Curved backgrounds and scattering amplitudes

Tim Adamo
University of Edinburgh

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with A. Cristofoli & P. Tourkine [2112.09113 + to appear]

also work with Casali, Gonzo, Ilderton, Klisch, Kol, MacLeod, Mason, Nekovar, Sharma

Motivation

Consider scattering amplitudes in a **curved** background (asymp. flat), an **exact**, **non-perturbative** solution e.g.,

- non-trivial gauge field configuration ($F_{\mu\nu} \neq 0$)
- curved space-time ($R_{\mu\nu\rho\sigma} \neq 0$)

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Many reasons to be interested in this:

- Playground where pert./non-pert. effects meet
- Myriad physical applications: lasers, heavy ions, grav. waves, black holes,...
- Proving ground for robustness of any amplitudes method

The bad news

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- Background field Feynman rules a nightmare
- *Functional* degrees of freedom in the background
- No momentum conservation, tree-amplitudes not rational functions
- No Huygens' principle $\Rightarrow$ tails [Friedlander, Harte, TA-Casali-Mason-Nekovar]
- Memory effect [Christodoulou, Bieri-Garfinkle-Yau,...]

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Tree-level frontier with textbook approach: 4-points (strong-field QED, plane wave background)

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1. we have many non-textbook approaches (the whole point of this conference!)
2. even low-mult./loops in curved backgrounds encode *lots* of information

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But others don't (spinor helicity, ambi/twistor theory) or might not (double copy, worldsheet-based unitarity)

Smoking gun: **all-multiplicity** gluon/graviton scattering in self-dual gauge fields/space-times [TA-Mason-Sharma]

Today

Explore what 2-point amplitudes in curved backgrounds can teach us about scattering in a flat background

1. Covariant approach to relativistic eikonal regime $\rightarrow$ interesting new formulae
2. Use this to answer questions like: ‘What is the massless limit of Kerr?’

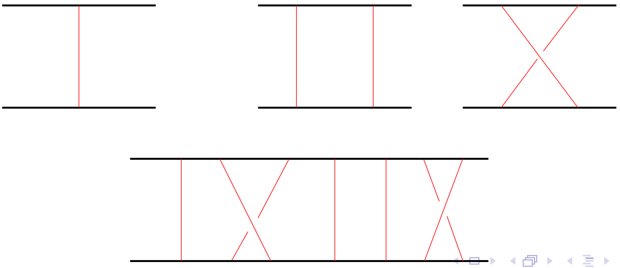
Eikonal exponentiation

Small-angle scattering can be remarkably simple:

- $2 \to 2$ scattering with $s \gg -t$
- Dominant ladder diagrams re-sum to give *eikonal amplitude*

$$\mathcal{M}_{\rm eik} \sim \int \mathrm{d}^2 x^\perp \, \mathrm{e}^{-\mathrm{i}\, q_\perp \cdot x^\perp} \left(\mathrm{e}^{\mathrm{i}\,\chi_1(x^\perp)} - 1 \right) \sim A_4(q)\, \mathrm{e}^{\mathrm{i}\varphi}$$

where $q_\perp$ is (small) exchanged momentum, *eikonal phase* χ_1 is inv. Fourier transform of tree-level exchange A_4



Properties

When eikonal exponentiation holds...

- $\chi_1 \sim s^{J-1} \Rightarrow$ highest spin exchange dominates ['t Hooft]
- Manifest classicality: $\mathcal{M}_{\text{eik}} \sim A_4(q)\, \mathrm{e}^{\mathrm{i}\varphi}$ and $\mathcal{M}_{\text{eik}} \sim \hbar^{-1}$
- φ carries info about classical bound states ['t Hooft, Damour,...]
- integrals can often be evaluated; e.g., gravitational scattering of mass m scalars ['t Hooft, Kabat-Ortiz]

$$\mathrm{i}\,\mathcal{M}_{\text{eik}}(q) = \frac{2\,\pi}{\mu^2}\,\sqrt{s(s-4m^2)}\,\frac{\Gamma(1-\mathrm{i}\,\alpha(s))}{\Gamma(\mathrm{i}\alpha(s))}\,\left(\frac{4\mu^2}{q_\perp^2}\right)^{1-\mathrm{i}\alpha(s)}$$

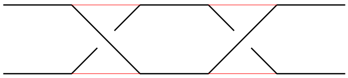
$$\alpha(s) := G\,\frac{(s-2m^2)^2-2m^4}{\sqrt{s(s-4m^2)}}$$

But...

Tricky to determine when eikonal exp. actually holds

Need to establish that ladders dominate to all orders

Counterexample $\rightarrow$ ϕ^3 scalar theory [Tiktopoulos-Treiman, Eichten-Jackiw]



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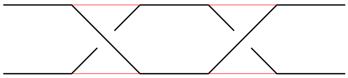
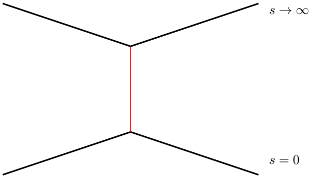


Requires ladders form geometric series $\xrightarrow{?}$ grav. scattering of massive spinning particles



[Guevara-Ochirov-Vines, Arkani-Hamed-Huang-O'Connell, Haddad, Chiodaroli-Johansson, ...]

Odd to need so much info (infinite no. of diagrams, non-trivial resummation) to get simple results...

Is there a better way?

Basic idea

At small momentum transfer, each particle looks like a fixed, classical object to the other

$2 \to 2$ eikonal scattering $\leftrightarrow$ $1 \to 1$ scattering in a curved background sourced by the other particle

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- Old idea: grav./electromag. scattering of massless scalars ['t Hooft, Jackiw-Kabat-Ortiz]
- Other ways to understand classicality of eikonal [Cristofoli-et al., TA-Gonzo-Kol,...]

Precise proposal

Consider $2 \to 2$ gravitational scattering with $s \gg -t$
incoming momenta p_μ, P_μ, outgoing momenta p'_μ, P'_μ

If eikonal exponentiation holds, then

1. Let particle P_μ be stationary source for Einstein equations
2. Compute $1 \to 1$ scattering amplitude M_2 of particle p_μ in this space-time (at large impact parameter, linear in G)
3. Eikonal amplitude given by:

$$M_2 = \frac{\hat{\delta}(p'_0 - p_0)}{4\,M}\,\mathcal{M}_{\text{eik}}$$

Need to clarify what we mean by $1 \to 1$ scattering amplitude in curved space-time...

...does **not** exist generically!

[Hawking, Gibbons, Woodhouse, Candelas,...]

Scattering in curved space-time

2-point amplitudes = quadratic action evaluated on-shell $\rightarrow$ boundary term

$$M_2 = \int_{\partial X} \mathrm{d}^3 y \, \sqrt{|h|} \, \bar{\phi}_{\rm in} \, n^\mu \, \nabla_\mu \, \phi_{\rm out}$$

- curved space-time (X, g)
- boundary ∂X w/ coords y^i, induced metric h, normal vector n^μ
- $\phi_{\rm in}$ incoming free field (Minkowski space)
- $\phi_{\rm out}$ outgoing field in curved space-time

Two cases to consider:

1. X admits S-matrix $\Rightarrow$ evaluate M_2 on all boundaries (finite + infinite)
2. X does not admit S-matrix $\Rightarrow$ evaluate M_2 only on linearised 'large-distance' boundaries (spatial or null infinity)

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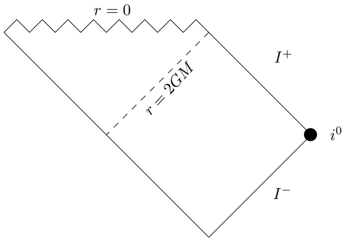
Ex: Schwarzschild black hole, event horizon has particle creation $\Rightarrow$ no S-matrix



Stationary backgrounds

Consider any large-distance, weakly curved stationary space-time $\Rightarrow$ linearised metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + h_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

Wave equation becomes

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\phi = h^{\mu\nu}\,\partial_\mu\partial_\nu\phi$$

Strategy

1. Make WKB ansatz for wave in z-direction

$$\phi_{\mathrm{WKB}} = \mathrm{e}^{\mathrm{i}\,\chi(x)/\hbar}\,, \qquad \chi(x) = \sum_{n=0}^{\infty} \chi_n(x)\,, \quad \chi_n(x) \sim G^n$$

subject to $\chi_0 = -p \cdot x$, $\chi_1(x^{\perp}, z = -\infty) = 0$

2. Solve for $\chi_1(x^{\perp}, z)$ in small angle approx.
3. Plug into M_2 at $r \to \infty$ boundary, find:

$$M_2 = -\frac{\mathrm{i}\, p_z\, \hat{\delta}(p_0' - p_0)}{\hbar^2} \int \mathrm{d}^2 x_{\perp} \mathrm{e}^{-\mathrm{i}\bar{q}_{\perp}\cdot x_{\perp}} \left(\mathrm{e}^{\mathrm{i}\,\chi_1(x^{\perp})/\hbar} - 1\right)$$

for $\chi_1(x^{\perp}) := \chi_1(x^{\perp}, z \to \infty)$

Upshot

$1 \to 1$ scattering on *any* stationary space-time structurally equivalent to eikonal amplitude:

$$M_2 = \frac{\hat{\delta}(p_0' - p_0)}{4\,M}\,\mathcal{M}_{\rm eik}$$

for M ADM mass of background, $P^\mu = M\,u^\mu$ momentum of background source

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Conjecture: If space-time has a source w/ 'particle-like' interpretation, this is a true equivalence.

Evidence

Passes the easy tests:

- Easy to show background diffeo invariance
- Shockwave background $\leftrightarrow$ massless scalar scattering (established long ago ['t Hooft])
- Schwarzschild background $\leftrightarrow$ massive scalar scattering
- Also detects cases that fail: scattering in pure ϕ^3 theory – $\phi \sim \frac{1}{r}$ not a solution with correct source

Scattering with spin

Scattering of mass m scalar with mass M *infinite spin* particle

Unclear how/if eikonal exponentiation occurs here
(cf., emergence from classical limit [Cristofoli-et al.])

Our prescription $\Rightarrow 1 \to 1$ scattering of mass m scalar in Kerr of mass M

Use harmonic coords, linear in G but all orders in spin [Vines]

Eikonal with spin

Gives exponentiation of GOV amplitude [Guevara-Ochirov-Vines, Chung-Huang-Kim-Lee, Arkani-Hamed-Huang-O'Connell]

$$\text{Eikonal phase} \;\; \chi_1 = -2\hbar \sum_{\pm} \alpha_\pm(s) \, \log(\mu \, |x_\perp \mp a_\perp|)$$

$$\text{with} \;\; \alpha_\pm(s) := \frac{G\, m\, M\, (1 \pm v)^2\, \gamma(v)}{2\hbar\, v}$$

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Eikonal amplitude surprisingly complicated:

factorized/KLT-like form involving products of confluent hypergeometric and Gamma functions

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The scattering on Kerr framework:

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- provides new, explicit expression for $\mathcal{M}_{\text{eik}}$

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The scattering on Kerr framework:

- suggests eikonal exponentiation holds with infinite spin multipole moments
- provides new, explicit expression for $\mathcal{M}_{\text{eik}}$
- surprising that amplitude has KLT-like structure (cf., [Verlinde-Verlinde])
- poles pick up complex part $\to$ instabilities in classical bound states? (cf., [Baumann-Chia-Stout-ter Haar])

Surprising application

What is the massless (ultraboosted) limit of the Kerr metric?

(sometimes called an impulsive 'gyraton' metric)

Lack of clarity in literature, many contradictory claims

[Ferrari-Pendenza, Balasin-Nachbagauer, Griffiths-Podolsky, Barrabes-Hogan, Frolov-Israel-Zelnikov,...]

More precisely...

Is there an *interesting* (i.e., with spin effects) massless limit of Kerr in the class

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathbb{M}} + G\,\delta(x^-)\,f(x^\perp)\,(\mathrm{d}x^-)^2?$$

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$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathbb{M}} + G\,\delta(x^-)\,f(x^\perp)\,(\mathrm{d}x^-)^2?$$

Spoiler alert: No.

Our prescription immediately implies

$$f(x^{\perp}) = -\frac{\chi_1(x^{\perp})}{p_+}$$

This gives a two way street:

1. Pick a metric, read off a 4-point amplitude from eikonal phase, see if it makes sense
2. Pick a 4-point amplitude, compute associated eikonal phase, look at associated metric

Results

Ultraboosting Kerr metric directly:

- along direction of spin $\rightarrow$ spin effects vanish
- perp. to spin $\rightarrow$ diffeo. equiv. to (non-spinning) shockwave
  - captures (naive) massless limit of GOV

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Ferrari-Pendenza gyraton:

- incorrect stress tensor (null dust)

Most interesting case is Balasin-Nachbagauer gyraton

obtained by ultraboosting *source* of Kerr [Israel]

$$f(x^{\perp}) = 8\,\log(\mu\,r) - 4\,\Theta(a-r)\left[2\,\log\left(\frac{r}{a+\sqrt{a^2-r^2}}\right) + \frac{\sqrt{a^2-r^2}}{a}\right]$$

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At large impact parameter $b \gg a \sim R_S$, no spin effects

but generically, incredibly simple 4-point amplitude

$$A_4(q) = G^2\,\frac{s^2}{q_{\perp}^2}\left(\frac{\sin(a\cdot q)}{a\cdot q} + \cos(a\cdot q)\right)$$

Lots to think about!

- Proof for this prescription
- Other backgrounds/eikonal amplitudes, beyond leading eikonal
- Exploring structure of Kerr/spinning eikonal: physics of poles, KLT-like structure...
- Extracting physical observables $\rightarrow$ *all-order* in G results [TA-Cristofoli-Ilderton]
- Higher-multiplicity $\rightarrow$ (part of) eikonal + emission [Lodone-Rychkov, Gruzinov-Veneziano, Ciafaloni-Colferai-Coradeschi-Veneziano, Di Vecchia-Heissenberg-Russo-Veneziano, TA-Ilderton-MacLeod,...]
- Applications to celestial holography [de Gioia-Raclariu, Gonzo-McLoughlin-Puhm]