Compatibility of Landau Singularities

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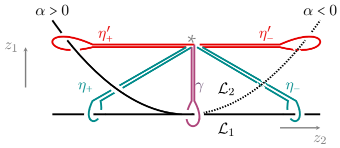
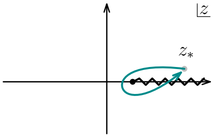


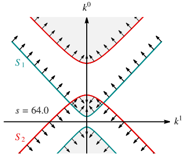
with Andrew McLeod, Matthew Schwartz, Cristian Vergu

Outline

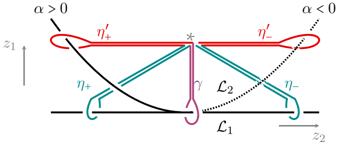
1. Introduction: Singularities of amplitudes



2. Discontinuities of amplitudes



3. Relations for sequential discontinuities



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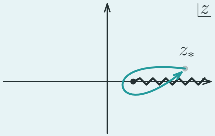

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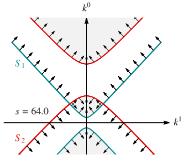

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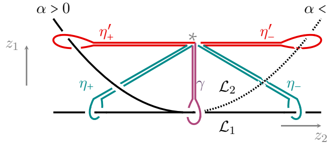


Motivation

How can we exploit the analyticity properties of Feynman integrals?

Introduction

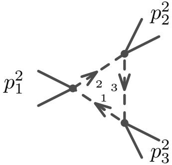
Analyticity properties of amplitudes: program dating back to pre 1960s

Atkinson, Bjorken, Boyling, Bros, Cahill, Coleman, Cutkosky, Eden, Epstein, Fotiadi, Froissart, Glaser, Karplus, Landau, Landshoff, Lascoux, Mandelstam, Nakanishi, Norton, Olive, Pham, Polkinghorne, Screaton, Sommerfield, Stapp, Steinmann, Wichmann, ...

Many modern applications
(*generalized unitarity, perturbative & non-perturbative bootstrap, dispersion relations, on-shell recursion relations, ...*)

Abreu, Arkani-Hamed, Bern, Bourjaily, Britto, Brown, Cachazo, Caron-Huot, Córdova, Correia, Dennen, Dixon, Drummond, Duhr, Dulat, Feng, Foster, Gardi, Golden, Goncharov, Gurdogan, He, He, Henn, Kosower, Liu, McLeod, Miczajka, Mizera, Papathanasiou, Paulos, Pennington, Postnikov, Schwartz, Sever, Smirnov, Smirnov, Spradlin, Telen, Tourkine, Trnka, Vergu, Viera, Volovich, Witten, von Hippel, Yang, Zhang, Zhiboedov, Zoia, ...

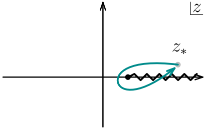
Example of analytic structure

$I =$ 

$$\propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

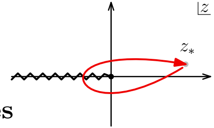
with $z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$

Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x} dx$ has a branch cut for $z > 1$:



$$\mathrm{Disc}_{z=1}\mathrm{Li}_2(z) = 2\pi i \int_1^z \frac{1}{x} dx = 2\pi i \ln(z)$$

Logarithm $\ln(z) = \int_1^z \frac{1}{x} dx$ has a branch cut for $z < 0$:



$$\mathrm{Disc}_{z=0}\ln(z) = 2\pi i$$

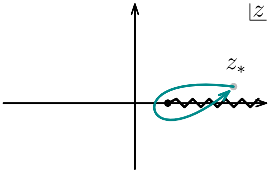
Structure & new branch points in **sequential discontinuities**

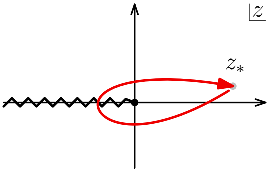
Analytic structure: Symbol

Symbol makes sequential-discontinuity structure manifest

$$\mathrm{Li}_2(z) \to -(1-z) \otimes z$$

Here: Investigate constraints on consecutive symbol entries





Goncharov, Spradlin, Vergu, Volovich 2010

See talks by Dixon, Schwartz, Wilhelm

Why constraints on discontinuities?

Learn about analytic structure from

i) Location and types of **singularities** (logarithmic/square root)
(*constrains symbol entries*)

See talk by McLeod

ii) **Sequential discontinuities** around branch points
(*constrains consecutive symbol entries*)

iii) Additional constraints in **physical regions**
(*fixes constant factors between terms*)

How to find branch points: Landau equations

Loop momenta — Set N=1

$$\mathcal{A}_G^\Gamma(p) \propto \int_0^\infty \frac{\mathrm{d}^E \alpha_e}{\mathrm{GL}(1)} \int_\Gamma \mathrm{d}^{Ld} k \, N \exp\left[\sum \alpha_e (q_e^2 - m_e^2 + i\varepsilon)\right]$$

Saddle-point analysis shows branch points when

$$\alpha_e(q_e^2 - m_e^2) = 0, \qquad \text{and} \qquad \sum_{\text{loop}} \alpha_{e'} q_{e'}^\mu = 0$$

See talks/posters by Córdova, Correia, He, Henn, McLeod, Pokraka, Zhiboedov

Solutions give codimension $\geqslant 1$ constraints on *external* kinematics

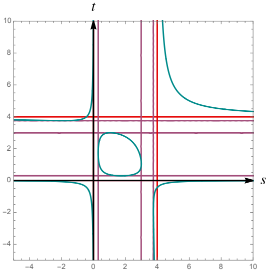
(As opposed to UV/IR singularities, for any kinematics)

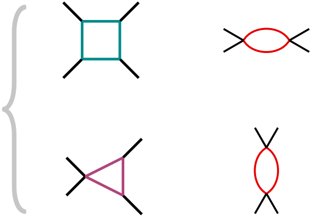
See talk by Mizera

Bjorken, Landau, Nakanishi, Brown, Mühlbauer, Klausen, Mizera, Telen

Landau diagrams

Correspond to any *subdiagrams* of original diagrams, where lines are on shell

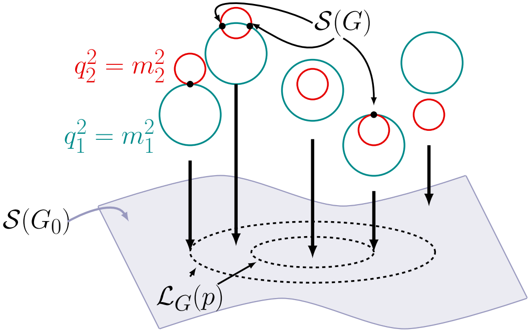




Singularities on **physical sheet** special: Only for solutions with $\alpha_e \geq 0$

Pseudothresholds: Solutions with at least one $\alpha_{e'} \ngeq 0$

Geometric interpretation of Landau equations



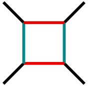
Reinterpretation:

Branch points occur when projection map from internal+external-variable space to external-variable space drops rank

$$\sum_e \alpha_e \mathrm{d}(q_e^2 - m_e^2)|_p = 0$$

Glimpse of results

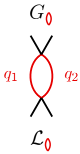
$G_{\square}$

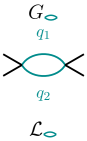
$I_{\square} =$  $= \int \frac{\mathrm{d}^{Ld}k}{\prod(q_e^2 - m_e^2 + i\varepsilon)}$

$\mathrm{Disc}_{\mathcal{L}_{\ominus}} \mathrm{Disc}_{\mathcal{L}_{0}} I_{\square} = 0$

Non-hierarchical singularities

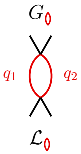
(Steinmann-type constraints)

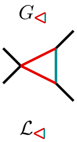




$\mathrm{Disc}_{\mathcal{L}_{\triangleleft}} \mathrm{Disc}_{\mathcal{L}_{0}} I_{\square} = \mathrm{Disc}_{\mathcal{L}_{\triangleleft}} I_{\square}$

Hierarchical singularities

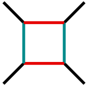




Landshoff, Pham, Stapp

Glimpse of results

$G_{\square}$ ← Diagram

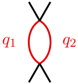
Feynman integral → $I_{\square} =$  $= \int \frac{\mathrm{d}^{Ld}k}{\prod(q_e^2 - m_e^2 + i\varepsilon)}$

$\mathrm{Disc}_{\mathcal{L}_{\ominus}} \mathrm{Disc}_{\mathcal{L}_{0}} I_{\square} = 0$

Non-hierarchical singularities

(Steinmann-type constraints)

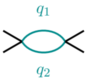
G_{0}



$\mathcal{L}_{0}$

Landau variety →

$G_{\ominus}$

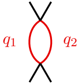


$\mathcal{L}_{\ominus}$

$\mathrm{Disc}_{\mathcal{L}_{\triangleleft}} \mathrm{Disc}_{\mathcal{L}_{0}} I_{\square} = \mathrm{Disc}_{\mathcal{L}_{\triangleleft}} I_{\square}$

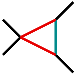
Hierarchical singularities

G_{0}



$\mathcal{L}_{0}$

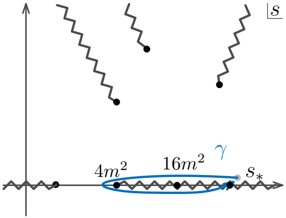
$G_{\triangleleft}$



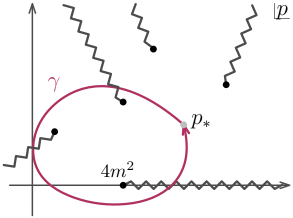
$\mathcal{L}_{\triangleleft}$

Landshoff, Pham, Stapp

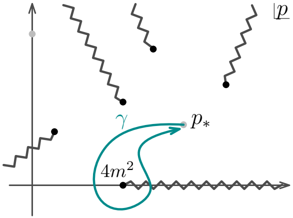
Difference between disc constraints



Total discontinuities



Time-ordered perturbation theory constraints



This talk

See talks/posters by Córdova, Correia, Mazáč, Zhiboedov, ...

Bourjaily, HSH, McLeod, Schwartz, Vergu 2020

Need sufficiently generic masses

Outline

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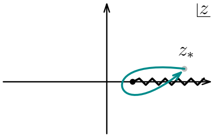
Singularities of amplitudes



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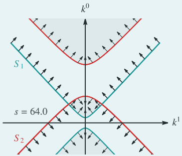
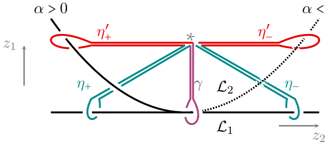
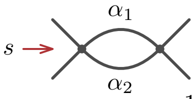


3. Relations for sequential discontinuities



Disc of Amplitudes: Introduction



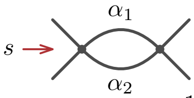
$$I(s) = \lim_{\varepsilon\to 0^+} i\pi \int_0^1 \mathrm{d}\alpha \frac{1}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\varepsilon}$$

$$= \frac{-2\pi}{\sqrt{-[s-(m_1-m_2)^2][s-(m_1+m_2)^2]}} \log\left(\frac{\sqrt{(m_1+m_2)^2-s}-i\sqrt{s-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-s}+i\sqrt{s-(m_1-m_2)^2}}\right) \text{ in } d=2$$

How do we compute discontinuities around branch points?

$$\mathrm{Disc}_{s=(m_1\pm m_2)^2} I(s)$$

Disc of Amplitudes: Introduction



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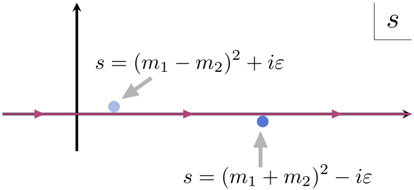
How do we compute discontinuities around branch points?

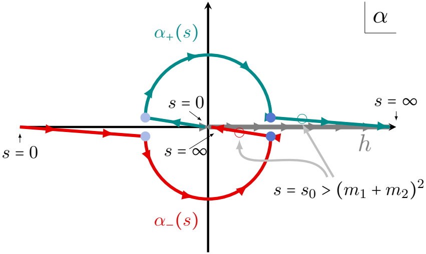
$$\mathrm{Disc}_{s=(m_1\pm m_2)^2} I(s)$$

1. Brute force, using full expression
2. Use integral representation
3. Use Picard-Lefschetz theorem

Integral representation for discontinuities

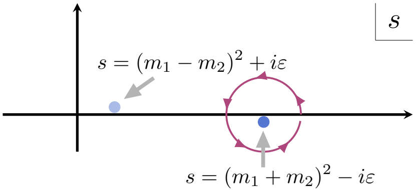
$$I(s) = \lim_{\varepsilon\to 0^+} i\pi \int_0^1 \mathrm{d}\alpha \frac{1}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\varepsilon}$$

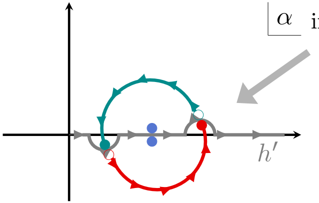




Integral representation for discontinuities

$$I(s) = \lim_{\varepsilon \to 0^+} i\pi \int_0^1 \mathrm{d}\alpha \frac{1}{s\alpha(1-\alpha) - m_1^2\alpha - m_2^2(1-\alpha) + i\varepsilon}$$



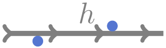


We can calculate the discontinuity using the new integration cycle h'

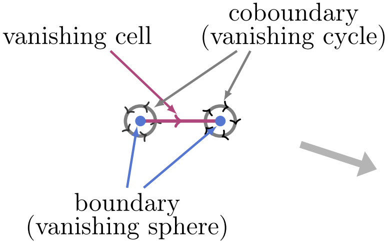
Introduction to Picard-Lefschetz

$\mathrm{Var}[h] =$

$-$



$=$



Shortcut to computing

$\mathrm{Disc}_{s=(m_1+m_2)^2} I(s)$

Picard-Lefschetz Theorem

Discontinuity around the Landau branch point corresponding to the on-shell surfaces $S_1, S_2, \ldots S_m$ given by

New integration contour: Coboundary of boundary of vanishing cell

$$\mathrm{Var}\colon H_n(X \setminus \bigcup_{i=1}^{m} S_i) \to H_n(U \setminus \bigcup_{i=1}^{m} S_i)$$

$$\mathrm{Var}\, h = (-1)^{\frac{(n+1)(n+2)}{2}} \langle e, h\rangle (\delta_1 \ldots \delta_m \partial_m \ldots \partial_1 e),$$

Challenging to compute

Integer-valued intersection number between contour and vanishing cell

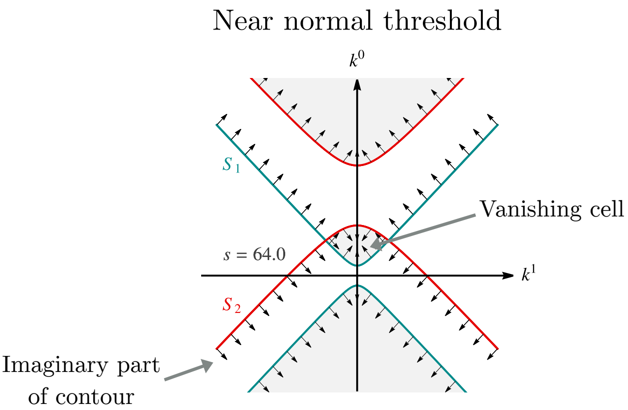
Pham

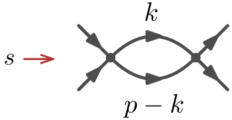
Recent applications:

Bönisch, Duhr, Fischbach, Klemm, Nega 2021

Bogner, Schweitzer, Weinzierl 2017-18

Back to Momentum space

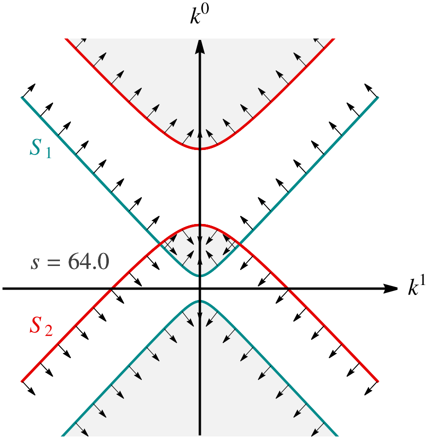




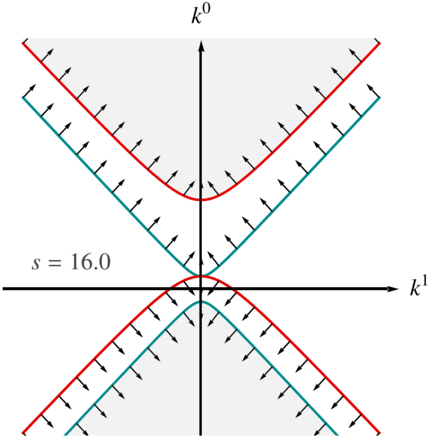
$$S_1 : (k^0)^2 - (k^1)^2 - m_1^2 = 0$$

$$S_2 : (p^0 - k^0)^2 - (k^1)^2 - m_2^2 = 0$$

Integration contour & vanishing cell

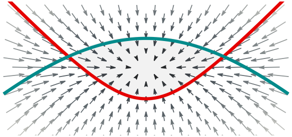


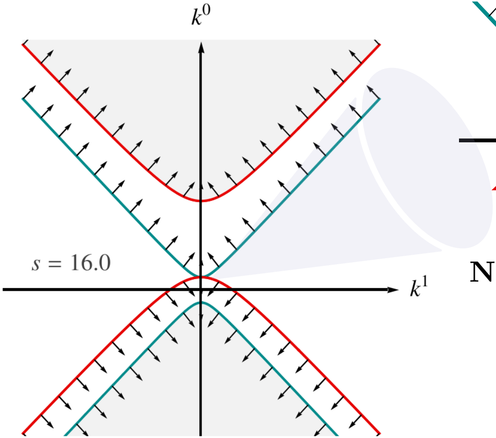
Integration contour & vanishing cell

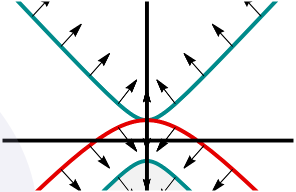


Integration contour & vanishing cell

Pinching of contour

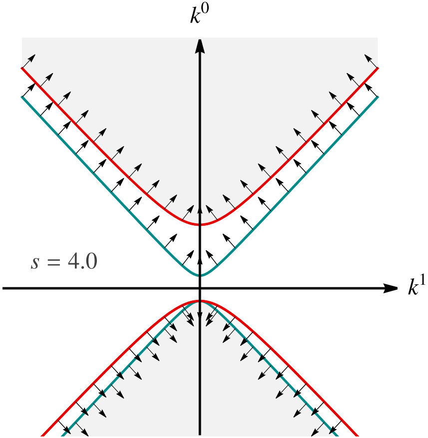




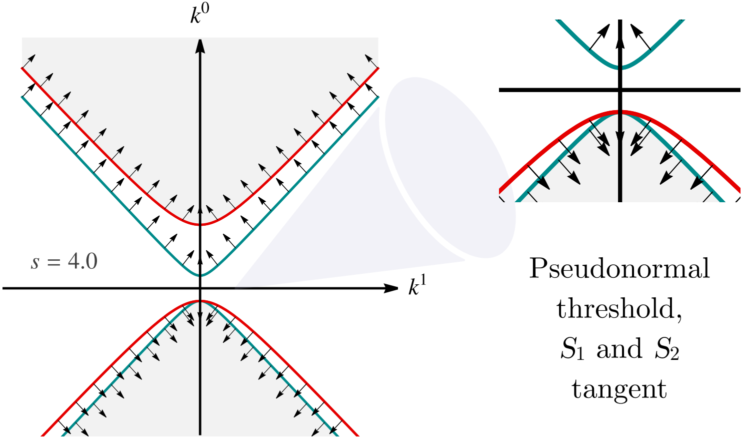


Normal threshold,
S_1 and S_2
tangent

Integration contour & vanishing cell



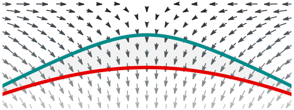
Integration contour & vanishing cell

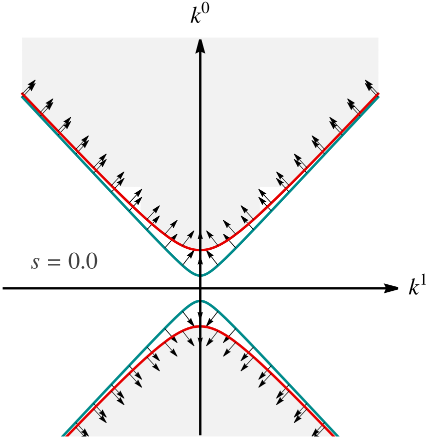


Pseudonormal threshold, S_1 and S_2 tangent

Integration contour & vanishing cell

No pinching of contour





Near pseudonormal threshold:

Vanishing cell exists, but no pinching of contour

CUTKOSKY'S FORMULA

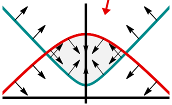
$$I_G(p) = \int_h \omega, \qquad \omega = \frac{\mathrm{d}^{Ld} k_c}{\prod (q_e^2 - m_e^2)}$$

Physical region important for getting delta functions

Applying Picard-Lefschetz gives Cutkosky's formula for encircling **individual** branch points

$$\mathrm{Disc}_{\mathcal{L}} I_G(p) = \int_{\delta_1 \cdots \delta_l \partial_l \cdots \partial_1 e} \omega = (-2\pi i)^l \int_{U_{\mathcal{L}}} \mathrm{d}^{Ld} k_c \frac{\delta(q_1^2 - m_1^2) \cdots \delta(q_l^2 - m_l^2)}{\prod_{e=l+1} (q_e^2 - m_e^2)},$$

One cut at a time: Not same as imaginary part



Cutkosky, Leray, Pham, Bloch, Kreimer, Mühlbauer

Recap

We have learned:

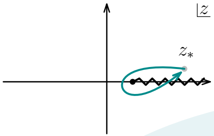
Picard-Lefschetz theorem proves Cutkosky's formula

*We have the **tools** to:*

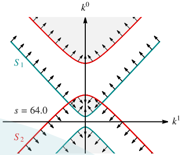
Generalize to sequential discontinuities

Outline

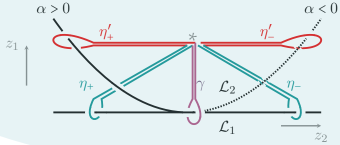
1. Introduction: Singularities of amplitudes



2. Discontinuities of amplitudes

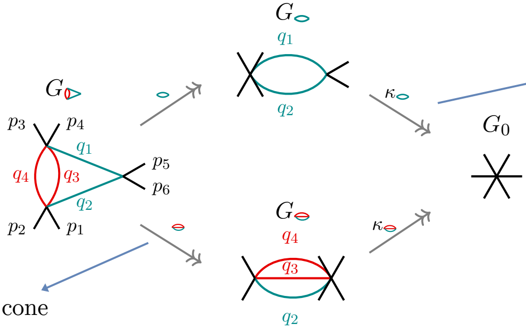


3. Relations for sequential discontinuities



Singularities of discontinuities

Since discontinuities are given by localization on on-shell surfaces, their singularities correspond to larger Landau diagrams

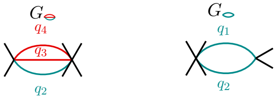


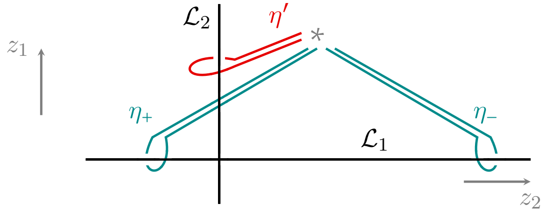
Ice cream cone diagram has bubble, sunrise and ice cream cone singularities

Discontinuity has a singularity at ice cream cone singularity

Non-hierarchical case

When Landau curves correspond to non-hierarchical diagrams, e.g.





- Curves intersect transversally
- Paths commute: $\eta_\pm \circ \eta' = \eta' \circ \eta_\pm$
- Sequential discontinuities in physical regions given by cutting all particles

Disc of Disc

Applying Picard-Lefschetz twice gives

$$\mathrm{Disc}_{\mathcal{L}_1}\mathrm{Disc}_{\mathcal{L}_2} I_G(p) \propto \langle e, h' \rangle \int_{h''} \omega$$

$$\propto \int_{U_{\mathcal{L}_1 \cup \mathcal{L}_2}} \mathrm{d}^{Ld} k_c \frac{\delta(q_1^2 - m_1^2) \cdots \delta(q_{l+k}^2 - m_{l+k}^2)}{\prod_{e=l+k+1}(q_e^2 - m_e^2)}$$

Fotiadi, Froissart, Lascoux, Pham

Disc of Disc

Applying Picard-Lefschetz twice gives

Vanishing cell — New integration contour after encircling $\mathcal{L}_2$

$$\mathrm{Disc}_{\mathcal{L}_1}\mathrm{Disc}_{\mathcal{L}_2} I_G(p) \propto \langle e, h' \rangle \int_{h''} \omega$$

$$\propto \int_{U_{\mathcal{L}_1 \cup \mathcal{L}_2}} \mathrm{d}^{Ld} k_c \frac{\delta(q_1^2 - m_1^2) \cdots \delta(q_{l+k}^2 - m_{l+k}^2)}{\prod_{e=l+k+1} (q_e^2 - m_e^2)}$$

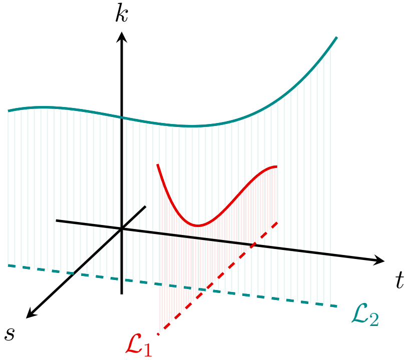
Corollary: If the Landau equations for $\mathcal{L}_1$ and $\mathcal{L}_2$ give incompatible conditions on the loop momenta,

$$\mathrm{Disc}_{\mathcal{L}_1}\mathrm{Disc}_{\mathcal{L}_2} I_G(p) = 0$$

Technical points:
Assume no on-shell loop with all $\alpha_e = 0$
Assume isolated Landau curves
Assume UV&IR finiteness

Fotiadi, Froissart, Lascoux, Pham

Disc of Disc

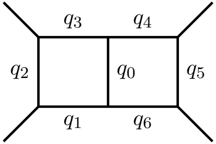


Whenever varieties do not intersect in *loop-momentum space*, the sequential discontinuity is zero

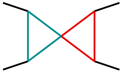
For a generic-mass configuration: **Landau diagram** must *factorize* to have a nonzero sequential discontinuity

Allowed: Double triangle singularity

$\mathcal{I}_{\square\square} =$

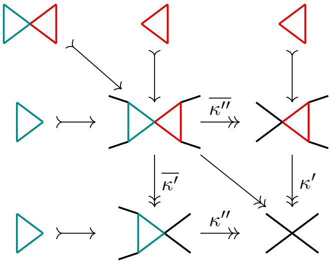


$G_{\mathcal{L}} =$

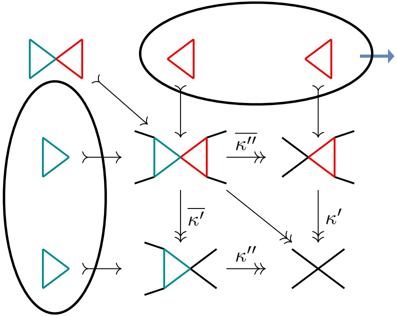


$$\mathrm{Disc}_{\triangleleft}\,\mathrm{Disc}_{\triangleright} I_{\square\square} \propto \langle e_{123}, h_{\triangleright}\rangle \langle e_{456}, h_{\triangleleft}\rangle \int \frac{\mathrm{d}^d k_{\triangleright}\,\mathrm{d}^d k_{\triangleleft}}{q_0^2 - m_0^2} \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6$$

Allowed Pham diagram



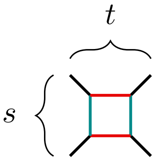
Allowed Pham diagram



Kernels of contractions must be the same

NOT ALLOWED: STEINMANN RELATIONS

No sequential discontinuities in partially overlapping channels in the physical region



Steinmann

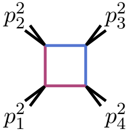
Here: $\mathcal{A}$ cannot have a term of the form $\log(s-4m^2)\log(t-4m^2)$

Important for bootstrapping amplitudes

Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdoğan,
von Hippel, McLeod, Liu, Papathanasiou, Wilhelm
See talks by Dixon, Wilhelm

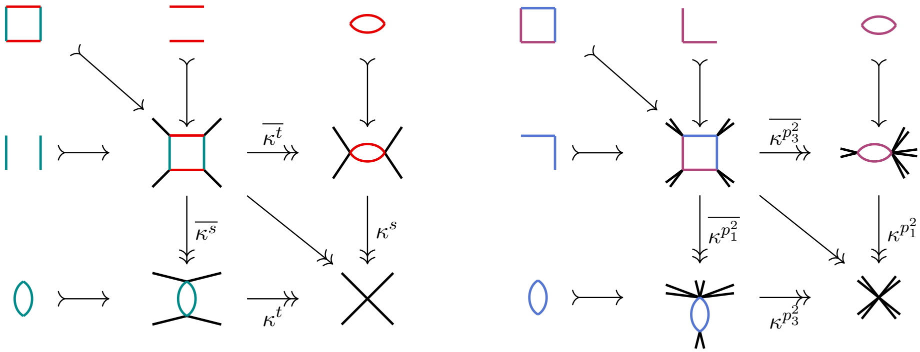
NOT ALLOWED: GENERALIZED RELATIONS

Same proof shows: no sequential discontinuities in channels with incompatible kernels in the physical region



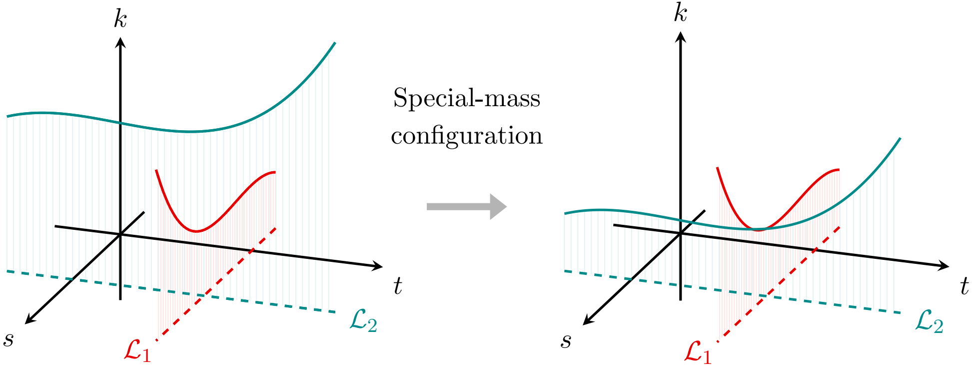
Here: $\mathcal{A}$ cannot have a term of the form $\log(p_1^2 - 4m^2)\log(p_3^2 - 4m^2)$

Not allowed: Generalized relations



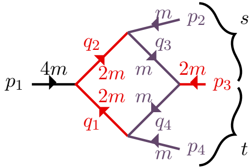
Steinmann "violation" in special kinematics

What happens if the masses are not generic?



STEINMANN "VIOLATION" IN SPECIAL KINEMATICS

For special mass configurations obtained by forcing Landau equations to be compatible, amplitude can have sequential monodromies:

$G_{\square} =$ 

Here: Bubble, triangle and box branch points collide in physical kinematics

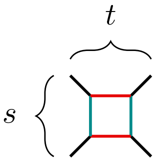
Includes massless case

Steinmann "violation" in special kinematics

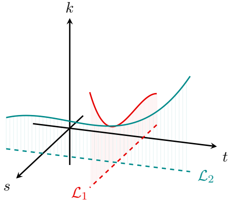
In $(2,2)$ signature when $M^2 = 2m^2$:

Compatible discontinuities in $s = 4m^2$ and $t = 4m^2$

Symbol **factorizes**:



$$\mathcal{S}(I_{\text{box}}) \supset \frac{y_{13} + \sqrt{y_{13}^2 - 1}}{y_{13} - \sqrt{y_{13}^2 - 1}} \otimes \frac{y_{24} + \sqrt{y_{24}^2 - 1}}{y_{24} - \sqrt{y_{24}^2 - 1}} + (y_{13} \leftrightarrow y_{24}),$$

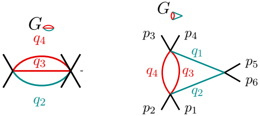


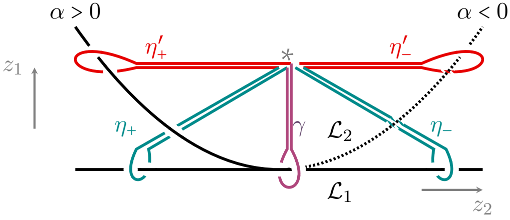
$$\text{with} \quad y_{ij} = \frac{p_{ij}^2 - m_i^2 - m_j^2}{2 m_i m_j}$$

Factorization characteristic of allowed sequential discontinuities

Hierarchical case

When Landau curves correspond to hierarchical diagrams, e.g.





- Curves intersect tangentially
- Paths satisfy:

$$\eta'_+ \circ \eta_+ = \eta'_- \circ \eta_- = \eta_+ \circ \eta'_- = \eta_- \circ \eta'_+$$

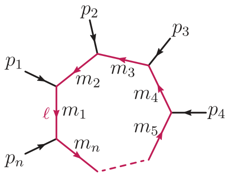
- Sequential discontinuities in physical regions: $\mathrm{Disc}_{\mathcal{L}_2}\mathrm{Disc}_{\mathcal{L}_1} I_G(p) = \mathrm{Disc}_{\mathcal{L}_2} I_G(p)$

Symbol of All mass n-gon in n-dimensions

$$\mathcal{S}\big(I_n^{\text{one loop}}\big) \propto \frac{1}{\sqrt{\det y}} \sum \omega^{\varnothing}_{\{i_1,i_2\}} \otimes \omega^{\{i_1,i_2\}}_{\{i_1,i_2,i_3,i_4\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}\,, \qquad \text{(even } n\text{)}$$

$$\mathcal{S}\big(I_n^{\text{one loop}}\big) \propto \frac{1}{\sqrt{\det y}} \sum \omega^{\{i_1\}}_{\{i_1,i_2,i_3\}} \otimes \omega^{\{i_1,i_2,i_3\}}_{\{i_1,i_2,i_3,i_4,i_5\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}\,, \qquad \text{(odd } n\text{)}$$

$$\omega^I_J = d\log \frac{-D^{I\cup\{i\}}_{I\cup\{j\}} + i\sqrt{D_I D_J}}{-D^{I\cup\{i\}}_{I\cup\{j\}} - i\sqrt{D_I D_J}}\,, \qquad \text{where } J = I\cup\{i,j\}.$$



D^J_I is the determinant of the minor of $y_{ij} = \frac{p_{ij}^2 - m_i^2 - m_j^2}{2m_i m_j}$, with rows in I and columns in J.

Bourjaily, Gardi, McLeod, Vergu 2019

All mass n-gon, "tail" terms and cuts

In Lorentzian signature,

$$\mathcal{S}\left(I_n^{\text{one loop}}\right) \propto \sum_{\omega_0,\cdots\omega_\nu} \sum_{\sigma=0}^{\nu} \frac{\left|S^{n-2\sigma}\right|}{2^{n+1-\sigma}} \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n$$

$$\text{with} \quad S^{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Cuts of n-gon:

$$\text{cut}_J\, I_n^{\text{one loop}}(y) = \frac{(2\pi i)^{|J|}}{\sqrt{\det y}} \sqrt{\det y'}\, I_{n-|J|}^{\text{one loop}}(y') \qquad \text{with} \quad y' = \frac{D_{J\cup\{j\}}^{J\cup\{i\}}}{D_J}$$

See Abreu, Britto, Duhr, Gardi 2017

Conclusions

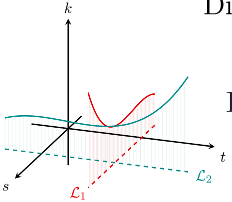
Explored two types of constraints on **sequential discontinuities** of massive amplitudes:

I. Non-hierarchical (*Steinmann-type*)

$$\text{Disc}_{\mathcal{L}_{\ominus}}\text{Disc}_{\mathcal{L}_{0}} I_{\square} = 0$$

II. Hierarchical

$$\text{Disc}_{\mathcal{L}_{\triangleleft}}\text{Disc}_{\mathcal{L}_{0}} I_{\square} = \text{Disc}_{\mathcal{L}_{\triangleleft}} I_{\square}$$



In **special-mass configurations**, sequential discontinuities may become allowed

Thanks!