The duals of Feynman Integrals

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(based on 2104.06898 and 2112.00055 with Simon Caron-Huot)

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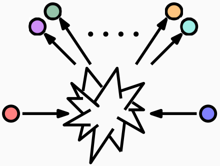
Outline

1. Background and motivation
    - What is integral reduction?
        - Integration-by-parts
        - Generalized unitarity
2. Intersection theory, relative twisted cohomology and dual forms
3. Workflow and key steps
    - (dual) DEQs
    - How to compute intersection numbers (c-map)
    - Where are we in the development phase? Roadblocks?
4. Conclusions and future directions

Scattering Amplitudes and Feynman/loop integrals

Scattering Amplitudes | Feynman/loop Integrals



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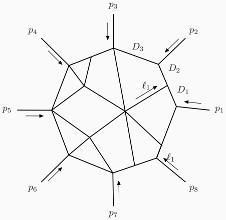


Feynman integrals (FI):

$$\mathcal{I}[\mathcal{N},\{D_j\}] = \left(\prod_{i=1}^{L}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d\ell_i}{\pi^{d/2}}\right)\frac{\mathcal{N}}{\prod_{j\in J} D_j}$$

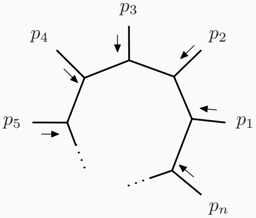
$\mathcal{N}$ ⟶ theory dependent (messy) numerator

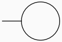
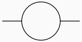
$$D_j = \left(\sum_{a\in A}\ell_a + \sum_{b\in B} p_b\right)^2 + m_j^2$$

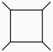
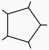
⟶ theory independent propagators

Integral reduction

$$\mathcal{A}_n^{\text{1-loop}}(d_{\text{int}} \in \mathbb{N}, \varepsilon) =$$



$$\mathcal{A}_n^{\text{1-loop}}(d_{\text{int}} - 2\varepsilon) = c_{\text{tad}}(\varepsilon)$$  $$+ c_{\text{bub}}(\varepsilon)$$  $$+ c_{\text{tri}}(\varepsilon)$$ 

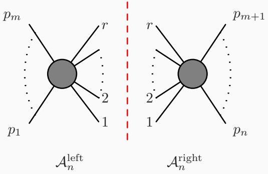
$$+ c_{\text{box}}(\varepsilon)$$  $$+ c_{\text{pent}}(\varepsilon)$$ 

How do we find the coefficients $c_\bullet$?

Generalized unitarity (GU)

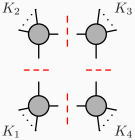
Fix $c_\bullet$'s by matching disc of FI and basis decomposition

Cuts (propagator residues) $\implies$ factorization:



$c_\bullet$ are built out of factorized amplitudes

$c_{\text{box}}(\varepsilon = 0) =$



$= \prod (\text{trees})$ [Britto, Cachazo, Feng 2004]

Generalized unitarity $\implies$ Analytic structure of FI determined by cuts!

Generalized unitarity (GU)

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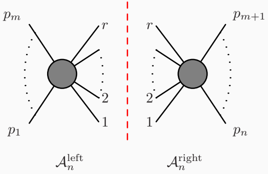

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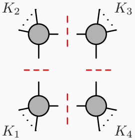
$c_{\text{box}}(\varepsilon = 0) =$  $= \prod(\text{trees})$

[Britto, Cachazo, Feng 2004]

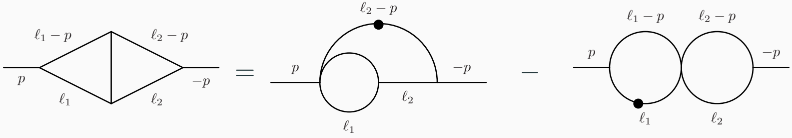
Generalized unitarity $\implies$ Analytic structure of FI determined by cuts!

☹ Formulas for sub-topologies (c_{tad}) hard to find

☹ Harder when $d = 4 - 2\varepsilon$

Integration-by-parts (IBP) identities

$$0 = \int d^d\ell_1 \, d^d\ell_2 \underbrace{\frac{\partial}{\partial \ell_2^\mu} \left(\frac{(\ell_2 - \ell_1)^\mu}{\ell_1^2 \, \ell_2^2 \, (\ell_2 - \ell_1)^2 \, (\ell_1 - p)^2 \, (\ell_2 - p)^2} \right)}_{\text{total derivative – not unique! [Chetyrkin 1981]}}$$



☹ Black box (despite efficient code)

☹ Additional formalism need to:

- Avoid unphysical squared propagators [Kosower 2018]
- Utilize (generalized) unitarity cuts

Alternative: Intersection theory

Intersection theory $\implies$ extraction of $c_\bullet$ without IBP identities

[Mastrolia, Mizera 2018; Frellesvig, et. al. 2019, 2020; Mizera, AP 2019; Caron-Huot, AP 2021; ...]

FIs unique up to shifts by total derivative (IBPs) $\implies$ cohomology

$$|\varphi\rangle \in H^\bullet = \frac{\nabla \text{ - closed forms (FIs)}}{\nabla \text{ - exact forms (IBPs)}}$$

$\exists$ Poincaré dual cohomology $\langle\varphi^\vee| \in (H^\bullet)^\vee$

Inner product $\langle\varphi^\vee|\varphi\rangle$ (intersection number)

$$\langle\varphi_i^\vee|\varphi_j\rangle = \delta_{ij} \quad \implies \quad |\Phi\rangle = \sum_i |\varphi_i\rangle \underbrace{\langle\varphi_i^\vee|\Phi\rangle}_{\text{e.g., } c_{\text{box}}}$$

Computation of $\langle\bullet|\bullet\rangle$ is algebraic (residues) $\implies$ echos what we like from GU

Feynman integrals in the language of intersection theory

multi-valued function → u; single-valued differential form → φ

$$\mathcal{I}[u,\varphi] = \int u \, \varphi \qquad \mathrm{d}^d\ell \propto \underbrace{(\ell_\perp^2)^{-\varepsilon}}_{\text{branch pts}} \underbrace{\frac{\mathrm{d}\ell_\perp^2 \wedge \mathrm{d}^4\ell}{\ell_\perp^2}}_{\text{5-form}}$$

Feynman Integrands: $u\,\varphi \propto \overbrace{(\ell_\perp^2)^{-\varepsilon}}^{u} \overbrace{\frac{\mathrm{d}\ell_\perp^2 \wedge \mathrm{d}^4\ell}{\ell_\perp^2 \; D_1^{\nu_1}\cdots D_n^{\nu_n}}}^{\varphi}$

regulated singularities → $(\ell_\perp^2)^{-\varepsilon}$

!!! poles: unregulated singularities !!! → $D_1^{\nu_1}\cdots D_n^{\nu_n}$

Intersection number: $c_\bullet = \langle \varphi^\vee | \varphi \rangle \propto \int_{\mathbb{C}^5} (u^\vee \varphi^\vee) \wedge (u\,\varphi)$

What object goes here?? → $(u^\vee \varphi^\vee)$

The undeformed dual space

Dual forms localized to GU cuts:

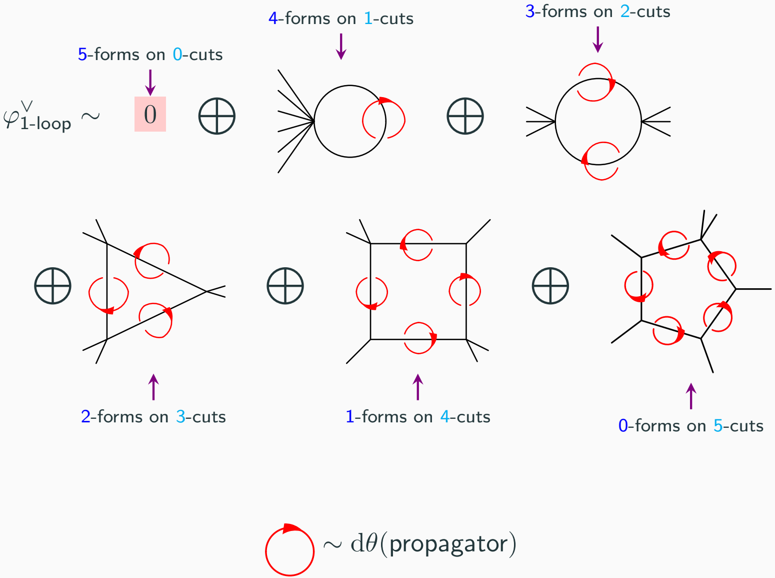
FIs: $\|\varphi\rangle$		dual forms: $\langle\varphi^\vee\|$
propagators/poles	$\leftrightarrow$	$\underbrace{\text{boundaries/zeros (on-shell)}}_{\text{compact support}}$

Dual cohomology spanned by holomorphic forms times:

$\theta(|\text{propagators}| > \epsilon)$: and/or $\mathrm{d}\theta(|\text{propagators}| > \epsilon)$:

θ or $\mathrm{d}\theta$ for each propagator

The undeformed dual space



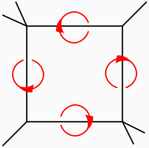
Streamlining the intersection number

$$\sim (\mathrm{d}\theta)^4 \wedge \left[\frac{\mathrm{d}^1\ell_\parallel}{(\ell_\perp^2|_{\text{box}})^1}\right]_{\text{c-map}}$$

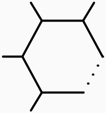
Localized/on-shell dual form

Any (1-loop) process

$c_{\text{box}}(\varepsilon) = \Big\langle$

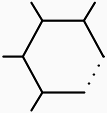


$\Big|$



$\Big\rangle$

$\sim \Big\langle\, \varphi^\vee_{\text{left-over}}(\ell_\perp^2|_{\text{box}}) \,\Big|\, \text{Res}_{\text{box}} \Big[$



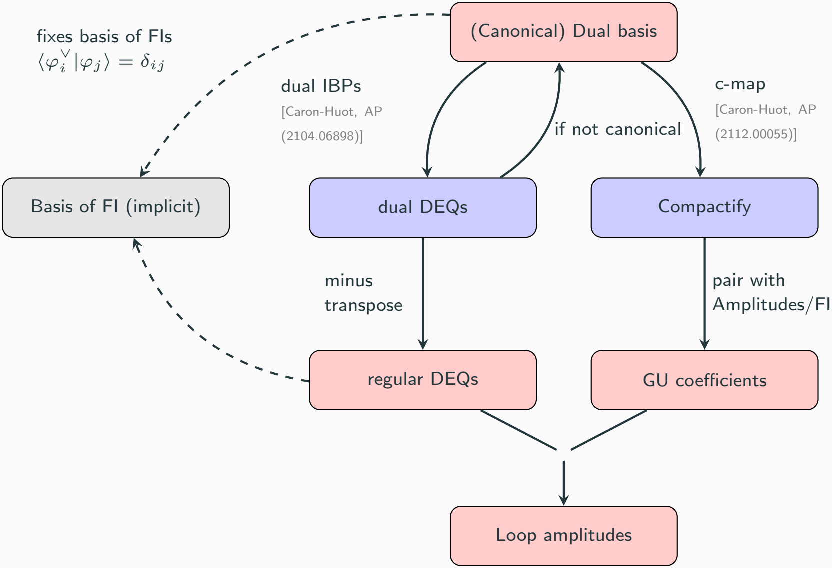
$\Big] \Big\rangle$

Factorizes

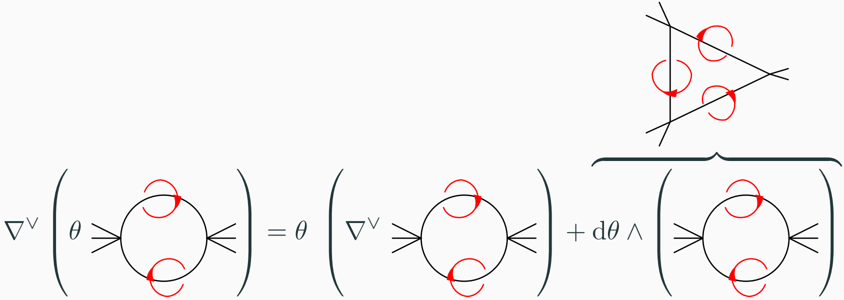
$=$ rational (algebraic) fn. of Mandelstams and ε

Left-over $\implies$ $\mathrm{d}\theta$'s at branch points ($\ell_\perp^2|_{\text{box}} = 0, \infty$) from c-map (essential for higher order terms in ε)

Workflow


(Canonical) Dual basis
fixes basis of FIs
$\langle \varphi_i^\vee | \varphi_j \rangle = \delta_{ij}$
dual IBPs
[Caron-Huot, AP (2104.06898)]
if not canonical
c-map
[Caron-Huot, AP (2112.00055)]
Basis of FI (implicit)
dual DEQs
Compactify
minus transpose
pair with Amplitudes/FI
regular DEQs
GU coefficients
Loop amplitudes

Dual DEQs are transposed



Dual IBP-forms are simple: supported on cuts and no squared propagators!

Simple canonical differential equations for 1-loop (dual) FI in **any** dimension

[Caron-Huot, AP (2104.06898); Bourjaily, Gardi, McLeod, Vergu (2020); Abreu, Britto, Duhr, Gardi (2017) Volovich, Spradlin (2011); Schläfli (1860)]

Compactifying (c-map)

$\varphi_c^\vee$ must have compact support for well defined intersection number

$$c_\bullet = \langle \varphi^\vee | \varphi \rangle \propto \int_{\mathbb{C}^5} (u^\vee \varphi_c^\vee) \wedge (u\ \varphi)$$

Compact support $\implies$ localization of intersection number (residues)

c-map: $\varphi^\vee$ with partial compact support $\exists\ \varphi_c^\vee : \varphi^\vee - \varphi_c^\vee = \nabla^\vee(\bullet)$
requires local primitives

$$\varphi_c^\vee = \theta(\text{props})\theta(\text{branch pts})\varphi^\vee - \mathrm{d}\theta(\text{props})\psi^\vee - \mathrm{d}\theta(\text{branch pts})\psi^\vee$$

$\nabla^\vee \psi^\vee \equiv \varphi^\vee$ near other propagators and branch pts

as a Laurent series

Trivial for 1-forms (integrate order by order)

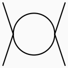
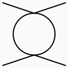
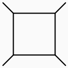
Harder for $(p > 1)$-forms

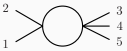
c-map in practice

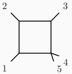
For $(p > 1)$-dual forms, there are 2 options:

- Fibration [Mizera 2019]: 1-variable at a time
  - ☺ Reuse simplicity of 1-form c-map
  - 😐 Matrix/vector valued
  - ☹ Construct $\varphi_c^\vee$ for dual basis all at once
  - ☹ Singularities at ∞ (from numerators) get progressively worse
  - 😐 Choice: coordinates and order of fibration
- Combinatoric [Matsumoto 80's]: deal with the multi-variate nature
  - ☺ Construct $\varphi_c^\vee$ individual dual forms
  - ☹ Need primitives for all ways of approaching a singularity
  - ☹ Need primitives for the primitives
  - 😐 Doable for $p = 2$
  - 😐 Choice: coordinates and which primitives are used where

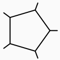
Putting things together: 1-Loop amplitudes

$$\mathcal{A}_4^{h_1 h_2 h_3 h_4} = c_{\mathsf{bub}_s} \Bigg(\ $$  $$\ \Bigg) + c_{\mathsf{bub}_t} \Bigg(\ $$  $$\ \Bigg) + c_{\mathsf{box}} \Bigg(\ $$  $$\ \Bigg)$$

$$\mathcal{A}_5^{h_1 h_2 h_3 h_4 h_5} = \overbrace{c_{\mathsf{bub}_{s_{12}}}}^{\times 5} \Bigg(\ $$ 

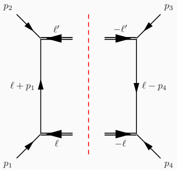
$$\ \Bigg) + \cdots + \overbrace{c_{\mathsf{box}_{45}}}^{\times 5} \Bigg(\ $$ 

$$\ \Bigg)$$

$$+ \cdots + \underbrace{c_{\mathsf{pent}}}_{\times 1} \Bigg(\ $$  $$\ \Bigg)$$

$$c_\bullet = \Big\langle \mathrm{d}\theta_{\textcolor{red}{\mathsf{cut}}}(\text{dual form}) \Big| \text{Feynman form} \Big\rangle = \Big\langle \text{dual form} \Big| \underbrace{\mathrm{Res}_{\textcolor{red}{\mathsf{cut}}}(\text{Feynman form})}_{} \Big\rangle$$

from cuts by glueing trees



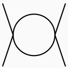
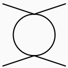
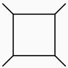
Massless 1-Loop amplitudes and 4-dimensional limits

Proof of concept: reproduced 4- and 5-point gluon amplitudes at 1-loop

[Bern, Dixon, Kosower (90's), see 2112.00055 for details]

More interesting, systematic study of 4d-limit

$$\frac{c_s^{(\mathsf{tri})}}{\varepsilon^2} + \frac{c_s^{(\mathsf{bub})}}{\varepsilon} + c_s^{(\mathsf{rat})} \qquad\qquad \frac{c^{(\mathsf{box})}}{\varepsilon^2}$$

$\mathcal{A}_4 = c_{\mathsf{bub}_s}(\varepsilon)$ () $+\ c_{\mathsf{bub}_t}(\varepsilon)$ () $+\ c_{\mathsf{box}}(\varepsilon)$ ()

$$-2 + 2\varepsilon \log(-s) + \varepsilon^2(\zeta_2 - \log^2(-s)) \qquad\qquad -\varepsilon^2\left(\pi^2 + \log^2(t/s)\right)$$

Massless 1-Loop amplitudes

$$c_{\mathrm{bub}_s}[\varphi](\varepsilon) = \sum_{(\ell_\perp^2, \ell_-, \ell_+)} \mathrm{Res}_{\ell_-} \mathrm{Res}_{\ell_+} \mathrm{Res}_{\ell_\perp^2} \Big[\underbrace{\psi^\vee_{(\ell_\perp^2, \ell_+, \ell_-)}}_{\substack{\text{triple primitive of bub-dual} \\ \text{from combinatoric c-map}}} (\mathrm{Res}_{s\text{-channel}} \varphi) \Big]$$

		$(0,\infty)$	(∞,∞)	$(D_{\mathrm{tri}}, \mathrm{soft})$	$(D_{\mathrm{tri}}, \infty)$
ε^{-2}	$\ell_\perp^2 = 0$			✓	
	$\ell_\perp^2 = \infty$				
ε^{-1}	$\ell_\perp^2 = 0$		✓		
	$\ell_\perp^2 = \infty$				
ε^{0}	$\ell_\perp^2 = 0$		✓		✓
	$\ell_\perp^2 = \infty$		✓		✓

Better coordinates? [Badger 2008]

Conclusions and Future Directions

Recovered generalized unitarity in arbitrary dimension from twisted relative cohomology

1-loop warm up: [S. Caron-Huot, AP (2021); S. Caron-Huot, AP (2021)]

- 1-loop basis of canonical dual forms
- Obtained DEQ for this basis in any dimension
- Constructed 1-loop 4- and 5-point gluon amplitudes from their cuts (coeffs all orders in ε)
- 4-dimensional limit extract rational terms (ε/ε)

Next goals:

- Loop-by-loop determination of 2-loop DEQs
  [See M. Giroux's poster: ε-form of 3-mass sunrise]
- Improve c-map efficiency (what are good choices?)
- Rational terms at 2-loops