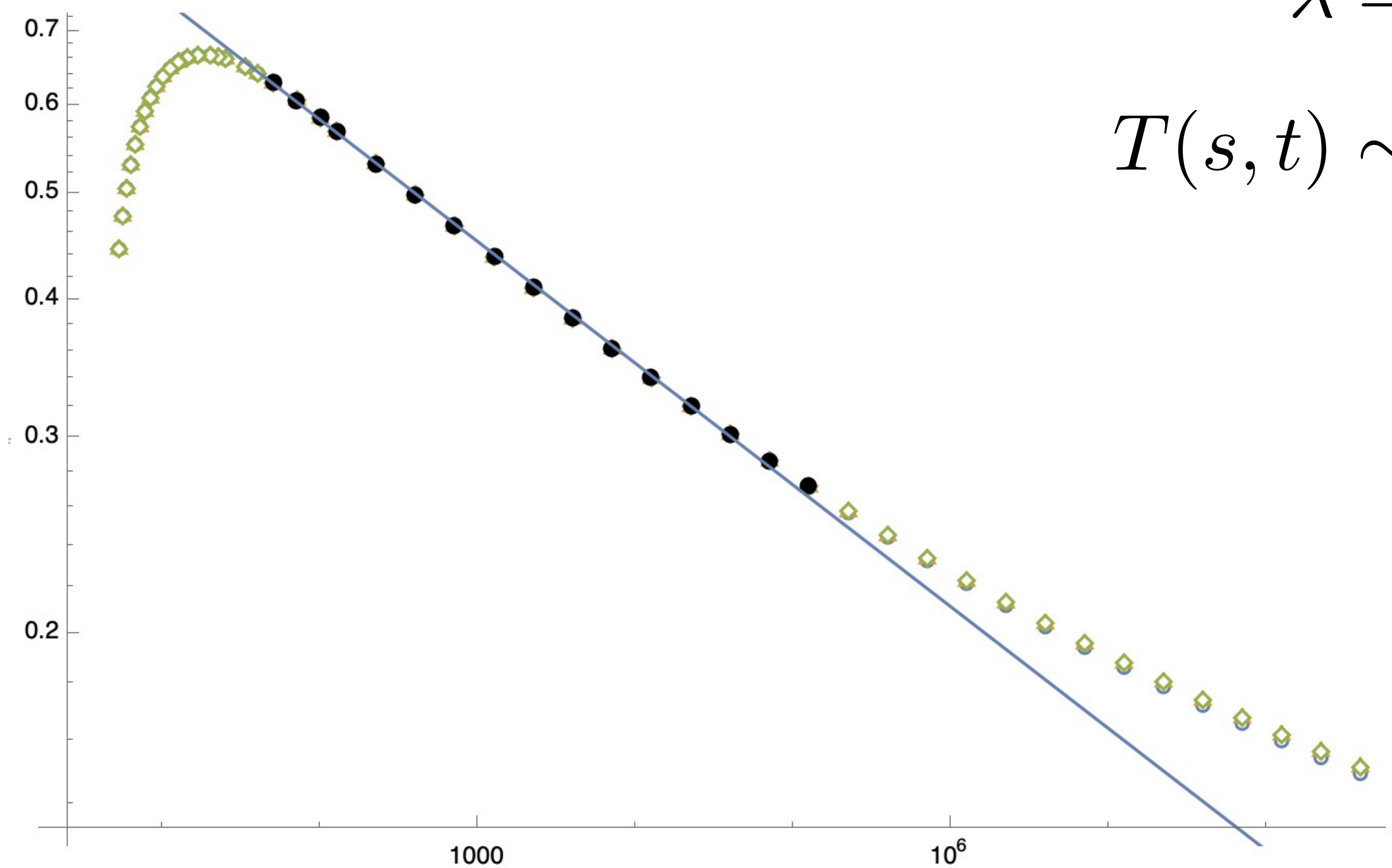


(primal bootstrap is inconclusive - more precision is needed)

Regge limit

$$\lambda = 3\pi$$

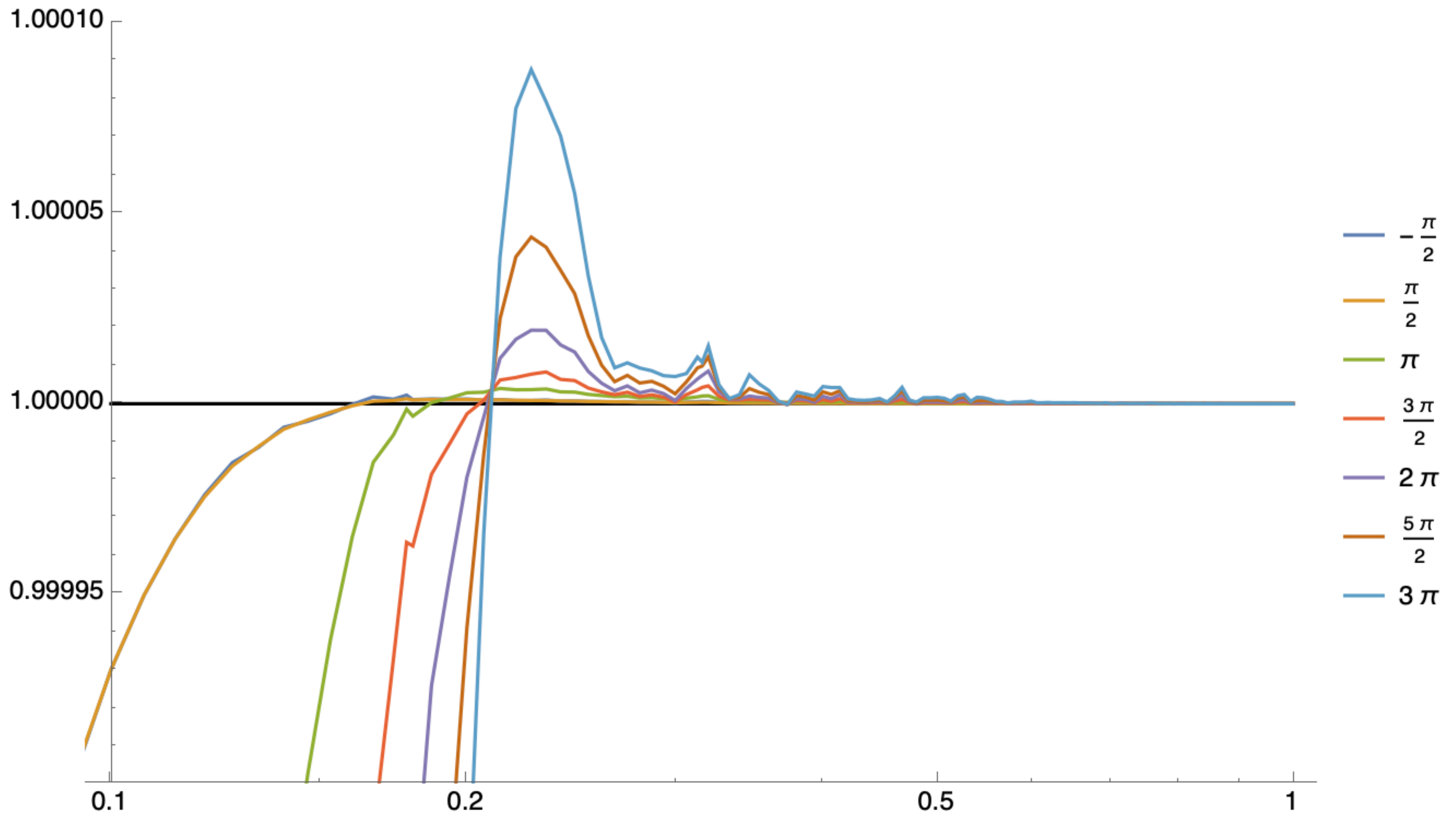
$$T(s,t) \sim s^{-0.1}$$



Results

- there is $\sim 10^{-10}$ violation of inelastic unitarity

$(|S_2| - 1) \times 10^6$

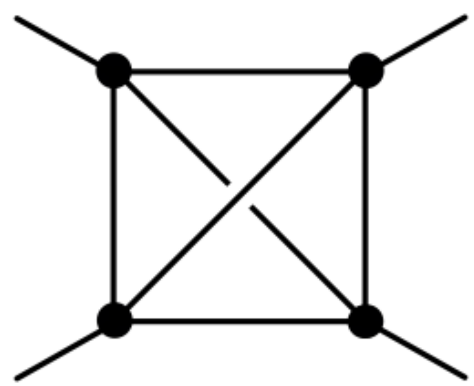


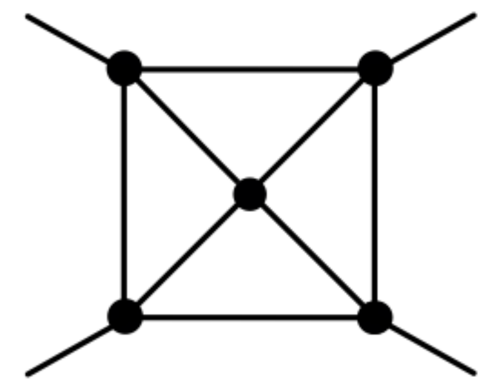
$x = 4m^2/s$

Curiously, it scales like λ^4, so at the moment we believe it is real.

Closing the system of equations

In the physical ϕ^4, $\rho_{\mathrm{MP}}(s,t) \neq 0$





in the full theory these graphs correspond to

$$\rho_{\mathrm{MP}} = \int K_{np} \Big(T_{2\to 2}\Big)^4 \qquad\qquad \rho_{\mathrm{MP}} = \int K_{pl} \Big(T_{2\to 2}\Big)^5$$

Deriving these kernels is an important open problem.

Conclusions

Result: a computational algorithm to explore the space of nonperturbative S-matrices based on dispersive representation of the amplitude and iterations of unitarity and crossing.
A lot to explore!

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thank you for your attention!

Warm-up: ϕ^4 iterations

Can we find the S-matrix by iterating the basic equations:

- Once-subtracted dispersion relations

$$T(s,t) = c_0(t) + \int_{4m^2}^{\infty} \frac{ds'}{\pi} K(s',s,t) T_s(s',t)$$

$$K(s',s,t) = \frac{(s-2m^2+t/2)^2}{(s'-2m^2+t/2)^2} \left(\frac{1}{s'-s} + \frac{1}{s'-u} \right)$$

- Unitarity

$$T_s(s,t) = \frac{(s-4m^2)^{1/2}}{8(4\pi)^2\sqrt{s}} \int_{-1}^{1} dz' dz'' \mathcal{P}(z,z',z'') T(s,z') T^*(s,z'') + \theta(s-s_{\mathrm{MP}}) \sigma_{\mathrm{inel}}(s,t)$$

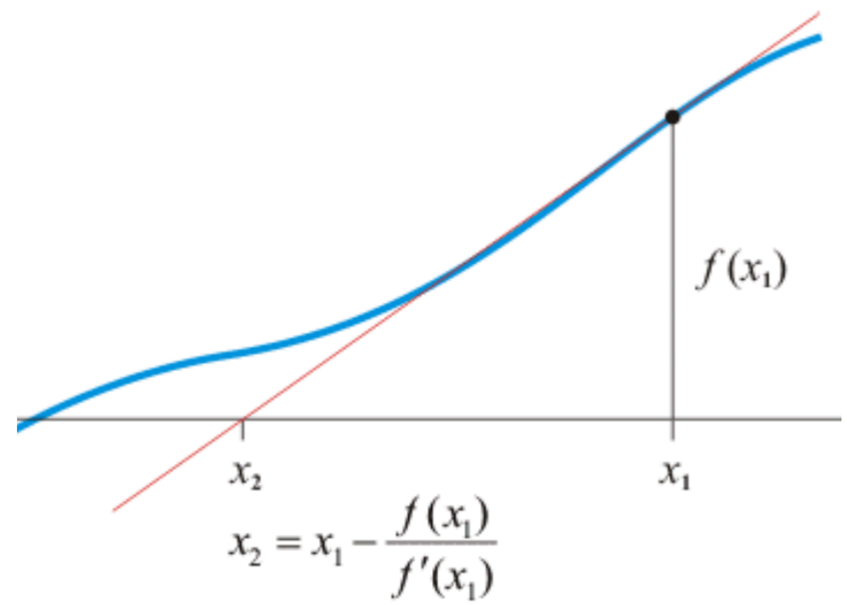
- Dispersive crossing

$$c_0(t) + \int_{4m^2}^{\infty} \frac{ds'}{\pi} K(s',s,t) T_s(s',t) = c_0(s) + \int_{4m^2}^{\infty} \frac{ds'}{\pi} K(s',t,s) T_s(s',s)$$

Newton-Kantorovich method

To improve the convergence we can also try using the Newton-Kantorovich method (really the Newton method)

$$x_n = x_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}$$



Introduce $f = 1 - \Phi$

$$\rho_n = \rho_{n-1} + [\Phi'[\rho_{n-1}]]^{-1}(1 - \Phi)[\rho_{n-1}]$$

x10 speed up

$$\Phi'[\rho_{n-1}](\rho_n - \rho_{n-1}) = (1 - \Phi)[\rho_{n-1}]$$

[Paulos, Penedones, Toledo, van Rees, Vieira '16]

Solution

This problem can be solved exactly. Consider a given inelasticity

$$S(s)S^*(s) = 1 - f_{inel}(s)$$

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The general solution to the this problem takes the form

$$S(s) = S_{\text{CDD}}(s) e^{\int_{4m^2}^{\infty} \frac{ds'}{2\pi i} \log[1-f_{inel}(s')] \frac{\sqrt{s(s-4m^2)}}{\sqrt{s'(s'-4m^2)}} \left(\frac{1}{s'-s} + \frac{1}{s'-(4m^2-s)} \right)}$$

purely elastic

[Paulos, Penedones, Toledo, van Rees, Vieira '16]

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purely elastic

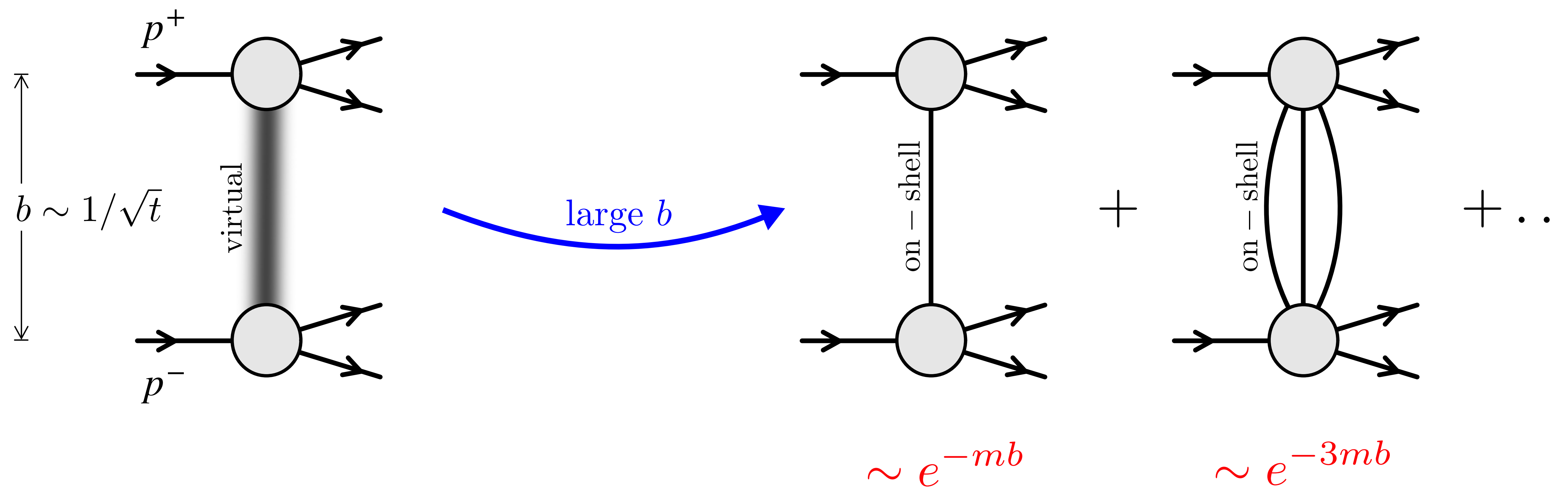
For example

$$S_{\text{CDD}}^{\text{pole}} = \frac{\sqrt{s(s-4m^2)} + \sqrt{m_p^2(4m^2 - m_p^2)}}{\sqrt{s(s-4m^2)} - \sqrt{m_p^2(4m^2 - m_p^2)}}$$

Scattering Implies Production

In $d \geq 3$

$$T_{2\to 2} \longrightarrow T_{2\to n>2}$$

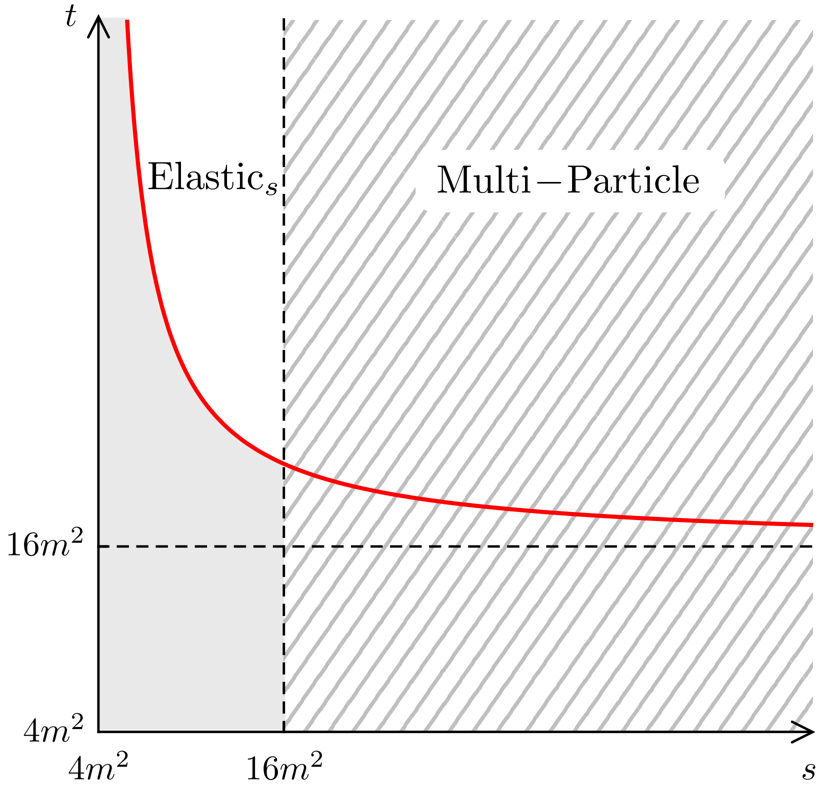


for two-to-two scattering, elastic unitarity:

An opportunity — was never implemented.

Aks Theorem

[Aks '65]



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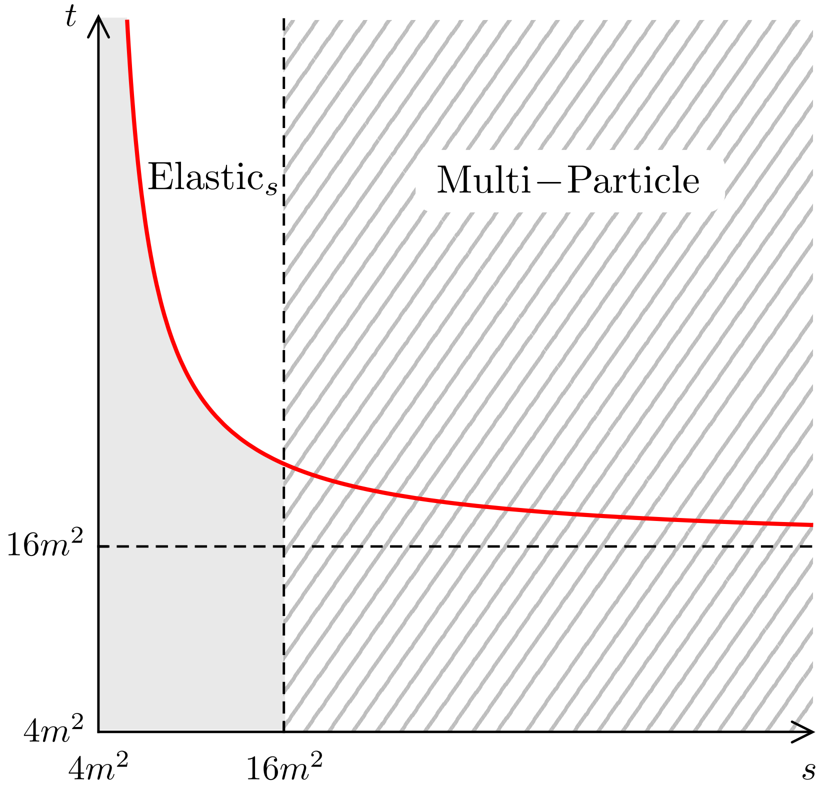
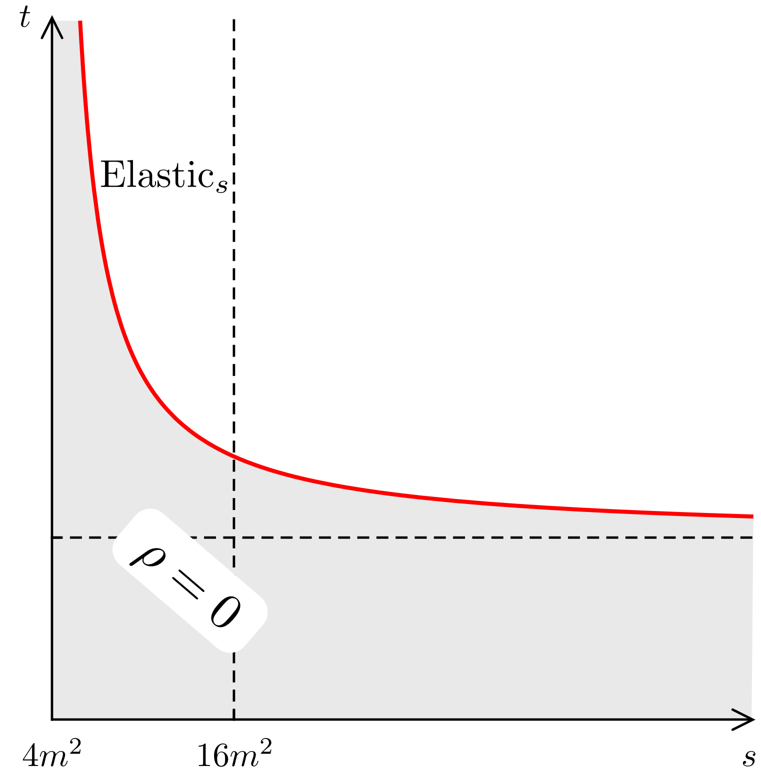


no particle production



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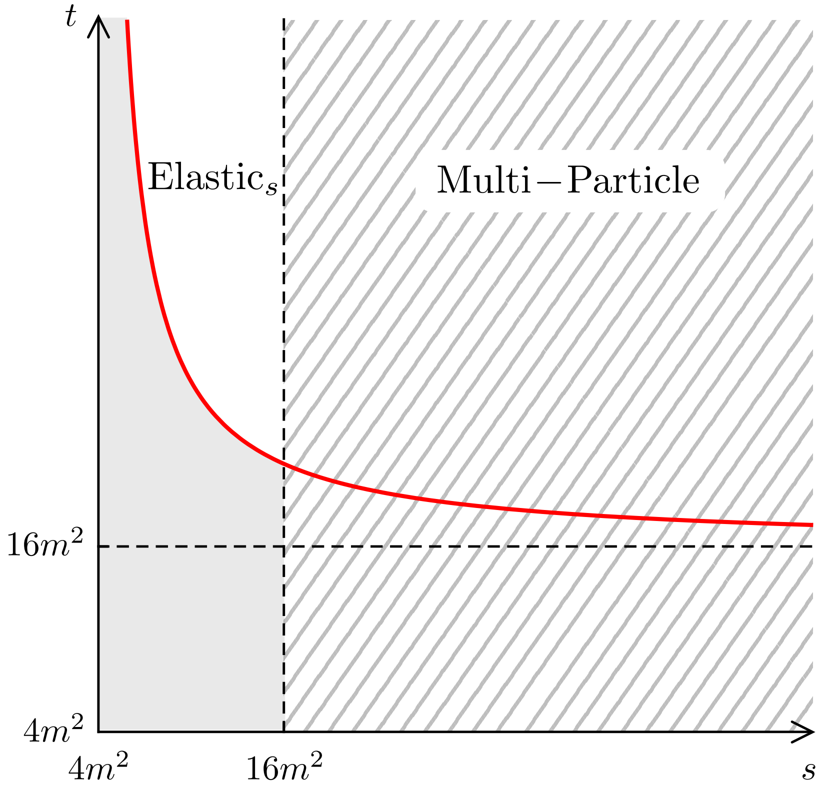

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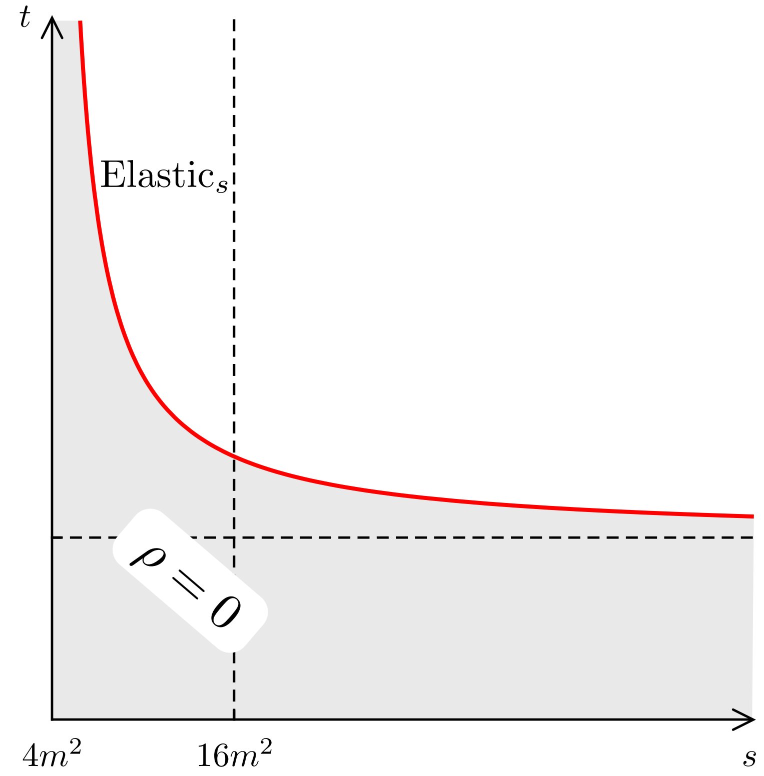

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$$+$$

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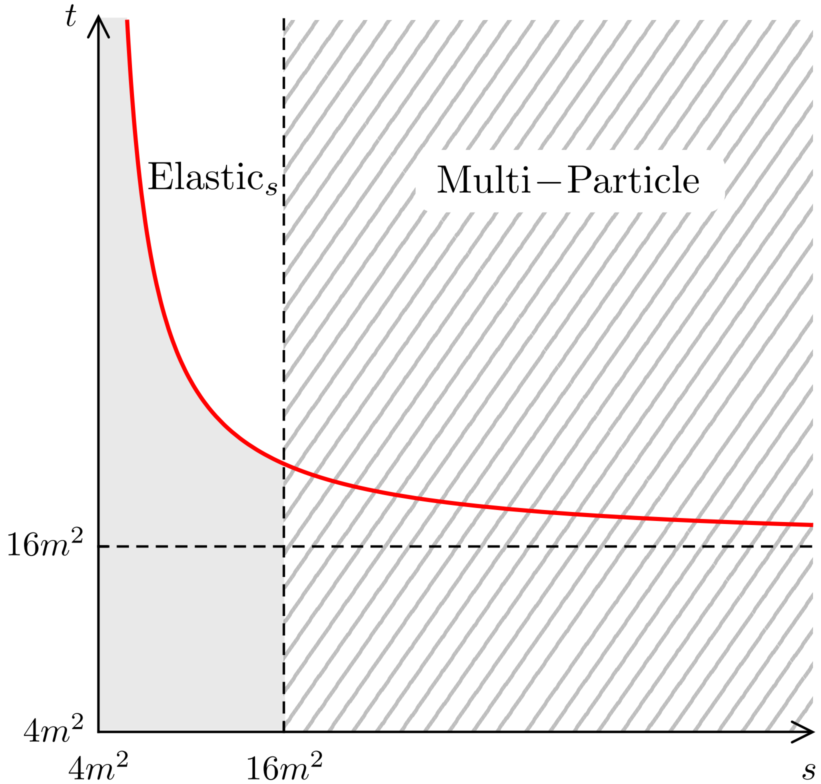

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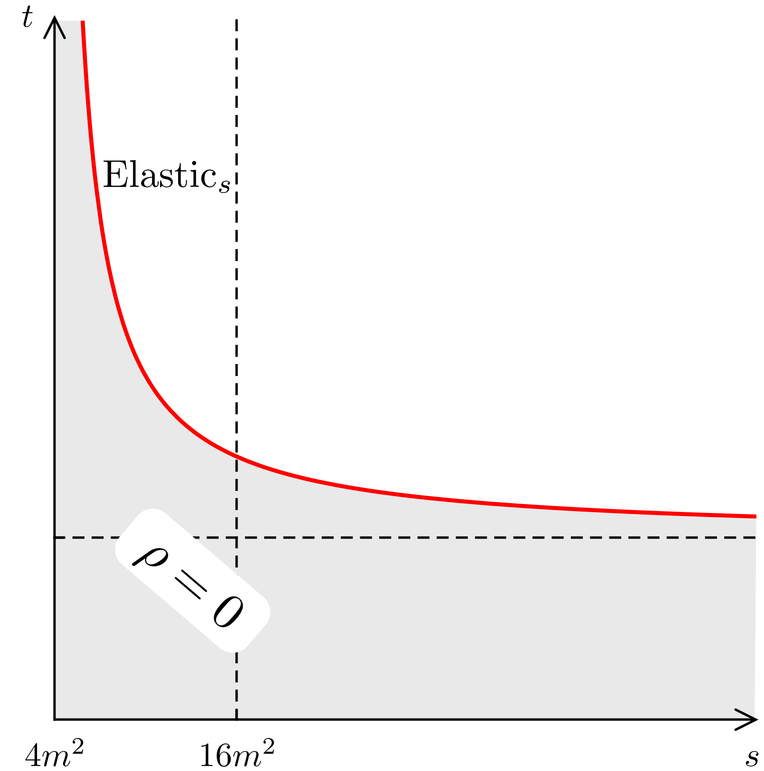

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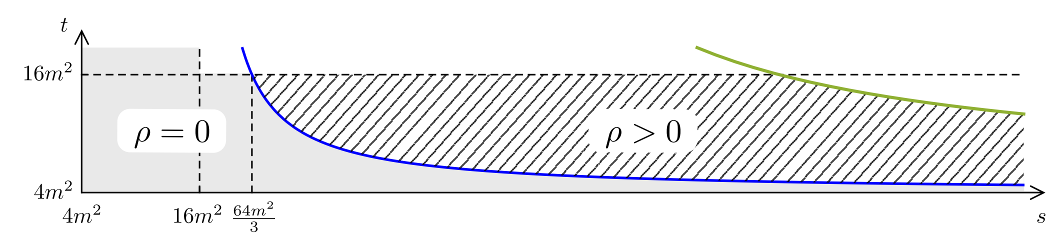

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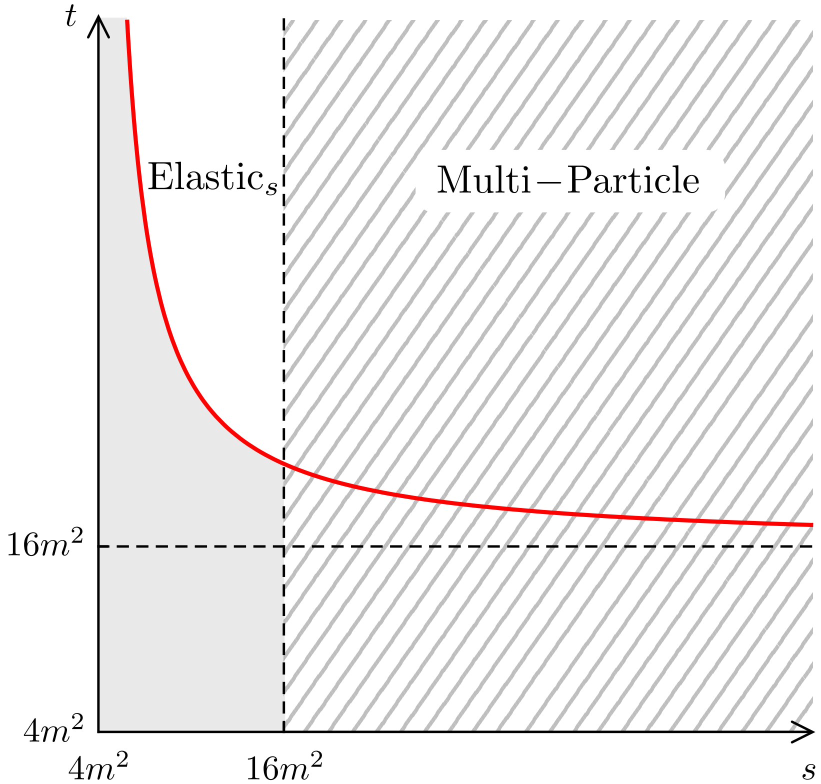

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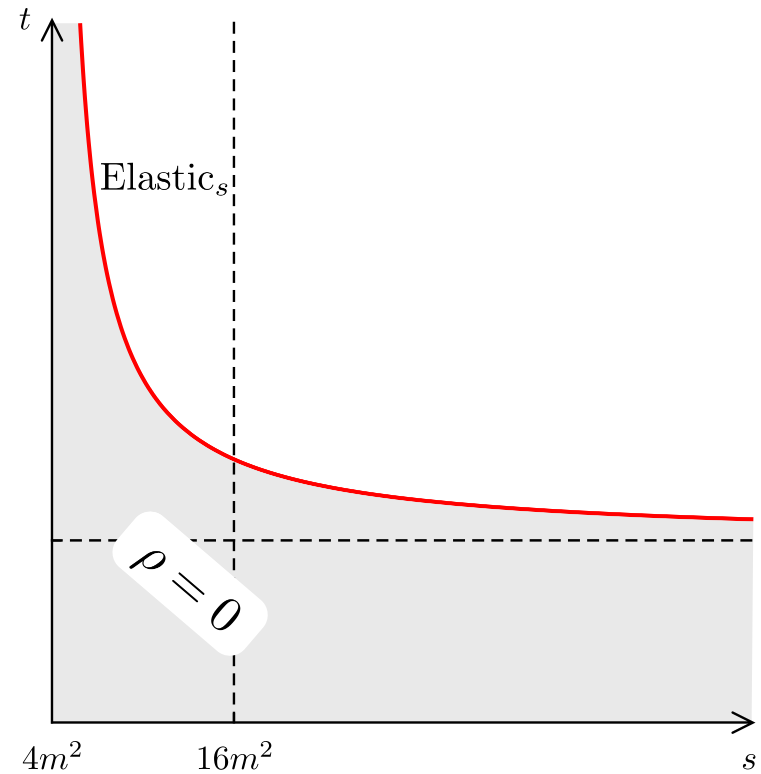

contradiction

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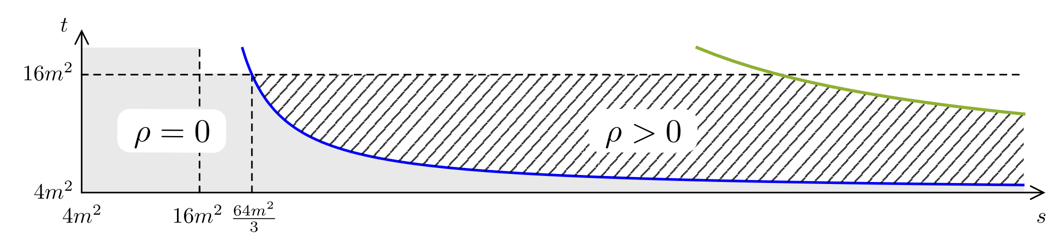

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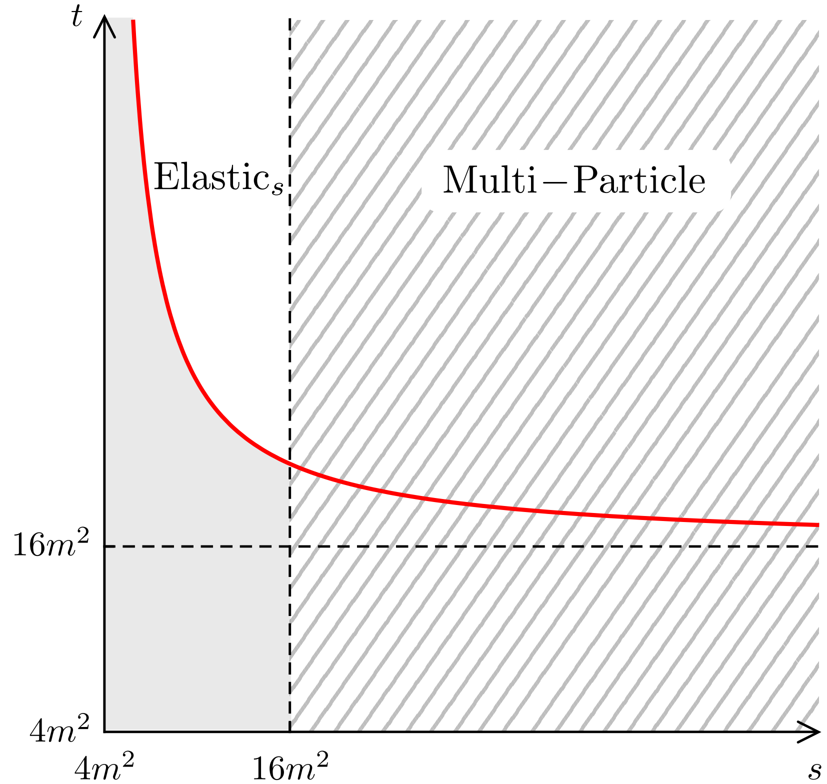

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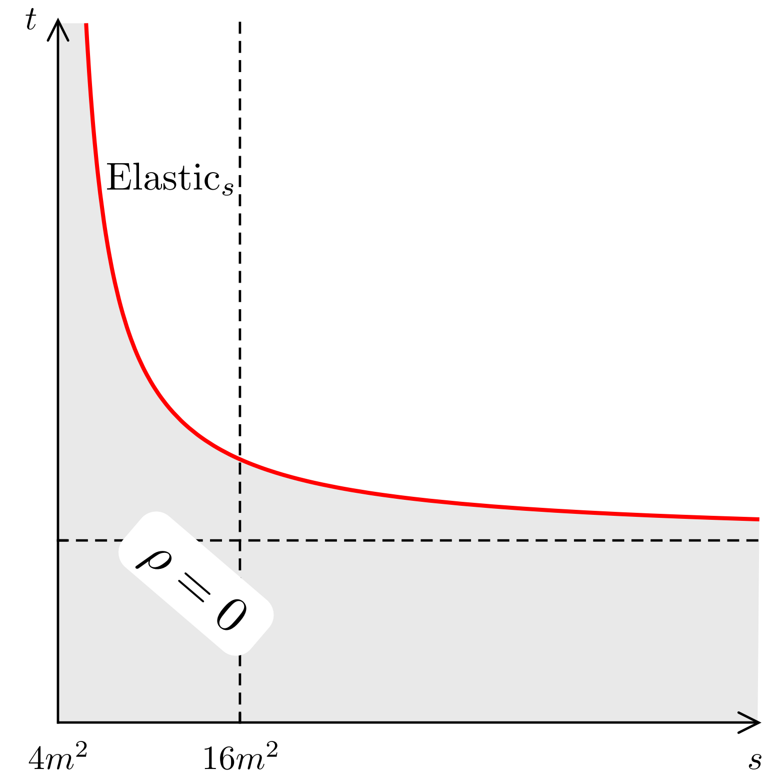

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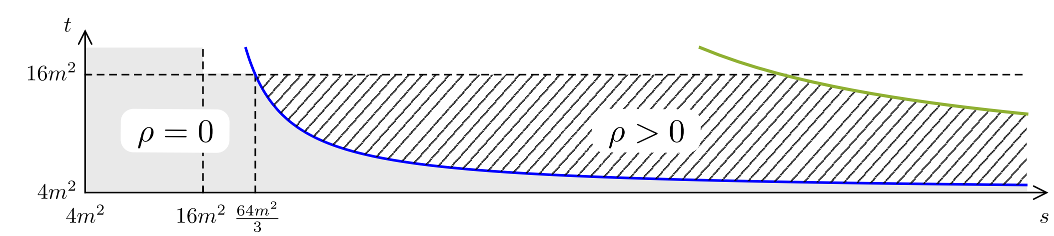

$$\int d\text{LIPS}_4 \times \text{Disc}_t T^{(+)}_{2\to 4}\, \text{Disc}_t T^{(-)}_{2\to 4} \times K^{2\to 4}_{\text{Mandelstam}}$$

$$= \frac{(t-4m^2)^{\frac{d-3}{2}}}{4\pi^2(4\pi)^{d-2}\sqrt{t}} \int_{\bar{z}_1}^{\infty} d\eta' \int_{\bar{z}_1}^{\infty} d\eta''\, \text{Disc}_s T^{(+)}_{2\to 2}(t,\eta')\text{Disc}_s T^{(-)}_{2\to 2}(t,\eta'')\text{Disc}_{\bar{z}} K_d(\bar{z},\eta',\eta'') > 0$$