A Tale of Tails
via Generalized Unitarity

MICHÈLE LEVI

U of OXFORD

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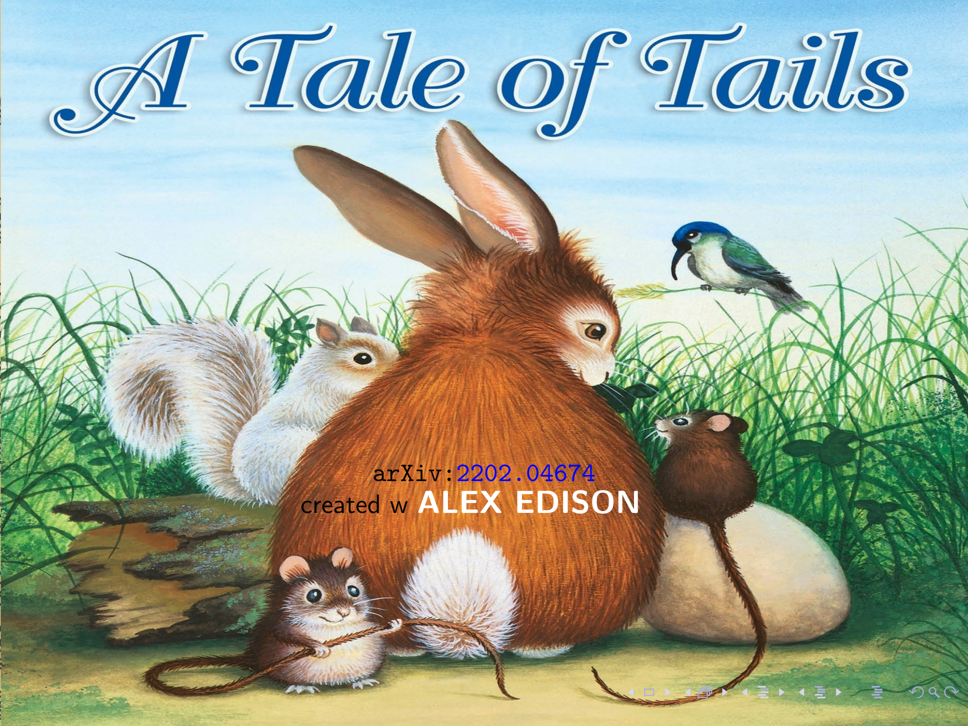

arXiv:2202.04674

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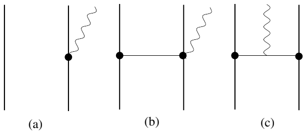
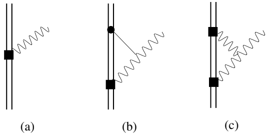
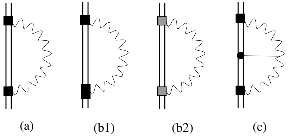
What We Are After...



$\Longrightarrow$



- Hereditary effects $\equiv$
  Scattering of radiation from binary off its own gravitational background
- Radiation-reaction starts at 2.5PN [nPN$\equiv (v/c)^{2n}$PN]
- Scattering starts at 4PN – affecting BOTH conservative dynamics and dissipative observables [4PN = current complete precision frontier]



Radiation-Reaction POV

Traditional GR

Trail of Tails

- Studied for decades. 1st paper in PN theory – *Blanchet & Damour* 1988
- Steady progress along the years led by *Luc Blanchet*, e.g. LRR 2014
- Latest milestone: Tail-of-Tail-of-Tail ("TTT") *Blanchet+* 2016 `arXiv:1607.07601` dissipation at +4.5(=7PN) though +4PN incomplete
- Some overlap with simple Self-Force theory results

Challenges/Strengths

- Laborious, slow, derivation "very" gauge-dependent...
- CONSISTENT & SOLID
- Hanldes separately either dissipative or conservative sides
- Output of direct use to LIGO+ (even mostly exclusive)

QFT Toolbox to the Rescue!

Goldberger & Ross 2009

- Formulated EFT for radiative sector, from *Goldberger & Rothstein* 2004
- Clear separation of scales and standard Feynman toolbox
- Background field method to handle mixing of field modes
- Targets only dissipative flux – but reproduced then state-of-the-art (TT)
- Significant milestone: Classical RG flow of quadrupole moment!

Galley+ 2009–17

- Time reversal is broken – invoked Closed-Time-Path (CTP) formalism!
- Defined proper conservative AND dissipative contributions
- Higher loops $\leftrightarrow$ Fewer graphs
- Reached only Tail level
- Several follow-ups by Foffa-Sturani+, Blumlein+

Our Mission

Goal: Set up Amplitudes-like framework to treat hereditary effects

Our Criteria:

1. Produce output for post-Newtonian binary inspirals – of use to LIGO+
2. Capture entirety of effects – both conservative + dissipative sides
3. Efficient & fast framework + machinery
4. Natural to generalize to all types of scattering + all PN orders

Other work in Amplitudes of different context and reach:

- Di-Vecchia+ `arXiv:2101.05772`, `arXiv:2104.03256`
- Bern+ `arXiv:2112.10750`
- Dlapa+ `arXiv:2112.11296`
- Jakobsen+ `arXiv:2207.00569`

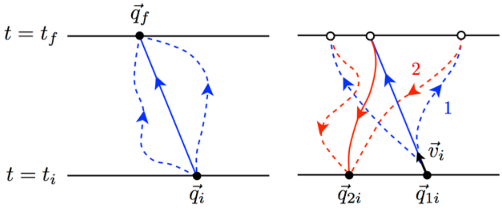
CTP 101

Closed-Time Path (CTP) Formalism

- Time reversal is broken
  $\rightarrow$ Initial-value problem ("in-in") instead of boundary-value problem (in-out)
- Non-equilibrium QFT – Schwinger/Keldysh formalism 1961/1964/5
- Degrees of freedom are formally doubled, and the action becomes:

$$S_{\text{CTP}}[\{\}_1,\{\}_2] \equiv S[\{\}_1] - S^*[\{\}_2]$$

- Worked out for classical context by Galley 2012, Galley+ arXiv:1412.3082



From *Galley*, arXiv:1210.2745, PRL 2013

CTP in Composite-Particle EFT

CTP in Worldline EFT

- Switch (from {1,2}) to {+,-} basis
- For generic fields: $\phi_+ \equiv (\phi_1 + \phi_2)/2$, $\phi_- \equiv \phi_1 - \phi_2$
- For discrete DOFs: $q_+ \equiv (q_1 + q_2)/2$, $q_- \equiv q_1 - q_2$
- Classical propagator-matrix is off-diagonal

$$G_{+-} = \begin{pmatrix} 0 & G_{\text{adv}} \\ G_{\text{ret}} & 0 \end{pmatrix}$$

- Retarded and advanced propagators

$$G_{ret/adv}(x - x') = \int \frac{d^D p}{(2\pi)^D} \frac{e^{-ip_\mu (x-x')^\mu}}{(p_0 \pm i\epsilon)^2 - \vec{p}^{\,2}}$$

$+i\epsilon$ for retarded, $D \equiv d + 1$ with d for number of spatial dimensions

Integrating Out via Generalized Unitarity (GU)

Starting from

$$S_{\text{eff(c)}}[g_{\mu\nu}, y_c^\mu, e_c{}^\mu{}_A] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R\,[g_{\mu\nu}] + \,S_{\text{pp(c)}}[g_{\mu\nu}(y_c), y_c^\mu, e_c{}^\mu{}_A](\sigma_c)$$

with

$$S_{\text{pp(c)}}[h_{\mu\nu}, y_c^\mu, e_c{}^\mu{}_A](t) = -\int dt\sqrt{g_{00}}\left[E(t) - \sum_{l=2}^{\infty}\frac{1}{l!}I^L(t)\nabla_{L-2}E_{i_{l-1}i_l} + \ldots\right.$$

we integrate out all field modes to get an effective theory of dynamical multipoles.

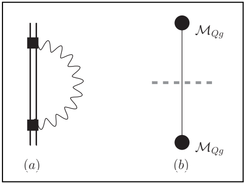
Integrating out via GU (instead of Feynman)

Multi-loop tools [*Smirnov*] + Generalized Unitarity:
Bern, Dixon, Dunabr, Kosower 1994, *Britto, Cachazo, Feng* 2004,...

$$S_{\text{eff}} = \int\frac{d\omega}{2\pi}\sum_{i\in\text{MI}} c_i(d,\omega)\,\mathcal{I}_i$$

$\{\mathcal{I}_i\}$ complete set of master integrals that span problem

Radiation Reaction



EFT and Amplitudes POVs

$$F^{(1)}(\lambda;\omega^2) = \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{1}{(-\ell_E^2+\omega^2)^\lambda} = R(d,\lambda,\omega^{2n(\lambda)})F^{(1)}(1;\omega^2)$$

$$S_{\mathrm{RR}} = \int \frac{\mathrm{d}\omega}{2\pi} c_{\mathrm{RR}}(\omega) F^{(1)}(1;\omega^2)$$

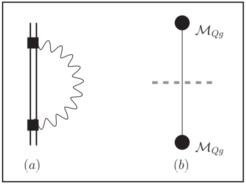
- Cuts assembled from tree amplitudes by public code `IncreasingTrees` [Edison+Teng 2020] via graviton state-sewing $\sum_{\text{states}} \varepsilon_k^{\mu\nu}\varepsilon_k^{\alpha\beta *} \equiv \mathcal{P}_k^{\mu\nu;\alpha\beta}$
- $\lambda_Q \equiv \sqrt{2\pi G_N}$

$$\mathcal{M}_{Qg} \equiv \lambda_Q J^{\mu\nu}\epsilon_{\mu\nu} \equiv \lambda_Q J^{\mu\nu}\varepsilon_\mu\varepsilon_\nu = -\lambda_Q I^{ab} \times \left(k_0 k_a \varepsilon_0 \varepsilon_b + k_0 k_b \varepsilon_0 \varepsilon_a - k_a k_b \varepsilon_0\varepsilon_0 - k_0 k_0 \varepsilon_a \varepsilon_b\right)$$

-

$$\mathcal{C}_{\mathrm{RR}} = \lambda_Q^2 J_1^{\mu\nu}\ \mathcal{P}^{\mu\nu;\alpha\beta} J_2^{\alpha\beta}\Big|_{P_\ell=\ell_E^2-\omega^2=0} = \delta(\ell_E^2-\omega^2)\lambda_Q^2\left(J_1^{\mu\nu}J_2^{\mu\nu} - \frac{J_1^{\mu\mu}J_2^{\nu\nu}}{d-1}\right)$$

Radiation Reaction



$$\mathcal{C}_{\rm RR} = \delta(P_\ell)\lambda_Q^2 \frac{(d+1)(d-2)}{(d+2)(d-1)}\omega^4 \kappa_{ab}(\omega)$$

with trace of CTP quadrupole DOFs
$\kappa_{ab}(\omega) = I_a^{ij}(-\omega) I_{ij,b}(\omega)$
with $a, b \in \{+,-\}$

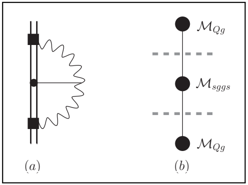
- $$S_{\rm RR} = \frac{2\pi G_N}{5}\int \frac{\mathrm{d}\omega}{2\pi}\,\omega^4 \sum_{a,b\in\{+,-\}} \kappa_{ab}(\omega) F^{(1)}(1_{ab})$$

  with retarded and advanced propagators $F^{(1)}(1_{-+}/1_{+-}) \equiv F^{(1)}(1; (\omega \pm i\epsilon)^2)$

- $$S_{\rm RR} = -i\frac{G_N}{5}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\omega^5 I_-^{ij}(-\omega) I_{+,ij}(\omega)$$

- ϵ_d contributions will matter for higher-order renormalization

Tail



3 momenta invariants but plugging into FIRE:

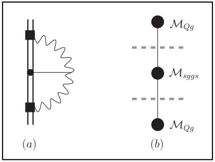
$$F^{(2)}(1_X, 1_Y, 0) = \int \frac{\mathrm{d}^d\ell_1\,\mathrm{d}^d\ell_2}{(2\pi)^{2d}} \frac{1}{(-\ell_1^2+\omega_X^2)(-\ell_2^2+\omega_Y^2)}$$
$$= F^{(1)}(1;\omega_X^2)F^{(1)}(1;\omega_Y^2)$$

- $$S_\mathrm{T} = \int \frac{\mathrm{d}\omega}{2\pi} c_\mathrm{T}(\omega) F^{(2)}(1_X, 1_Y, 0)$$
- Now we need an amplitude of 2 massive scalars and 2 gravitons. It is extracted from `IncreasingTrees`, and expanded in the large-mass limit:

$$\mathcal{M}_{sggs}(m_s\to\infty) = \frac{\lambda_g\lambda_E}{\omega_{k_2}^2}\frac{1}{2(k_2^\mu k_{3,\mu})}\Big[(k_2^\mu k_{3,\mu})\varepsilon_2^0\varepsilon_3^0 + \omega_{k_2}((\varepsilon_3^\mu k_{2,\mu})\varepsilon_2^0 - (\varepsilon_2^\mu k_{3,\mu})\varepsilon_3^0) - \omega_{k_2}^2(\varepsilon_2^\mu\varepsilon_{3,\mu})\Big]^2 + \mathcal{O}(m_s^{-1})$$

- $\mathcal{M}_{sgs} \equiv \lambda_E(p^\mu p^\nu/m_s^2)\varepsilon^\mu\varepsilon^\nu$ fixes $\lambda_E \equiv -\sqrt{8\pi G_N}E$, similarly $\lambda_g \equiv -\sqrt{32\pi G_N}$

Tail



- $$\mathcal{C}^{(2)}_{1,1,0} = \sum_{\text{states}} \mathcal{M}_{Qg(-\omega)} \mathcal{M}_{sggs} \mathcal{M}_{Qg(\omega)} \Big|_{\substack{P_{\ell_1}=0, P_{\ell_2}=0 \\ m_s \to \infty}}$$
$$= \lambda_Q^2\, \delta(P_{\ell_1}) \delta(P_{\ell_2})$$
$$\times\, J^{\mu\nu}_{I(-\omega)} P^{\mu\nu;\alpha\beta} \mathcal{M}^{\alpha\beta;\gamma\sigma}_{sggs} P^{\gamma\sigma;\rho\tau} J^{\rho\tau}_{I(\omega)} \Big|_{m_s \to \infty}$$

- Reducing again and evaluating integrals with appropriate CTP prescriptions:

$$S_{\mathrm{T}} = \frac{2}{5} G_N^2 E \int \frac{\mathrm{d}\omega}{2\pi} \omega^6 \kappa_{-+}(\omega) \left[\frac{1}{\epsilon_d} + \log\left(\frac{\omega^2}{\mu^2} \right) - i\pi\, \mathrm{sgn}(\omega) \right]$$

- In general, poles will cancel out from renormalization and lower-order zeros

Derivation of Physics

Conservtaive Dynamics

From the causal effective actions radiation-reaction forces are obtained by varying wrt CTP DOFs, $\{q^i_\pm\}$, then taking the physical limit, $q^i_+ \to q^i$ and $q^i_- \to 0$.

Dissipative Observables

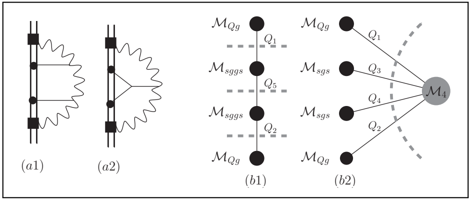
- No proper derivation of dissipative observables from CTP effective actions existed in the literature.
- We worked out generic derivation starting from generalized Noether theorem:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\partial L}{\partial t} + \dot{q}^i \left[\frac{\partial K}{\partial q^i_-}\right]_{\mathrm{PL}} + \ddot{q}^i \left[\frac{\partial K}{\partial \dot{q}^i_-}\right]_{\mathrm{PL}} + \dots$$

- L conservative potential = one of time histories
- K non-conservative potential = 2 time histories CANNOT be separated
- CTP quadrupoles as generalized DOFs

$$\int \mathrm{d}t \frac{\mathrm{d}E}{\mathrm{d}t} = \Delta E = \int \mathrm{d}\omega \frac{\mathrm{d}E}{\mathrm{d}\omega}$$

Tail-of-Tail (TT)



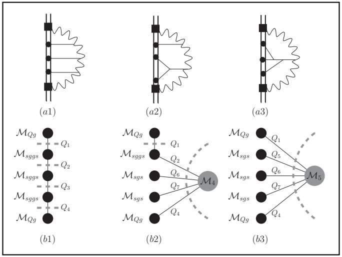
- After reduction of the integral family:

$$S_{\rm TT} = \int \frac{\mathrm{d}\omega}{2\pi} \left[c_1(\omega) F^{(3)}(1,1,0,0,1,0) + c_2(\omega) F^{(3)}(1,1,1,1,0,0) \right]$$

- We need the 4-graviton amplitude but no special kinematics needed there
- No effective action previously derived

$$S_{\rm TT} = \frac{107}{175}\, G_N^3 E^2 \int \frac{\mathrm{d}\omega}{2\pi}\, \omega^7 \kappa_{-+}(\omega) \left[\pi\, \mathrm{sgn}(\omega) + i \left[\frac{2}{3\,\epsilon_d} + \log\left(\frac{\omega^2}{\mu_1^2}\right)\right]\right]$$

Tail-of-Tail-of-Tail (TTT)



$$S_{\rm TTT} = \int \frac{\mathrm{d}\omega}{2\pi} \Big[c_1\, F^{(4)}_{1,1,1,1,0,0,0,0,0,0,0} + c_2\, F^{(4)}_{1,0,0,1,1,1,1,0,0,0} + \big(c_3\, F^{(4)}_{1,0,1,1,1,1,0,0,0,0} + c_4\, F^{(4)}_{1,1,0,1,0,1,1,0,0,0} \big) \Big]$$

- No effective action previously derived
- Result is quite lengthy with ζ_2 and $\zeta_3 \equiv$Apéry's constant

Dissipative Observables

Power Spectra

With $\kappa(\omega) \equiv I^{ij}(-\omega) I_{ij}(\omega)$

$$\begin{aligned}
\frac{\mathrm{d}E_{\mathrm{RR}}}{\mathrm{d}\omega} &= -\frac{G_N}{5\pi}\omega^6 \kappa(\omega) \\
\frac{\mathrm{d}E_{\mathrm{T}}}{\mathrm{d}\omega} &= -\frac{2}{5} G_N^2 E\, \omega^7 \kappa(\omega) \\
\frac{\mathrm{d}E_{\mathrm{TT}}}{\mathrm{d}\omega} &= \frac{428}{525\pi} G_N^3 E^2 \omega^8 \log(\omega/\mu_1) \kappa(\omega) \\
\frac{\mathrm{d}E_{\mathrm{TTT}}}{\mathrm{d}\omega} &= \frac{856}{525} G_N^4 E^3 \omega^9 \log(\omega/\mu_2) \kappa(\omega)
\end{aligned}$$

For T and TT – agreement w Bini+Geralico 2021, TTT – new

Flux for Circular Orbits

We can specialize to circular orbit with orbital frequency Ω.
Perfect agreement of our results with traditional Self-Force/PN theory.

Summary & Outlook

- Effective actions that encapsulate all conservative and dissipative physics, including TT, TTT – never previously derived
- Derivation of dissipative observables matches with results where available
- Set up Amplitudes-like framework + machinery for hereditary effects: Directly applicable to PN binary inspirals, proven to 4-loop, efficient & fast
- Competitive with traditional GR and standard EFT methods

Prospects:

- Scattering from generic multipole source off any multipole source background
- Extension to all PN orders

Both entail tree amplitudes with massive particles of any spin!