Classical general relativity from the double copy and the kinematic algebra of YM theory

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with

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+ to appear also with Graham Brown, Stefano De Angelis, Josh Gowdy

Amplitudes 2022, Charles University, Prague, 11th August 2022

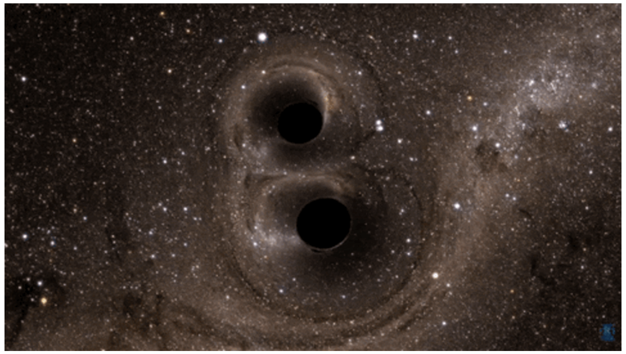
Queen Mary University of London

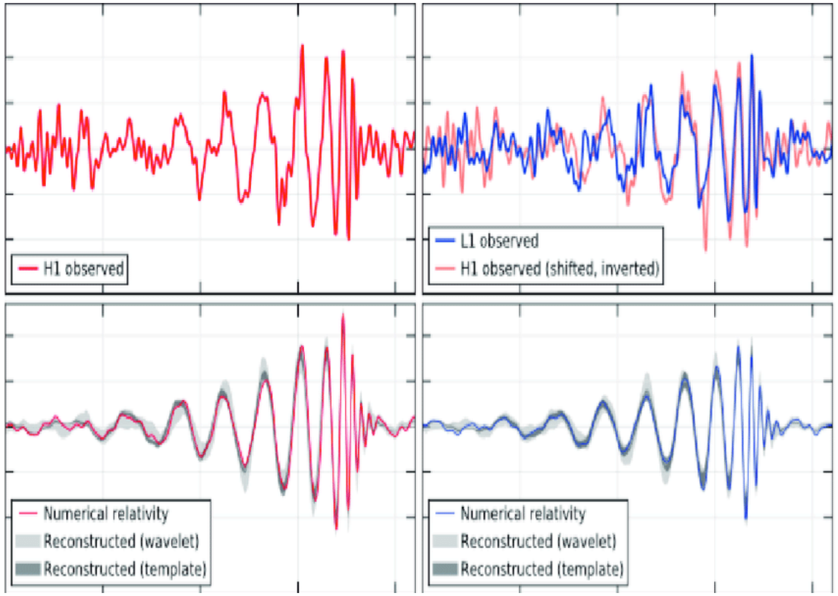
SAGEX
Scattering Amplitudes: from Geometry to Experiment

Motivation

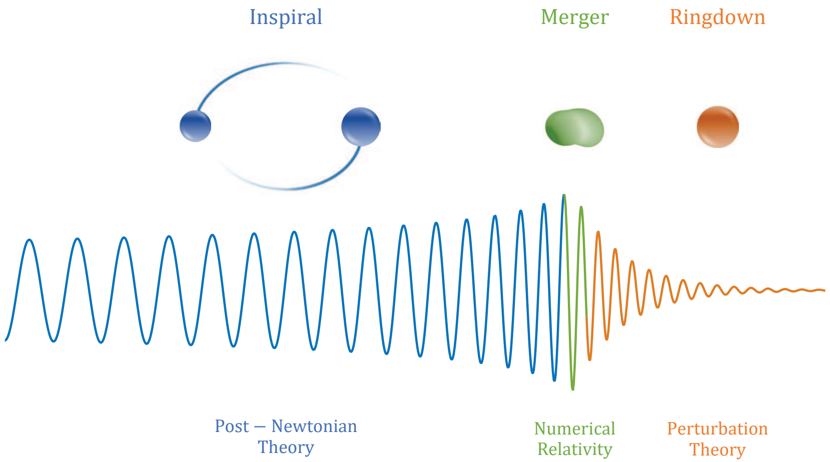
Exciting talks tomorrow!

- First direct observation of gravitational waves
  - LIGO and Virgo
  - Binary black hole merger





- Three phases
  - Inspiral
  - Merge
  - Ringdown



from Antelis and Moreno 1610.03567

Menu

- Black hole scattering in GR from a Heavy-mass Effective Field Theory (HEFT)
  - Two heavy scalars + gluons or gravitons from a novel colour/kinematics duality
  - Two-loop (3PM) scattering angle
- Uncovering the hidden "kinematic algebra" of amplitudes
  - a quasi-shuffle Hopf algebra for BCJ numerators
  - discovered in the HEFT…
  - …applies also to pure Yang-Mills theory after decoupling the heavy scalars

Newton potential from amplitudes

Newton potential from amplitudes

- Extract from elastic scattering of two heavy particles
  - Connection to amplitudes first suggested by Iwasaki in 1971, computed classical + quantum correction at $O(G^2)$ from one-loop Feynman diagrams
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- Loops contribute to classical physics!
(Iwasaki; Donoghue & Holstein; ….; Kosower, Maybee, O'Connell)
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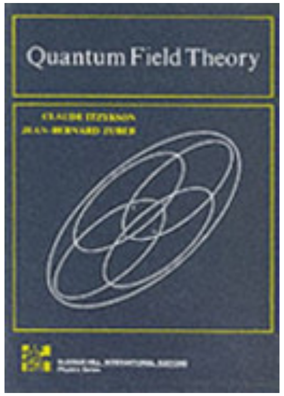
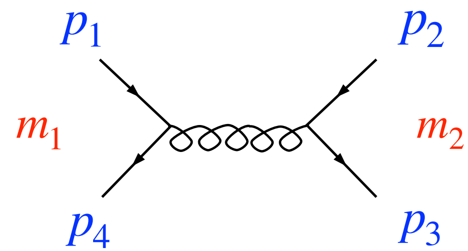
  - Loop expansion is not an $\hbar$ expansion



- Static amplitude: $A =$ 

$= i\,G\,\dfrac{16\pi m_1^2 m_2^2}{\vec{q}^{\,2}}$

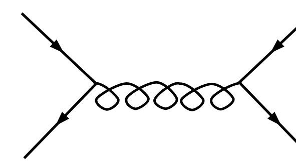
$\vec{q} = \vec{p}_3 - \vec{p}_2$

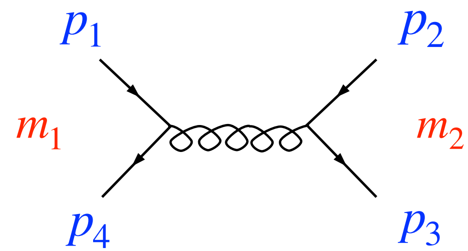
momentum tranfer

- Static potential: $\widetilde{V}_{\rm static}(\vec{q}) := i\,\dfrac{A(\vec{q})}{4m_1 m_2}$

Newton's law from scattering amplitudes!

$$V_{\rm static} = -\frac{G m_1 m_2}{r}$$

- Single-graviton exchange gives Newton potential 

- Static amplitude: $A =$ 

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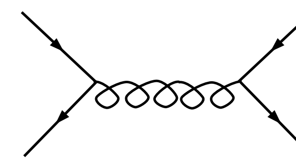
- Single-graviton exchange gives Newton potential 

- Amplitude-based derivations
  - classical (Neill & Rothstein 2013), classical + quantum (Bjerrum-Bohr, Donoghue, Vanhove 2013)
  - no sums over many diagrams, fully relativistic computation

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- Non-local/non-analytic effects from low-energy theory
  - GR is non-renormalisable, however….
  - …IR theory: low-energy gravitons that can propagate long distances
  - From momentum space to position space:

$$\frac{1}{q^2} \to \frac{1}{r}\,, \quad \frac{1}{q^2}\sqrt{-q^2} \to \frac{1}{r^2}\,, \quad \frac{1}{q^2}\, q^2\, \log(-q^2) \to \frac{1}{r^3}\,, \quad \frac{1}{q^2} q^2 \to \delta^{(3)}(\vec{x})$$

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  - non-analytic terms cannot be reabsorbed by a local counterterm, i.e. they cannot be modified by any high-energy change in the theory
  - from long-range propagation of massless particles at low energy
  - Ideal for unitarity-based techniques

Black hole scattering from HEFT

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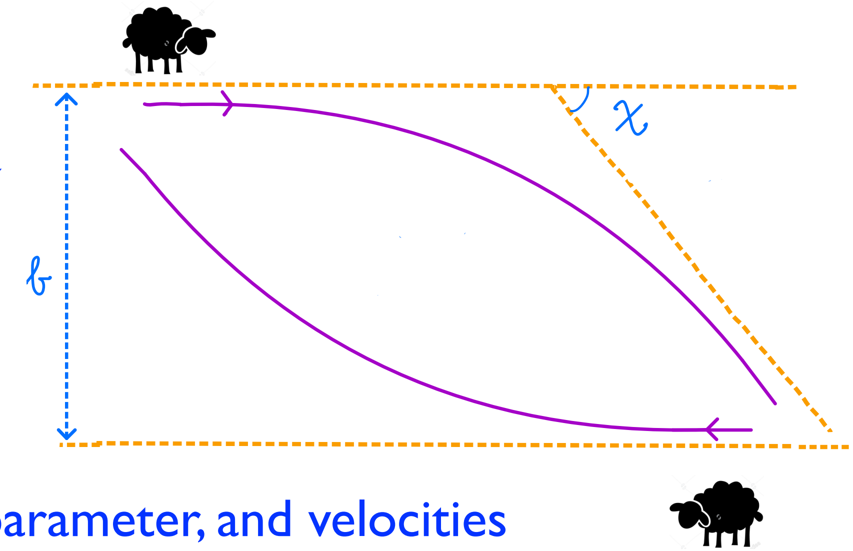
  - coordinate-independent quantity
  - spinless black holes, PM expansion
  - as a function of the masses, impact parameter, and velocities



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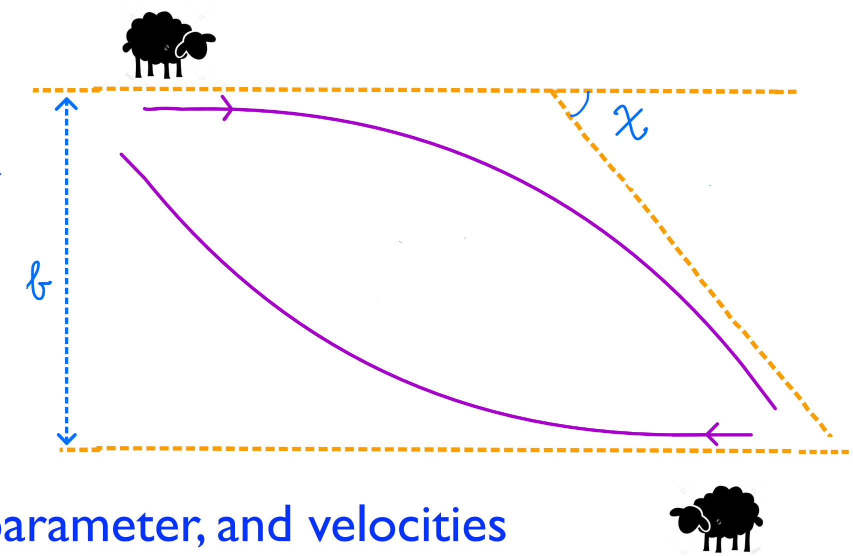

- Compute scattering angle χ
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  - as a function of the masses, impact parameter, and velocities
- Preliminary observations:
  - black holes exchange momenta that are much smaller than their masses
  - similar to Heavy-Quark Effective Theory (Georgi)
  - small momentum expansion same as $\hbar$ expansion, need only $\hbar \to 0$ limit!
  - two massive scalars + gluons/gravitons (see also Aoude, Damgaard, Haddad Helset)

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- Formulate Heavy-mass Effective Field Theory (HEFT): massive scalars + gluons (YM) (Brandhuber, Chen, GT, Wen)
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- Compute the scattering angle at 3PM
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- Bonus: find the kinematic algebra of HEFT & pure YM theory! (Brandhuber, Chen, GT, Johansson, Wen)
  - all-multiplicity formulae

Examples

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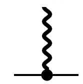
- Three point: $A_3^{\rm GR} =$ 

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- Four points: $A_4^{\rm GR} \to -(i\pi)\, m^3\, \delta(v\cdot \ell_1)(v\cdot \varepsilon_1)^2(v\cdot\varepsilon_2)^2 + \frac{m^2}{\ell_{12}^2}\Big(\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot \ell_2}\Big)^2 + \cdots$

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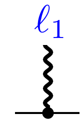
- Three point: $A_3^{\mathrm{GR}} =$ 

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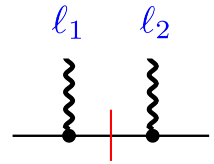
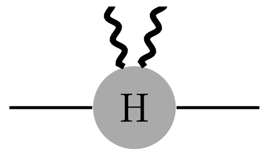
$=$ 

$+$ 

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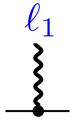
- **Three point:** $A_3^{\rm GR} =$  $= m^2(\epsilon_1 \cdot v)^2$

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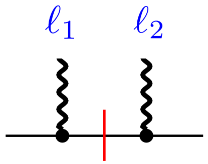
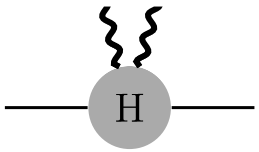
$=$ 

$+$ 

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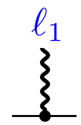
- Three point: $A_3^{\rm GR} =$  $= m^2(\epsilon_1 \cdot v)^2$

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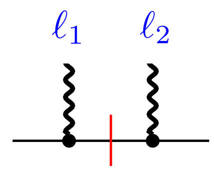
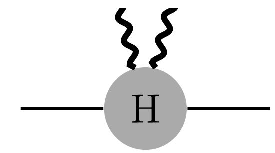
$=$ 

$+$ 

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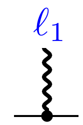
- **Three point:** $A_3^{\mathrm{GR}} =$  $= m^2(\epsilon_1 \cdot v)^2$

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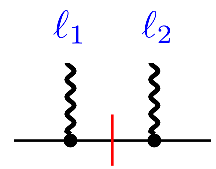
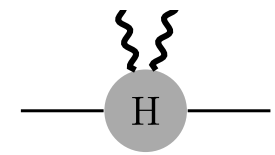
$=$  $+$ 

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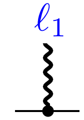
- **Three point:** $A_3^{\rm GR} =$  $= m^2(\epsilon_1 \cdot v)^2$

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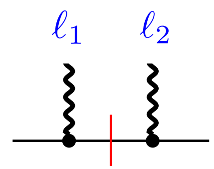
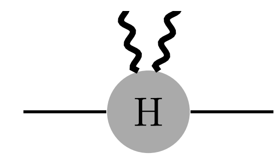
$=$ 

$+$ 

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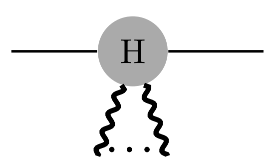
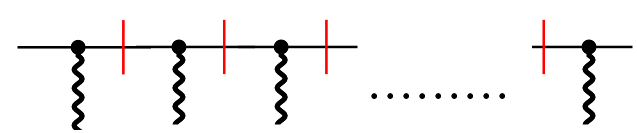
- **Generalises to higher points:**

$A_n^{\rm GR} \to$ 

$+ \cdots +$ 

  - sum of delta functions and HEFT amplitudes (more HEFT examples later)

One loop

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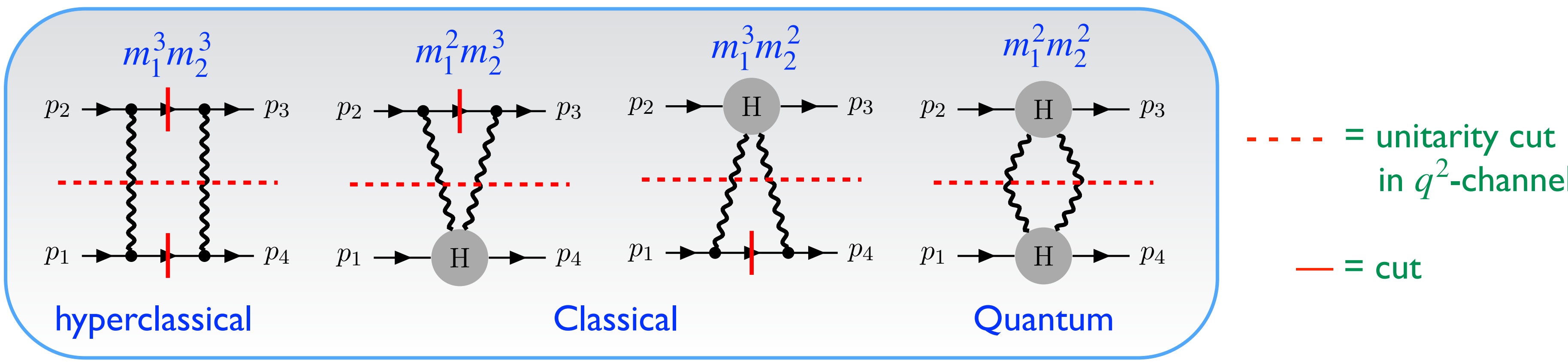
- Expansion of the one-loop amplitude in the masses:



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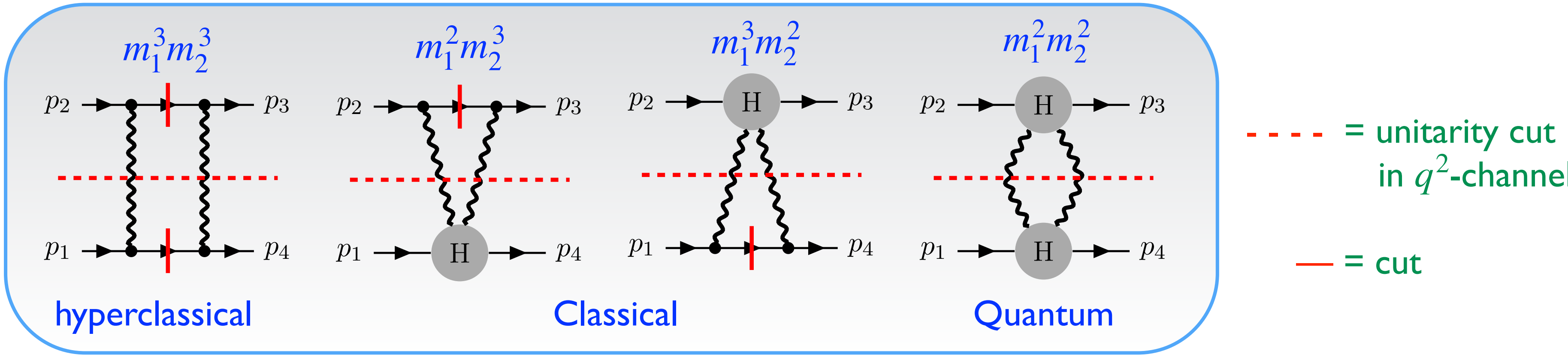

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- Further simplifications to extract classical physics!

Exponentiation

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- Conservative amplitude is a phase in impact parameter space (Glauber; Levi & Sucher; ….; Amati, Ciafaloni & Veneziano; Kabat & Ortiz)

$$\widetilde{S} = 1 + i\widetilde{\mathcal{M}} = e^{i\delta_{\text{HEFT}}} \qquad \chi = -\frac{\partial}{\partial J}\,\text{Re}\;\delta_{\text{HEFT}}$$

$$\widetilde{\mathcal{M}}(b) := \int \frac{d^D q}{(2\pi)^{D-2}}\delta(2\bar{p}_1\cdot q)\delta(2\bar{p}_2\cdot q)e^{-i\vec{q}\cdot\vec{b}}\mathcal{M}(q) \;=\; \frac{1}{\bar{\mathcal{J}}}\int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}e^{-i\vec{q}\cdot\vec{b}}\mathcal{M}(q)$$

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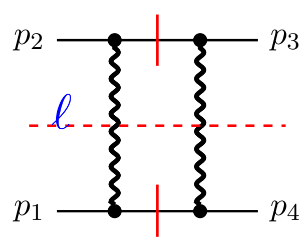

$$\sim \int \frac{d^D\ell}{(2\pi)^D}(-2\pi i)^2\delta(\bar{p}_1\cdot\ell)\delta(\bar{p}_2\cdot\ell)\mathcal{M}_{\rm L}(\ell)\mathcal{M}_{\rm R}(q-\ell) \xrightarrow{\rm IPS} -\widetilde{\mathcal{M}}_{\rm L}(b)\widetilde{\mathcal{M}}_{\rm R}(b)$$

hyperclassical

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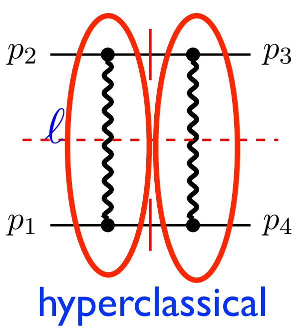

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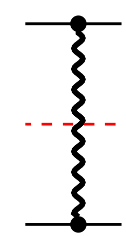
- Already accounted for from exponentiating tree level !



- $\delta_{\rm HEFT}$ directly obtained from 2-massive-particle-irreducible (2MPI) diagrams

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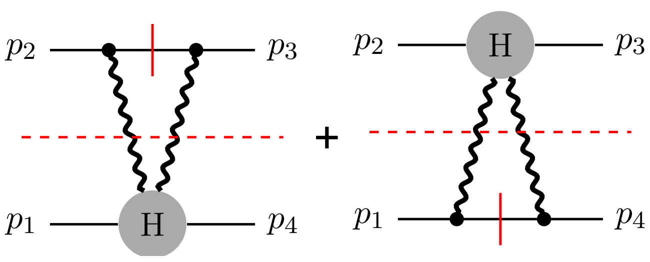
- Great simplification of the diagrammatics/computation!

$$\delta^{(1)}_{\rm HEFT} \sim$$



Two loops

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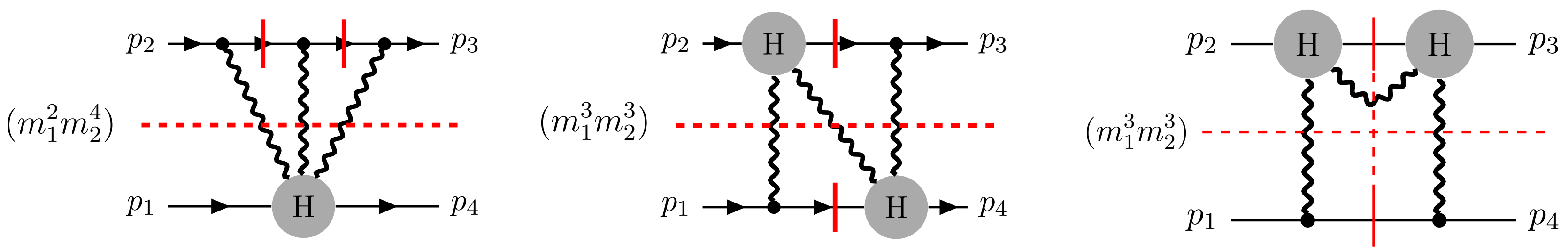
- Write down 2MPI HEFT diagrams only:



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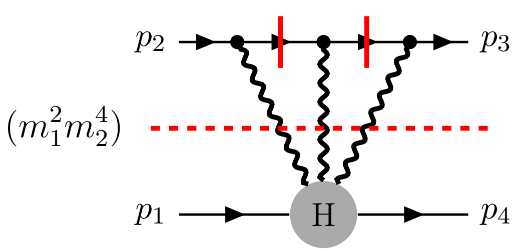
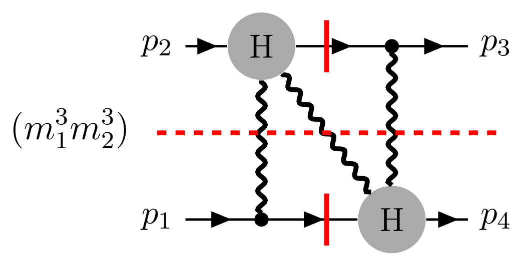
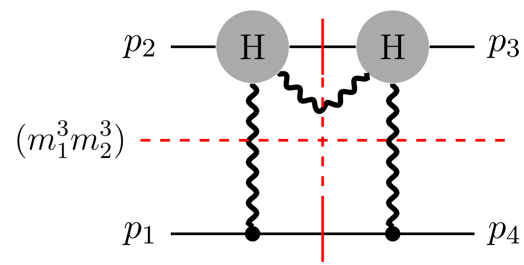


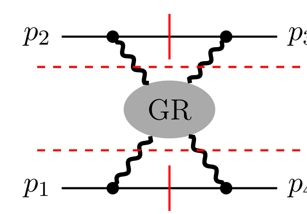


"conservative part"



"radiation reaction"

- Probe limit, "zig-zag", radiation reaction
- "Washing machine" does not add new contributions



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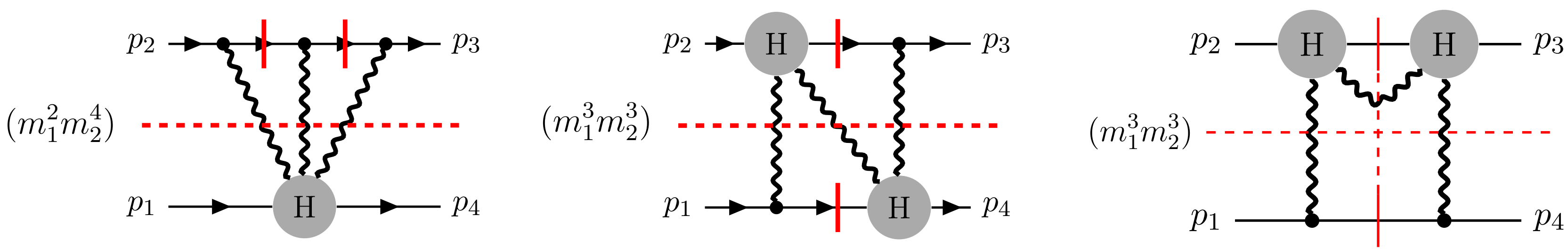
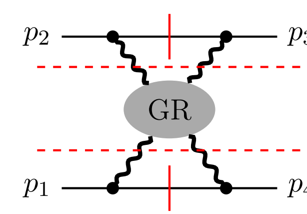


"conservative part" "radiation reaction"

  - Probe limit, "zig-zag", radiation reaction
  - "Washing machine" does not add new contributions



- Advantages of HEFT:
  - HEFT trees are compact, gauge invariant, and contain linear propagators
  - Expansion done at an earlier stage (contrast to other approaches)

Results

Results

- Standard variables:

$$y := \frac{p_1 \cdot p_2}{m_1 m_2} = v_1 \cdot v_2 = \frac{1}{\sqrt{1 - v^2}}$$

$$P = \frac{\sqrt{y^2 - 1}}{E}, \quad J = Pb$$

  - Static limit: $y \to 1$, ultra-relativistic limit: $y \to \infty$

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- HEFT phase computed from

$$\delta_{\mathrm{HEFT}} = \frac{1}{4m_1 m_2 \sqrt{y^2-1}} \int \frac{d^{D-2}q}{(2\pi)^{D-2}} e^{-i\vec{q}\cdot\vec{b}} \mathcal{M}^{\mathrm{2MPI}}_{\mathrm{HEFT}}$$

- Scattering angle $\chi = -\frac{\partial}{\partial J} \mathrm{Re}\, \delta_{\mathrm{HEFT}}$

Final result for the scattering angle

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- Up to 3PM:

$$\chi = \frac{G}{J}\,\frac{2m_1m_2\left(2y^2-1\right)}{\sqrt{y^2-1}} + \frac{G^2}{J^2}\,\frac{3\pi}{4\sqrt{s}}m_1^2m_2^2(m_1+m_2)\left(5y^2-1\right)$$
$$+\frac{G^3}{J^3}\,\frac{m_1m_2\sqrt{y^2-1}}{\pi s}\Bigg\{m_1^2m_2^2(m_1^2+m_2^2)\frac{2\pi\left(64y^6-120y^4+60y^2-5\right)}{3\left(y^2-1\right)^2}$$
$$+m_1^3m_2^3(-8\pi)\Bigg[\frac{\left(5y^2-8\right)\left(1-2y^2\right)^2}{6\left(y^2-1\right)^{3/2}}-\frac{y\left(2y^2-3\right)\left(1-2y^2\right)^2\operatorname{arccosh}(y)}{2\left(y^2-1\right)^2}$$
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Probe limit

Radiation reaction (half-PN orders!)

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Probe limit

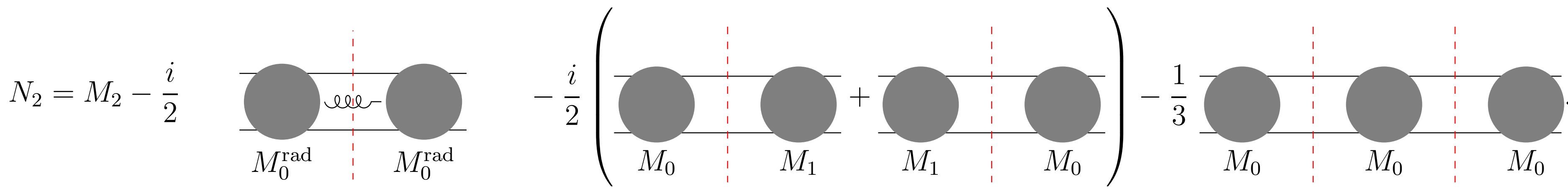
Radiation reaction (half-PN orders!)

- Agreement with previous 3PM calculations:
    - Conservative part: Bern, Cheung, Roiban, Shen, Solon, Zeng (2019), Cheung & Solon (2020)
    - Radiation reaction: Di Vecchia, Heissenberg, Russo, Veneziano; Herrmann, Parra-Martinez, Rif, Zeng; Bjerrum-Bohr, Damgaard, Planté, Vanhove

Comments

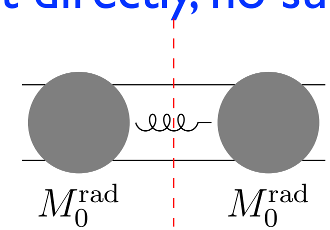
Comments

- Connection to Damgaard, Planté & Vanhove's work (2107.12891)
  - They write $S = \exp(i\,N)$ with N hermitian
  - They extract matrix element of N at two loops:



Iteration terms, simply not present in our approach

  - Simplicity hidden in N
  - Our $\mathrm{Re}\,\delta_{\text{HEFT}}$ agrees with the matrix element of their N, with our method providing a way to compute it directly, no subtraction needed
  - Imaginary part arising from



The kinematic algebra of HEFT and YM

(Brandhuber, Chen, GT, Johansson, Wen)

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- Two goals, one practical and one conceptual:
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- Kinematic algebra known in the self-dual sector of YM: area-preserving diffeomorphisms (Monteiro & O'Connell)
  - Self-dual Yang-Mills is somewhat trivial. What about the full interacting theory?
- Work in HEFT, then extract YM results as follows:
  - Label the two scalars as $n{-}1$ and n, with gluons being $1, 2, \ldots, n{-}2$
  - Then replace scalar velocity v with ϵ_{n-1} and impose $(p_1 + p_2 + \cdots + p_{n-2})^2 \to 0$

- General HEFT amplitudes in YM (A) and GR (M):

$$A(1\ldots n-2;v)=\sum_{\Gamma\in\rho}\frac{\mathcal{N}(\Gamma,v)}{d_\Gamma}\,,\qquad M(1\ldots n-2;v)=\sum_{\Gamma\in\tilde\rho}\frac{\left[\mathcal{N}(\Gamma,v)\right]^2}{d_\Gamma}$$

  - $\rho(\tilde\rho)$ =(un)ordered nested commutators of $n-2$ gluon (graviton) labels, leftmost fixed to 1
  - $n=4:\quad \rho=\tilde\rho=\{[1,2]\}$

    $n=5:\quad \rho=\{[[1,2],3],\,[1,[2,3]]\},\qquad \tilde\rho=\{[[1,2],3],\,[[1,3],2],\,[1,[2,3]]\}$
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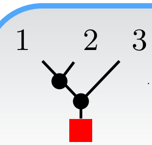
Example:



$$\longleftrightarrow\ \mathcal{N}([[1,2],3],v)\,,\quad d_{[[1,2],3]}=p_{12}^2p_{123}^2$$

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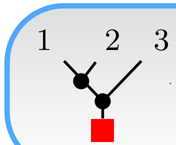
Example: 

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- BCJ numerators from "pre-numerators" (Chen, Johansson, Teng, Wang)
  - For instance $\mathcal{N}([[1,2],3],v)=\mathcal{N}(123,v)-\mathcal{N}(132,v)-\mathcal{N}(231,v)+\mathcal{N}(321,v)$
  - Jacobi relations automatically satisfied, e. g. $\mathcal{N}([1,[2,3]],v)=\mathcal{N}([[1,2],3],v)-\mathcal{N}([[1,3],2],v)$

Examples

Examples

- $n = 4$: $\mathcal{N}(12, v) = -\frac{v\cdot F_1\cdot F_2\cdot v}{2v\cdot p_1}$
  - Particles 3 and 4 are the massive scalars, and $F_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu$
  - set m=1 from now on

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- $n = 5$: $\mathcal{N}(123, v) = \dfrac{v\cdot F_1\cdot F_2\cdot F_3\cdot v}{3v\cdot p_1} - \dfrac{v\cdot F_1\cdot F_2\cdot V_{12}\cdot F_3\cdot v}{3v\cdot p_1 v\cdot p_{12}} - \dfrac{v\cdot F_1\cdot F_3\cdot V_1\cdot F_2\cdot v}{3v\cdot p_1 v\cdot p_{13}}$
  - Also define $V_\tau^{\mu\nu} = v^\mu \sum_{i\in\tau} p_j^\nu = v^\mu p_\tau^\nu$

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- Comments:
  - BCJ (pre-)numerators are manifestly gauge invariant (written in terms of field strengths)
  - They appear to have a clearly identifiable structure
  - Number of terms: $1, 3, 13, 75\ldots$ for $n = 4, 5, 6, 7\ldots.$

- Idea: construct pre-numerators by multiplying generators of an abstract algebra with a fusion product $\star$

$$\mathcal{N}(1\ldots n-2, v) = \langle T_1 \star \cdots \star T_{n-2} \rangle$$

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  - 6 points: $\langle T_{(123)} \star T_{(4)} \rangle = -\langle T_{(1234)} \rangle + \langle T_{(123),(4)} \rangle + \langle T_{(14),(23)} \rangle$

$$\langle T_{(12),(3)} \star T_{(4)} \rangle = -\langle T_{(12),(34)} \rangle - \langle T_{(124),(3)} \rangle + \langle T_{(12),(3),(4)} \rangle + \langle T_{(12),(4),(3)} \rangle + \langle T_{(14),(2),(3)} \rangle$$

$$\langle T_{(13),(2)} \star T_{(4)} \rangle = -\langle T_{(13),(24)} \rangle - T_{(134),(2)} \rangle + \langle T_{(13),(2),(4)} \rangle + \langle T_{(13),(4),(2)} \rangle + \langle T_{(14),(3),(2)} \rangle$$

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- Count number of terms in pre-numerator:
  - 4 points: 1 5 points: 3 6 points: 3+5+5=13

You are welcome to input
"sequence 1, 3, 13, 75"
into Google…

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- $n = 5$: $\langle T_{(123)} \rangle = \frac{v\cdot F_1 \cdot F_2 \cdot F_3 \cdot v}{3 v\cdot p_1}\,, \quad \langle T_{(12),(3)} \rangle = \frac{v\cdot F_1 \cdot F_2 \cdot V_{12} \cdot F_3 \cdot v}{3 v\cdot p_1 v\cdot p_{12}}\,, \quad \langle T_{(13),(2)} \rangle = \frac{v\cdot F_1 \cdot F_3 \cdot V_1 \cdot F_2 \cdot v}{3 v\cdot p_1 v\cdot p_{13}}$

with $F_i^{\mu\nu} := p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu\,, \quad V_\tau^{\mu\nu} := v^\mu \sum_{j\in\tau} p_j^\nu = v^\mu p_\tau^\nu$

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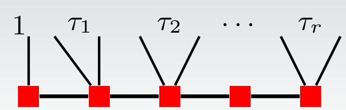
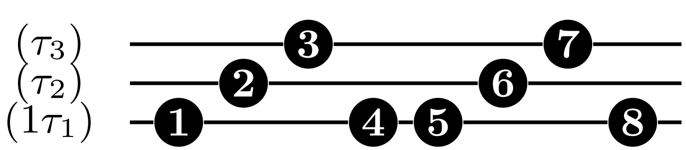
- Consistent with general formula:

$$\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle :=$$



$$= \frac{v\cdot F_{1\tau_1}\cdot V_{\Theta(\tau_2)}\cdot F_{\tau_2}\cdots V_{\Theta(\tau_r)}\cdot F_{\tau_r}\cdot v}{(n-2)v\cdot p_1 v\cdot p_{1\tau_1}\cdots v\cdot p_{1\tau_1\tau_2\cdots\tau_{r-1}}}$$

  - $\tau_1\cup\tau_2\cup\cdots\cup\tau_r = \{2,3,\ldots,n-2\} \qquad \tau_i\cap\tau_j=\varnothing$
  - $\Theta(\tau_i) = (\{1\}\cup\tau_1\cup\cdots\cup\tau_{i-1})\cap\{1,\ldots,\tau_{i[1]}\}$
  - Example: for $\langle T_{(1458),(26),(37)}\rangle = \dfrac{v\cdot F_{1458}\cdot V_1\cdot F_{26}\cdot V_{12}\cdot F_{37}\cdot v}{v\cdot p_1\, v\cdot p_{1458}\, v\cdot p_{124568}}$ draw musical diagram



$\Theta(26)=\{1\},\quad \Theta(37)=\{1,2\}$

- Stare at fusion rules (index 1 always a spectator):
  - Five points:

$$T_{(12)} \star T_{(3)} = T_{(12),(3)} + T_{(13),(2)} - T_{(123)}$$

$$(A)(B) = (A, B) + (B, A) + (A \sqcup B)$$

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  - Quasi-shuffle $$(A)(B) = (A,B) + (B,A) + (A \sqcup B)$$

Extra terms

- Search the internet!

ANNALES

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Tome 67, n° 3 (2017), p. 911-945.

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- Hopf algebras of quasi-shuffles

- Kinematic algebra is a quasi-shuffle Hopf algebra that generates all ordered partitions of a given set

  - Combinatorial Hopf algebras of shuffles and quasi-shuffles
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- General fusion product:

shuffling — stuffing

$$T_{(1\tau_1),\dots,(\tau_r)} \star T_{(j)} = \sum_{\sigma\in\{(\tau_1),\dots,(\tau_r)\}\sqcup\!\sqcup\{(j)\}} \overbrace{T_{(1\sigma_1),\dots,(\sigma_{r+1})}}^{\text{shuffling}} - \sum_{i=1}^{r} T_{(1\tau_1)\overbrace{,\dots,(\tau_{i-1}),(\tau_i j),(\tau_{i+1}),\dots,}^{\text{stuffing}}(\tau_r)}$$

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  - Shuffle product, e.g. $\{A,B\}\sqcup\!\sqcup\{C\}=\{ABC,ACB,CAB\}$
  - The subscripts τ of the Ts are all possible ordered partitions of $\{2,3,\ldots,n{-}2\}$
  - These Hopf algebras are endowed with a product that is commutative and associative, and with a coproduct, counit and antipode (Hoffman; Cartier; Ihara, Kaneohe & Zagier)

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- Closed formula for general pre-numerator:

$$\mathcal{N}(1\dots n{-}2, v) = \sum_{r=1}^{n-3} \sum_{\tau\in\mathbf{P}^{(r)}_{\{2,\cdots,n-2\}}} (-1)^{n+r} \langle T_{(1\tau_1),\dots,(\tau_r)} \rangle$$

  - $\mathbf{P}^{(r)}_{\{2,\ldots,n-2\}}$ = ordered partitions of $\{2,\ldots,n{-}2\}$ into r subsets, their number is $\left\{ \begin{matrix} n-3 \\ r \end{matrix} \right\}$

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  - Total number is $\sum_{r=1}^{n-3} r! \left\{ {n-3 \atop r} \right\} = \mathsf{F}_{n-3}$
  - # terms in a pre-numerator with n–2 gluons: Fubini number n–3 (ordered Bell numbers)
    # terms in a YM pre-numerator with n gluons: Fubini number n–2

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Guido Fubini

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n	3	4	5	6	7	8	9	10
F_{n-3}	1	1	3	13	75	541	4683	47293

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- What do Fubini numbers do for a living?
  - Count the possible outcomes of a horse race including ties!

  - E.g. for 2 horses we have three possible outcomes (either one can win, or there is a tie)
  - Also count number of permutohedron faces, Cayley trees, Cayley permutations…

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- Further explorations of the Hopf algebra structure
  (Brandhuber, Brown, Chen, Gowdy, GT, Wen to appear)
  - Components of the coproduct corresponds to factorisations
  - Antipodal map inverts the order of gluons in a BCJ numerator

Lance's talk yesterday

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Lance's talk yesterday

- Comparison to other works:
  - Cheung + Mangan (2021): BCJ numerators contain $2\, F_{n\text{-}2}$ terms for an n-point gluon amplitude, but no algebraic structure was identified (and twice the number of terms)
  - Common features: their BCJ numerators also depend on one reference momentum

Conclusions & open problems

- Heavy-mass Effective Field Theory and its double copy
  - Scattering angle of two massive black holes at 1, 2 and 3 PM
  - All multiplicity, manifestly gauge-invariant HEFT amplitudes
  - Quasi-shuffle Hopf kinematic algebra in HEFT and YM theory
- (Some) open issues
  - Apply HEFT to higher PM orders calculations…
  - …and to radiation, and connection to gravitational waveforms
  - Add spin, add higher-derivative interactions….
  - What does the kinematic Hopf algebra teach us about amplitudes?
  - Explicit expressions for generators?

…and many more…