From Scattering Amplitudes to Classical Gravitational *N-body* Potentials

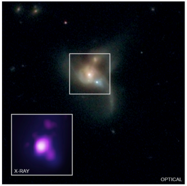
Callum R. T. Jones

based on **[2208.02281]** with Mikhail Solon

UCLA

Motivation

$N \geq 3$-body systems are commonplace in our universe...



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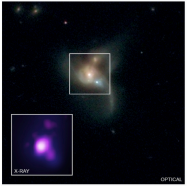

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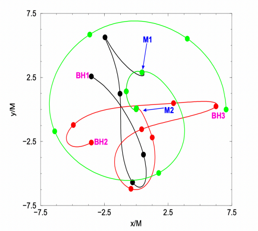
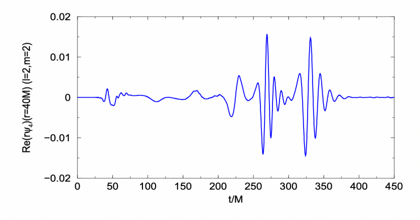
... and exhibit incredibly rich physical properties





[Campanelli, Lousto, Zlochower 0710.0879]

Many-Body Gravitational Systems

Newtonian gravity is linear: *"more is the same"*

$$V_{\text{N-body}}\left(\{\vec{p},\vec{x}\}\right) = -\sum_{i<j} \frac{G m_i m_j}{r_{ij}}$$

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Basic Question: what is the gravitational potential between N compact bodies?

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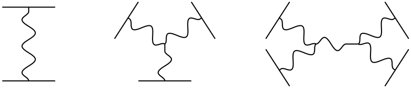
- Astrophysics of N-body systems, *dense star clusters, hierarchical triples, calculating gravitational waveforms...*

PM and PN for N-bodies



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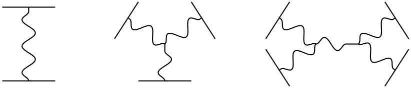

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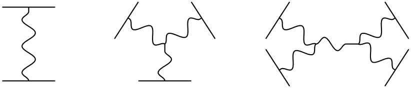

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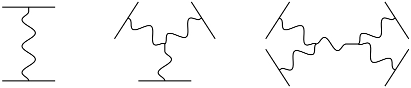

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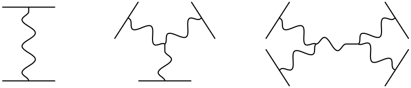

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[Newton 1687]

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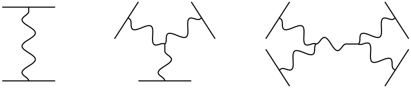

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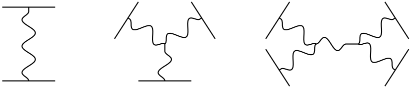

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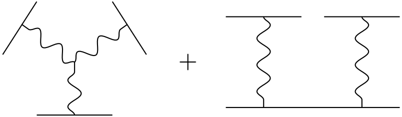
$$V^{2\text{PN}}_{\text{N-body}}(\vec{x},\vec{p}) = \underbrace{\sum_{ij} V^{(2)}_{ij}(\vec{x},\vec{p})}_{\text{two-loop}} + \underbrace{\sum_{ijk} V^{(3)}_{ijk}(\vec{x},\vec{p})}_{\text{one-loop}} + \underbrace{\sum_{ijkl} V^{(4)}_{ijkl}(\vec{x},\vec{p})}_{\text{tree-level}}$$ [Ohta et al. 1974; Schafer 1987; Chu 2008]

Can *scattering amplitudes* methods help push the state-of-the-art for N-body dynamics in GR?

Potentials from Scattering Amplitudes

Potential is calculated by matching **Feynman** amplitudes (*full theory*)



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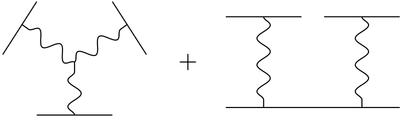

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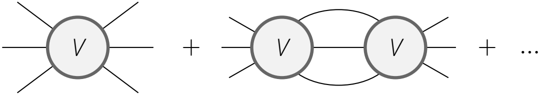

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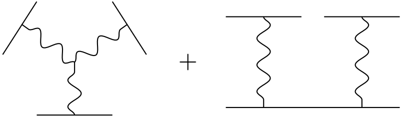

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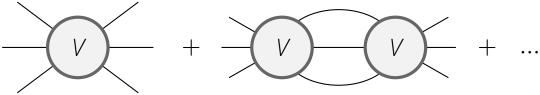
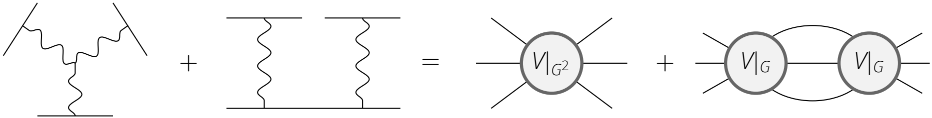


in the limit of small momentum transfer $|\vec{q}_1| \sim |\vec{q}_2| \sim |\vec{q}_3| \sim 0$.

Potentials from Scattering Amplitudes

Example: Three-body potential at $\mathcal{O}\left(G^2 v^\infty\right)$ (2PM)



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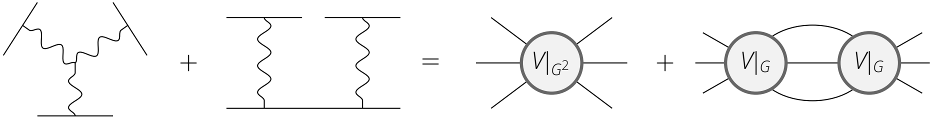

Born series solution of *relativistic* Lippmann-Schwinger equation

$$T|_{G^2}\left(\vec{p}_i, \vec{q}_i\right) \stackrel{!}{=} -V|_{G^2}\left(\vec{p}_i, \vec{q}_i\right) - \int_{\vec{k}_i} \frac{V|_G\left(\vec{k}_i, \vec{p}_i' - \vec{k}_i\right) V|_G\left(\vec{p}_i, \vec{k}_i - \vec{p}_i\right)}{\sum_i \left[E_i\left(\vec{p}_i\right) - E_i\left(\vec{k}_i\right)\right] + i\epsilon} + \mathcal{O}\left(G^3\right)$$

where $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$.

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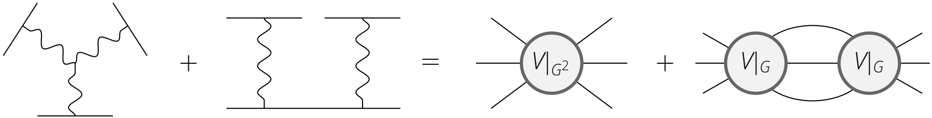

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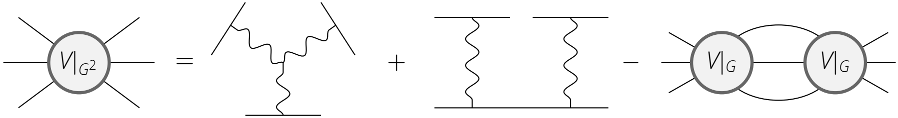

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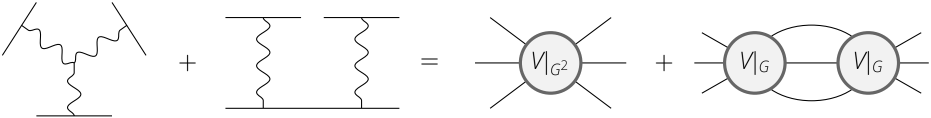

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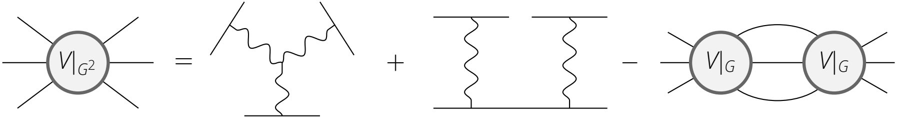

Take $\hbar \to 0$ limit $\Leftrightarrow$ expand to $\mathcal{O}\left(|\vec{q}|^{-4}\right)$.

More is different

Iteration Contributions

Momentum space 3-body Hamiltonian:

$$H(\vec{p},\vec{q}) = \sum_i \sqrt{|\vec{p}_i|^2 + M_i^2} + \sum_{ijk} \delta^{(3)}\left(\vec{q}_i + \vec{q}_j\right) \delta^{(3)}\left(\vec{q}_k\right) V_{ij}^{(2)}\left(\vec{p},\vec{q}\right) + \delta^{(3)}\left(\vec{q}_1 + \vec{q}_2 + \vec{q}_3\right) V^{(3)}\left(\vec{p},\vec{q}\right)$$

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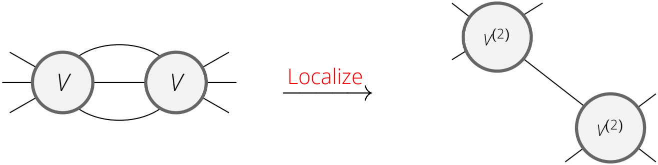

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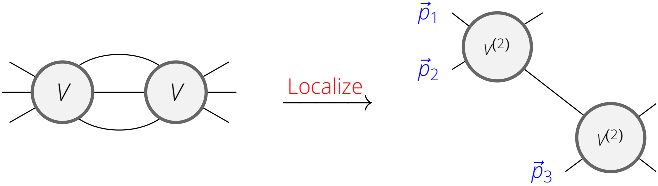

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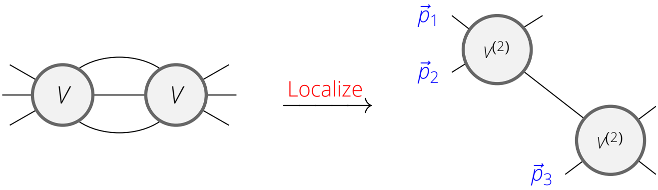

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k-body potential, $k \leq N$ in *general frame* needed as input for *N*-body calculation.

Gauge Dependence and Energy Conservation

Why is the effective potential not gauge invariant?

$$M_2\left(\vec{p}_i, \vec{q}_i\right) \stackrel{!}{=} -V^{(2)}\left(\vec{p}_i, \vec{q}_i\right) + \mathcal{O}\left(G^2\right).$$

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$$V^{(2)}\left(\vec{p}_i,\vec{q}_i\right) \sim V^{(2)}\left(\vec{p}_i,\vec{q}_i\right) + \frac{G}{|\vec{q}_1|^4}\left(\frac{\vec{p}_1\cdot\vec{q}_1}{E_1(\vec{p}_1)} - \frac{\vec{p}_2\cdot\vec{q}_1}{E_2(\vec{p}_2)}\right)^2.$$

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$$\vec{q}_1 + \vec{q}_2 = 0, \qquad \sum_{i=1}^{2}\left[E_i(\vec{p}_i) - E_i(\vec{p}_i - \vec{q}_i)\right] = \frac{\vec{p}_1 \cdot \vec{q}_1}{E_1(\vec{p}_1)} - \frac{\vec{p}_2 \cdot \vec{q}_1}{E_2(\vec{p}_2)} + \mathcal{O}\left(|\vec{q}|^2\right) = 0.$$

Potential defined as continuation from **energy-conservation surface.** *Example:*

$$V^{(2)}\left(\vec{p}_i, \vec{q}_i\right) \sim V^{(2)}\left(\vec{p}_i, \vec{q}_i\right) + \frac{G}{|\vec{q}_1|^4}\left(\frac{\vec{p}_1 \cdot \vec{q}_1}{E_1(\vec{p}_1)} - \frac{\vec{p}_2 \cdot \vec{q}_1}{E_2(\vec{p}_2)}\right)^2.$$

Canonical way to use this freedom

$$\vec{p}_i \cdot \vec{q}_j \overset{!}{=} \frac{E_i(\vec{p}_1)\left((\vec{p}_1 + \vec{p}_2) \cdot \vec{q}_j\right)}{E_1(\vec{p}_1) + E_2(\vec{p}_2)} + \mathcal{O}\left(|\vec{q}|^2\right)$$

Isotropic gauge: $\vec{p}_1 + \vec{p}_2 = 0$, can remove all $\vec{p} \cdot \vec{q}$

$$V^{(2)}_{\text{iso}}\left(p^2, q^2\right) = \frac{G}{q^2} c_1(p^2) + \frac{G^2}{q} c_2(p^2) + G^3 \log(q) c_3(p^2) + G^4 q c_4(p^2) + \ldots$$

Gauge Dependence and Energy Conservation

Why is the effective potential not gauge invariant?

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(Generalized) isotropic gauge: $\vec{p}_1+\vec{p}_2\neq 0$, can rewrite all $\vec{p}\cdot\vec{q}$ as $\left(\vec{p}_1+\vec{p}_2\right)\cdot\vec{q}_1$.

2-body 1PM Hamiltonian in general frame (generalized isotropic gauge):

$$V^{(2)}_{\text{gen-iso}}\left(\vec{p}_i,\vec{q}_i\right) = -\frac{4\pi G\left(m_1^2m_2^2-2\left(\vec{p}_1\cdot\vec{p}_2-E_1(\vec{p}_1)E_2(\vec{p}_2)\right)^2\right)}{E_1(\vec{p}_1)E_2(\vec{p}_2)\left[\underbrace{\left(\frac{(\vec{p}_1+\vec{p}_2)\cdot\vec{q}_1}{E_1(\vec{p}_1)+E_2(\vec{p}_2)}\right)^2}_{\text{"Energy transfer"}}-|\vec{q}_1|^2\right]}.$$

Gauge Dependence and Energy Conservation

Why is the effective potential not gauge invariant?

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$$\vec{q}_1+\vec{q}_2=0, \qquad \sum_{i=1}^{2}\left[E_i(\vec{p}_i)-E_i(\vec{p}_i-\vec{q}_i)\right] = \frac{\vec{p}_1\cdot\vec{q}_1}{E_1(\vec{p}_1)} - \frac{\vec{p}_2\cdot\vec{q}_1}{E_2(\vec{p}_2)} + \mathcal{O}\left(|\vec{q}|^2\right) = 0.$$

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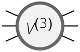
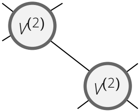
Gauge Dependence and Energy Conservation

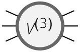
$M_3\left(\vec{p}_i, \vec{p}_i'\right) \overset{!}{=} -$ 

$-$ 

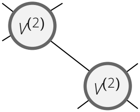
$+ \quad \dots \quad + \quad \mathcal{O}\left(G^3\right)$

Gauge Dependence and Energy Conservation

$$M_3\left(\vec{p}_i, \vec{p}_i'\right) \overset{!}{=} -$$



$$-$$



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3-body (classical) conservation of energy ambiguity. *Example:*

$$V^{(3)}\left(\vec{p}_i, \vec{q}_i\right) \sim V^{(3)}\left(\vec{p}_i, \vec{q}_i\right) + \frac{G^2}{|\vec{q}_1|^2|\vec{q}_2|^2|\vec{q}_3|^2} \underbrace{\left(\frac{\vec{p}_1 \cdot \vec{q}_1}{E_1(\vec{p}_1)} + \frac{\vec{p}_2 \cdot \vec{q}_2}{E_2(\vec{p}_2)} + \frac{\vec{p}_3 \cdot \vec{q}_3}{E_3(\vec{p}_3)}\right)^2}_{\sim \sum_i \left[E_i(\vec{p}_i) - E_i(\vec{p}_i')\right]} .$$

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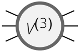
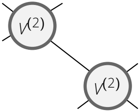
$M_3(\vec{p}_i, \vec{p}_i') \overset{!}{=}$ − 

− 

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How do we compare potentials calculated in different gauges?

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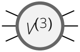
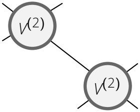
$M_3\left(\vec{p}_i, \vec{p}_i'\right) \overset{!}{=} -$ 

$-$ 

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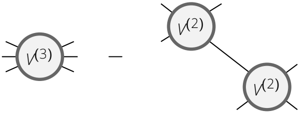
How do we compare potentials calculated in different gauges?

- Calculate explicit canonical transformations.

Gauge Dependence and Energy Conservation

$$M_3\left(\vec{p}_i, \vec{p}_i'\right) \overset{!}{=} \quad - \quad [V^{(3)}] \quad - \quad [V^{(2)} - V^{(2)}] \quad + \quad \cdots \quad + \quad \mathcal{O}\left(G^3\right)$$



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How do we compare potentials calculated in different gauges?

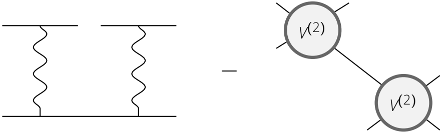
- Calculate explicit canonical transformations.
- Calculate gauge invariant physical observables

$$\Delta\mathcal{O} \overset{\hbar\to 0}{=} \langle \mathrm{in}|S^\dagger \mathbb{O} S|\mathrm{in}\rangle - \langle \mathrm{in}|\mathbb{O}|\mathrm{in}\rangle$$

classical scattering amplitude is an integral kernel for classical observables, *e.g.* Δp_i^μ and $\Delta J_i^{\mu\nu}$ [Kosower, Maybee, O'Connell 2018].

Iteration Subtraction

Imperfect cancellation between *full theory* and *EFT iteration*



$$- \quad \neq \quad 0$$

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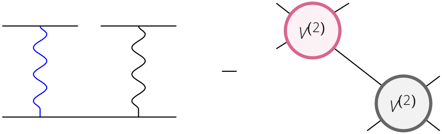

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Misalignment of graviton poles:

$$\frac{\mathcal{N}_{\text{Full}}}{\left[\left(\frac{\vec{p}_1\cdot\vec{q}_1}{E_1}\right)^2 - \vec{q}_1^{\,2}\right](p_2\cdot q_1)\left[\left(\frac{\vec{p}_3\cdot\vec{q}_3}{E_3}\right)^2 - \vec{q}_3^{\,2}\right]} - \frac{\mathcal{N}_{\text{EFT}}}{\left[\left(\frac{(\vec{p}_1+\vec{p}_2)\cdot\vec{q}_1}{E_1+E_2}\right)^2 - \vec{q}_1^{\,2}\right](p_2\cdot q_1)\left[\left(\frac{(\vec{p}_2+\vec{p}_3)\cdot\vec{q}_3}{E_2+E_3}\right)^2 - \vec{q}_3^{\,2}\right]}.$$

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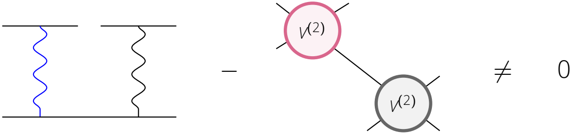

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Physically acceptable 3-body gauge fixing:

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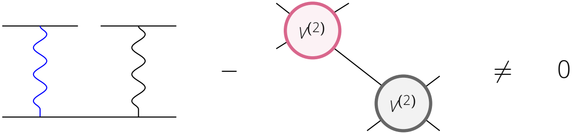

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Physically acceptable 3-body gauge fixing:

1. Cancellation of spurious *matter singularities*

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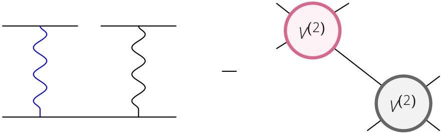

$\neq 0$

Misalignment of graviton poles:

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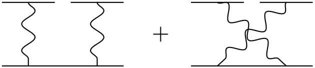
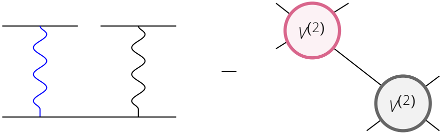
2. Cancellation of *super-classical* $\mathcal{O}\left(q^{-5}\right)$ terms



$= \mathcal{O}\left(q^{-4}\right)$

Iteration Subtraction

Imperfect cancellation between *full theory* and *EFT iteration*



$\neq \quad 0$

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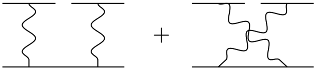

$= \mathcal{O}\left(q^{-4}\right)$

3. Manifest S_N permutation symmetry $\{\vec{p}_i, \vec{q}_i, m_i, Q_i\}$.

N-body PN Fourier Transforms

Really want the potential in *position space*, e.g. for N-body simulation.

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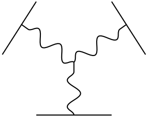

$$\prod_{i=1}^{3}\left[\int \frac{\mathrm{d}^3\vec{q}_i}{(2\pi)^3} e^{i\vec{x}_i\cdot\vec{q}_i}\right] \frac{(2\pi)^3\delta^{(3)}\left(\vec{q}_1+\vec{q}_2+\vec{q}_3\right)}{|\vec{q}_1|^2|\vec{q}_2|^2|\vec{q}_3|^2}$$

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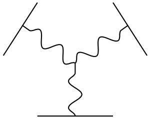

$$\prod_{i=1}^{3}\left[\int \frac{\mathrm{d}^3\vec{q}_i}{(2\pi)^3} e^{i\vec{x}_i\cdot\vec{q}_i}\right] \frac{(2\pi)^3\delta^{(3)}\left(\vec{q}_1+\vec{q}_2+\vec{q}_3\right)}{|\vec{q}_1|^2|\vec{q}_2|^2|\vec{q}_3|^2} \propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \frac{1}{|\vec{x}_0-\vec{x}_1||\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{x}_3|}$$

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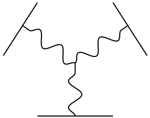

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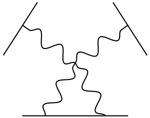

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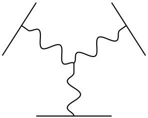

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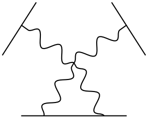
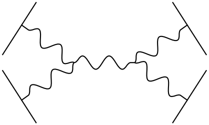


$$\propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0\,\frac{1}{|\vec{x}_0-\vec{x}_1|^2|\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{x}_3|}$$

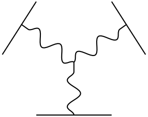


$$\propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \int \mathrm{d}^{3-2\epsilon}\vec{y}_0\,\frac{1}{|\vec{x}_0-\vec{x}_1||\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{y}_0||\vec{y}_0-\vec{x}_3||\vec{y}_0-\vec{x}_4|}$$

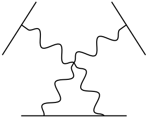
N-body PN Fourier Transforms

Really want the potential in *position space*, e.g. for N-body simulation.

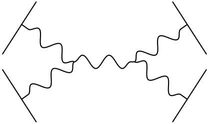
Post-Minkowskian N-body Fourier transforms may be too difficult. Best we can do is velocity expand before integration.



$$\prod_{i=1}^{3}\left[\int \frac{\mathrm{d}^3\vec{q}_i}{(2\pi)^3} e^{i\vec{x}_i\cdot\vec{q}_i}\right] \frac{(2\pi)^3\delta^{(3)}\left(\vec{q}_1+\vec{q}_2+\vec{q}_3\right)}{|\vec{q}_1|^2|\vec{q}_2|^2|\vec{q}_3|^2} \propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \frac{1}{|\vec{x}_0-\vec{x}_1||\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{x}_3|}$$



$$\propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \frac{1}{|\vec{x}_0-\vec{x}_1|^2|\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{x}_3|}$$



$$\propto \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \int \mathrm{d}^{3-2\epsilon}\vec{y}_0 \frac{1}{|\vec{x}_0-\vec{x}_1||\vec{x}_0-\vec{x}_2||\vec{x}_0-\vec{y}_0||\vec{y}_0-\vec{x}_3||\vec{y}_0-\vec{x}_4|}$$

Leading 4-body potential is not known in closed form! **[Chu 2008]**

N-body PN Fourier Transforms

Needed 3-body integrals

$$I_3^{d=3-2\epsilon}[\alpha_1,\alpha_2,\alpha_3] \equiv \int \mathrm{d}^{3-2\epsilon}\vec{x}_0 \frac{1}{|\vec{x}_0-\vec{x}_1|^{2\alpha_1}|\vec{x}_0-\vec{x}_2|^{2\alpha_2}|\vec{x}_0-\vec{x}_3|^{2\alpha_3}}.$$

N-body PN Fourier Transforms

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At $\mathcal{O}\left(G^2\right)$ need $\alpha_i \in \mathbb{Z} + \frac{1}{2}$: what is the space of functions?

N-body PN Fourier Transforms

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$$I_3^{d=3-2\epsilon}\left[\alpha_1,\alpha_2,\alpha_3\right] = \frac{A}{\epsilon} + B + C\log\left(r_{12}+r_{13}+r_{23}\right) + \mathcal{O}\left(\epsilon\right)$$

where A, B, C are polynomials in r_{ij} **[Loebbert, Plefka, Shi, Wang 2020]**.

N-body PN Fourier Transforms

Needed 3-body integrals

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Why is the answer so simple?

N-body PN Fourier Transforms

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where A, B, C are polynomials in r_{ij} [Loebbert, Plefka, Shi, Wang 2020].

Why is the answer so simple?

Reducible to *triple-K* integrals [Bzowski, McFadden, Skenderis 2013]

$$I_3^d[\alpha_1,\alpha_2,\alpha_3] \propto \int_0^\infty \mathrm{d}x x^{\frac{d}{2}-1} K_{\frac{d}{2}-\alpha_2-\alpha_3}(r_{23}x)\, K_{\frac{d}{2}-\alpha_1-\alpha_3}(r_{13}x)\, K_{\frac{d}{2}-\alpha_1-\alpha_2}(r_{12}x)$$

N-body PN Fourier Transforms

N-body PN Fourier Transforms

General α_i: formal hypergeometric series expression [Boos, Davydychev 1991]

$$I_3^{d=3-2\epsilon}[\alpha_1, \alpha_2, \alpha_3] \sim F_4\left[\ldots; \frac{r_{12}^2}{r_{23}^2}, \frac{r_{13}^2}{r_{23}^2}\right] + 3 \text{ more terms},$$

has a region of convergence $r_{12} + r_{13} < r_{23}$

N-body PN Fourier Transforms

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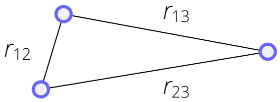
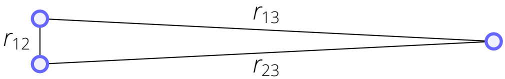


the *reverse triangle inequality!* Converges nowhere in physical region.

N-body PN Fourier Transforms

Construct new, convergent series expansion in *hierarchical limit* $r_{12} \ll r_{13} \sim r_{23}$



N-body PN Fourier Transforms

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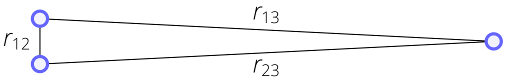

Evaluate order-by-order using *method of regions* [Beneke, Smirnov 1998]

$$I_3^{3-2\epsilon}\left[1,\frac{1}{2},\frac{1}{2}\right] = \frac{2\pi(2r_{13}^2 - \vec{x}_{12}\cdot\vec{x}_{13})}{3r_{13}^3\epsilon} + \frac{4\pi}{9r_{13}^3}\left[3(\vec{x}_{12}\cdot\vec{x}_{13} - 3r_{13}^2)\log(r_{12}) + \vec{x}_{12}\cdot\vec{x}_{13} + 3r_{13}^2\log(r_{13})\right] + \ldots$$

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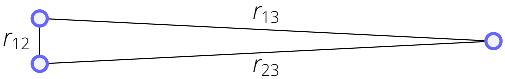

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Converges in the physical region for generic α_i.

N-body PN Fourier Transforms

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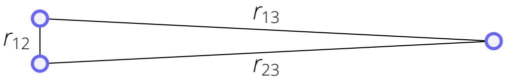

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Converges in the physical region for generic α_i.

Provides all necessary Fourier transform integrals at $\mathcal{O}\left(G^3\right)$ in hierarchical limit.

Putting it all together

Main Result: 3-body Hamiltonian at 2PM

$$H(\vec{p},\vec{q}) = \sum_i \sqrt{|\vec{p}_i|^2 + M_i^2} + \sum_{(ijk)\in S_3} \left[(2\pi)^6 \delta^{(3)}\left(\vec{q}_i + \vec{q}_j\right) \delta^{(3)}\left(\vec{q}_k\right) \frac{1}{2} V_{ij}^{(2)}(\vec{p},\vec{q}) + (2\pi)^3 \delta^{(3)}\left(\vec{q}_1 + \vec{q}_2 + \vec{q}_3\right) V_{ijk}^{(3)}(\vec{p},\vec{q}) \right]$$

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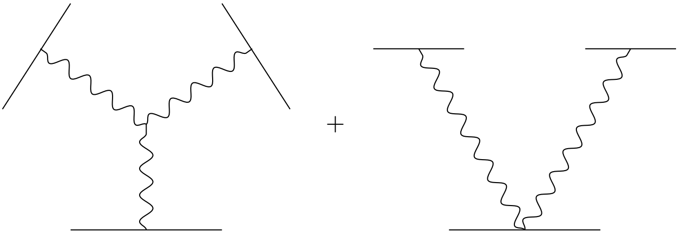
$$V_{ijk}^{(3)}(\vec{p},\vec{q}) = {\color{red} V_{Y,ijk}^{(3)}(\vec{p},\vec{q}) + V_{V,ijk}^{(3)}(\vec{p},\vec{q})} + V_{\text{anti-matter},ijk}^{(3)}(\vec{p},\vec{q}) + V_{\text{subtraction},ijk}^{(3)}(\vec{p},\vec{q})$$



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$$\begin{aligned} V_{\text{Y},ijk}^{(3)} = \frac{1}{8E_iE_jE_k} \frac{256\pi^2 G^2}{q_i^2 q_j^2 q_k^2} \Big[& \left(q_i \cdot q_j\right) \left(2m_k^2 (p_i \cdot p_j)^2 + 2m_i^2 (p_j \cdot p_k)^2 - 4\left(p_i \cdot p_j\right)(p_i \cdot p_k)\left(p_j \cdot p_k\right) - m_i^2 m_j^2 m_k^2\right) \\ & + (p_k \cdot q_i)\left(p_k \cdot q_j\right)\left(m_i^2 m_j^2 - 2(p_i \cdot p_j)^2\right) - 2m_k^2 \left(p_i \cdot p_j\right)\left(p_i \cdot q_j\right)\left(p_j \cdot q_i\right) \\ & + \left(p_j \cdot q_i\right)\left(p_k \cdot q_j\right)\left(4\left(p_i \cdot p_j\right)(p_i \cdot p_k) - 2m_i^2\left(p_j \cdot p_k\right)\right) \\ & + Q_iQ_j\Big(\frac{1}{2}m_k^2\left(p_i \cdot q_j\right)\left(p_j \cdot q_i\right) - \left(q_i \cdot q_j\right)\left(\frac{1}{2}m_k^2\left(p_i \cdot p_j\right) - (p_i \cdot p_k)\left(p_j \cdot p_k\right)\right) \\ & \qquad - 2\left(p_j \cdot p_k\right)\left(p_i \cdot q_j\right)(p_k \cdot q_i) + \left(p_i \cdot p_j\right)(p_k \cdot q_i)\left(p_k \cdot q_j\right)\Big)\Big], \end{aligned}$$

$$V_{\text{V},ijk}^{(3)} = \frac{1}{8E_iE_jE_k} \frac{64\pi^2 G^2}{q_i^2 q_k^2} \left[8m_i^2 (p_j \cdot p_k)^2 - 16\left(p_i \cdot p_j\right)(p_i \cdot p_k)\left(p_j \cdot p_k\right) - Q_iQ_j^2Q_k(p_i \cdot p_k) + 8Q_iQ_j(p_i \cdot p_k)\left(p_j \cdot p_k\right)\right].$$

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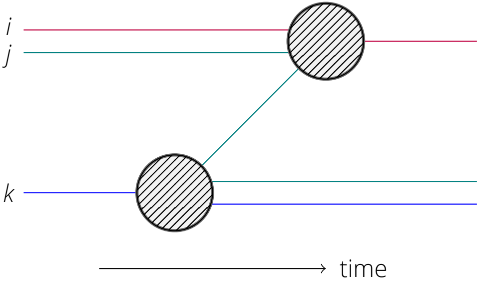
$$V_{ijk}^{(3)}(\vec{p},\vec{q}) = V_{Y,ijk}^{(3)}(\vec{p},\vec{q}) + V_{V,ijk}^{(3)}(\vec{p},\vec{q}) + V_{\text{anti-matter},ijk}^{(3)}(\vec{p},\vec{q}) + V_{\text{subtraction},ijk}^{(3)}(\vec{p},\vec{q})$$



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$$V_{\text{anti-matter},ijk}^{(3)} = -\frac{1}{32E_iE_j^3E_k}\frac{256\pi^2G^2\left(\left(p_i\cdot p_j\right)\left(Q_iQ_j-2\left(p_i\cdot p_j\right)\right)+m_i^2m_j^2\right)\left(\left(p_j\cdot p_k\right)\left(Q_jQ_k-2\left(p_j\cdot p_k\right)\right)+m_j^2m_k^2\right)}{q_i^2q_k^2}$$

Anti-matter singularity far away from classical region.

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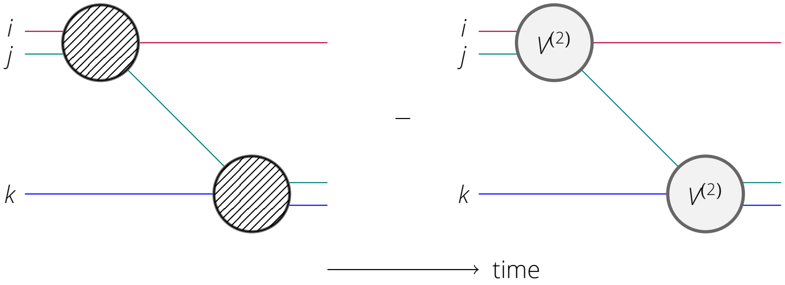

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$$H\left(\vec{p},\vec{q}\right)=\sum_i\sqrt{|\vec{p}_i|^2+M_i^2}+\sum_{(ijk)\in S_3}\left[(2\pi)^6\delta^{(3)}\left(\vec{q}_i+\vec{q}_j\right)\delta^{(3)}\left(\vec{q}_k\right)\frac{1}{2}V_{ij}^{(2)}\left(\vec{p},\vec{q}\right)+(2\pi)^3\delta^{(3)}\left(\vec{q}_1+\vec{q}_2+\vec{q}_3\right)V_{ijk}^{(3)}\left(\vec{p},\vec{q}\right)\right]$$

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$$V_{\text{subtraction},ijk}^{(3)}=-\frac{\mathcal{N}_{ijk}^{(L)}\delta\mathcal{N}_{ijk}^{(R)}+\mathcal{N}_{ijk}^{(R)}\delta\mathcal{N}_{ijk}^{(L)}}{\hat{q}_{ijk}^{(L)}\hat{q}_{ijk}^{(R)}}+\frac{\delta q_{ijk}^{(L)}\mathcal{N}_{ijk}^{(L)}\mathcal{N}_{ijk}^{(R)}}{\hat{q}_{ijk}^{(L)}q_{ijk}^{(L)}\hat{q}_{ijk}^{(R)}}+\frac{\delta q_{ijk}^{(R)}\mathcal{N}_{ijk}^{(L)}\mathcal{N}_{ijk}^{(R)}}{\hat{q}_{ijk}^{(L)}\hat{q}_{ijk}^{(R)}q_{ijk}^{(R)}}+\frac{\Delta E_{ijk}\delta q_{ijk}^{(L)}\delta q_{ijk}^{(R)}\mathcal{N}_{ijk}^{(L)}\mathcal{N}_{ijk}^{(R)}}{\hat{q}_{ijk}^{(L)}q_{ijk}^{(L)}\hat{q}_{ijk}^{(R)}q_{ijk}^{(R)}}$$

Non-universal (theory dependent) functions:

$$\mathcal{N}_{ijk}^{(L)}=\frac{4\pi G}{E_iE_j}\left[m_i^2m_j^2-Q_iQ_jE_iE_j(\rho_{ij}-1)-2E_i^2E_j^2(\rho_{ij}-1)^2\right]+\frac{2\pi G}{E_i^2E_j^2}\left[m_i^2m_j^2(E_j\tau_{ii}-E_i\tau_{ji})+Q_iQ_jE_iE_j(E_i(\rho_{ij}-1)\tau_{ji}+E_j(\tau_{ji}-\tau_{ii}\rho_{ij}))\right.$$
$$\left.-2E_i^2E_j^2(\rho_{ij}-1)(E_j(\tau_{ii}(\rho_{ij}+1)-2\tau_{ji})-E_i(\rho_{ij}-1)\tau_{ji})\right]$$

$$\mathcal{N}_{ijk}^{(R)}=\frac{4\pi G}{E_jE_k}\left[m_j^2m_k^2-Q_jQ_kE_jE_k(\rho_{jk}-1)-2E_j^2E_k^2(\rho_{jk}-1)^2\right]$$
$$+\frac{2\pi G}{E_j^2E_k^2}\left[\frac{1}{2}m_j^2m_k^2(E_k(-2\tau_{ii}+\tau_{jk}-2\tau_{kk})+E_j\tau_{kk})+\frac{1}{2}Q_jQ_kE_jE_k(E_k(\rho_{jk}(2\tau_{ii}-\tau_{jk}+2\tau_{kk})-2(\tau_{ki}+\tau_{kk})+\tau_{jk})+E_j(\tau_{jk}-\tau_{kk}\rho_{jk}))\right.$$
$$\left.+E_j^2E_k^2(\rho_{jk}-1)(E_k(2\tau_{ii}(\rho_{jk}+1)-4\tau_{ki}-\rho_{jk}\tau_{jk}+2\tau_{kk}(\rho_{jk}-1)+\tau_{jk})+E_j(2\tau_{jk}-\tau_{kk}(\rho_{jk}+1)))\right]$$

$$\delta\mathcal{N}_{ijk}^{(L)}=\frac{2\pi G}{E_iE_j}E_j\left[4E_iE_j(\rho_{ij}-1)+Q_iQ_j\right],\qquad\delta\mathcal{N}_{ijk}^{(R)}=-\frac{2\pi G}{E_jE_k}(E_j+2E_k)\left[4E_jE_k(\rho_{jk}-1)+Q_jQ_k\right]$$

$$E_i=\sqrt{|\vec{p}_i|^2+m_i^2},\qquad\rho_{ij}=\frac{\vec{p}_i\cdot\vec{p}_j}{E_iE_j},\qquad\tau_{ij}=\frac{\vec{p}_i\cdot\vec{q}_j}{E_i}.$$

Main Result: 3-body Hamiltonian at 2PM

$$H\left(\vec{p},\vec{q}\right) = \sum_i \sqrt{|\vec{p}_i|^2 + M_i^2} + \sum_{(ijk)\in S_3} \left[(2\pi)^6 \delta^{(3)}\left(\vec{q}_i + \vec{q}_j\right) \delta^{(3)}\left(\vec{q}_k\right) \frac{1}{2} V^{(2)}_{ij}\left(\vec{p},\vec{q}\right) + (2\pi)^3 \delta^{(3)}\left(\vec{q}_1 + \vec{q}_2 + \vec{q}_3\right) V^{(3)}_{ijk}\left(\vec{p},\vec{q}\right) \right]$$

$$V^{(3)}_{ijk}\left(\vec{p},\vec{q}\right) = V^{(3)}_{Y,ijk}\left(\vec{p},\vec{q}\right) + V^{(3)}_{V,ijk}\left(\vec{p},\vec{q}\right) + V^{(3)}_{\text{anti-matter},ijk}\left(\vec{p},\vec{q}\right) + V^{(3)}_{\text{subtraction},ijk}\left(\vec{p},\vec{q}\right)$$

$$V^{(3)}_{\text{subtraction},ijk} = -\frac{\mathcal{N}^{(L)}_{ijk}\,\delta\mathcal{N}^{(R)}_{ijk} + \mathcal{N}^{(R)}_{ijk}\,\delta\mathcal{N}^{(L)}_{ijk}}{\hat{q}^{(L)}_{ijk}\hat{q}^{(R)}_{ijk}} + \frac{\delta q^{(L)}_{ijk}\mathcal{N}^{(L)}_{ijk}\mathcal{N}^{(R)}_{ijk}}{\hat{q}^{(L)}_{ijk}q^{(L)}_{ijk}\hat{q}^{(R)}_{ijk}} + \frac{\delta q^{(R)}_{ijk}\mathcal{N}^{(L)}_{ijk}\mathcal{N}^{(R)}_{ijk}}{\hat{q}^{(L)}_{ijk}\hat{q}^{(R)}_{ijk}q^{(R)}_{ijk}} + \frac{\Delta E_{ijk}\,\delta q^{(L)}_{ijk}\,\delta q^{(R)}_{ijk}\mathcal{N}^{(L)}_{ijk}\mathcal{N}^{(R)}_{ijk}}{\hat{q}^{(L)}_{ijk}q^{(L)}_{ijk}\hat{q}^{(R)}_{ijk}q^{(R)}_{ijk}}$$

Cancellation of super classical $\mathcal{O}\left(q^{-5}\right)$ terms:

$$\begin{aligned}
\mathcal{N}^{(L/R)}_{ijk} &= \mathcal{N}^{(R/L)}_{kji} + \mathcal{O}\left(q^1\right)\\
q^{(L/R)}_{ijk} &= q^{(R/L)}_{kji} + \mathcal{O}\left(q^3\right)\\
\hat{q}^{(L/R)}_{ijk} &= \hat{q}^{(R/L)}_{kji} + \mathcal{O}\left(q^3\right)\\
\delta q^{(L/R)}_{ijk} &= -\delta q^{(R/L)}_{kji} + \mathcal{O}\left(q^2\right)\\
\Delta E_{ijk} &= -\Delta E_{kji} + \mathcal{O}\left(q^2\right).
\end{aligned}$$

How do we know this result is correct?

Check #1: 3-Body PN Hamiltonians in GR

Compare with known PN 3-body results (classical world-line methods)

$$
\begin{aligned}
L^{(3)}_{2\mathrm{PN}} = \sum_i \sum_{j\neq i} \sum_{k\neq i} \Bigg[& \frac{1}{8 r_{ij} r_{ik}} \left(4(\vec{n}_{ij}\cdot\vec{v}_i)^2 + 18\vec{v}_i^2 - 16\vec{v}_j^2 - 32\vec{v}_i\cdot\vec{v}_j + 32\vec{v}_j\cdot\vec{v}_k \right) \\
& + \frac{1}{8 r_{ij}^2} \left(14(\vec{n}_{ij}\cdot\vec{v}_k)(\vec{n}_{ij}\cdot\vec{v}_k) - 12(\vec{n}_{ij}\cdot\vec{v}_i)(\vec{n}_{ik}\cdot\vec{v}_k) + (\vec{n}_{ij}\cdot\vec{n}_{ik})(\vec{n}_{ik}\cdot\vec{v}_k)^2 - (\vec{n}_{ij}\cdot\vec{n}_{ik})\vec{v}_k^2 \right) \\
& + \frac{2(\vec{n}_{ij}-\vec{n}_{jk})\cdot\vec{v}_{ij}}{(r_{12}+r_{13}+r_{23})^2} \left(4(\vec{n}_{ij}+\vec{n}_{ik})\cdot\vec{v}_{ij} + (\vec{n}_{ik}+\vec{n}_{jk})\cdot\vec{v}_{ik} \right) \\
& + \frac{9(\vec{n}_{ij}\cdot\vec{v}_{ij})^2 - 9\vec{v}_{ij}^2 - 2(\vec{n}_{ij}\cdot\vec{v}_{ik})^2 + 2\vec{v}_{ik}^2}{r_{ij}(r_{ij}+r_{ik}+r_{jk})} \Bigg] .
\end{aligned}
$$

state-of-the-art $\mathcal{O}\left(G^2 v^4\right)$, $\sim$ 5000 terms [Loebbert, Plefka, Shi, Wang 2020].

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$$L_{2PN}^{(3)} = \sum_i \sum_{j\neq i} \sum_{k\neq i} \Bigg[\frac{1}{8 r_{ij} r_{ik}} \left(4(\vec{n}_{ij}\cdot\vec{v}_i)^2 + 18\vec{v}_i^2 - 16\vec{v}_j^2 - 32\vec{v}_i\cdot\vec{v}_j + 32\vec{v}_j\cdot\vec{v}_k \right)$$

$$+ \frac{1}{8 r_{ij}^2} \left(14(\vec{n}_{ij}\cdot\vec{v}_k)(\vec{n}_{ij}\cdot\vec{v}_k) - 12(\vec{n}_{ij}\cdot\vec{v}_i)(\vec{n}_{ik}\cdot\vec{v}_k) + (\vec{n}_{ij}\cdot\vec{n}_{ik})(\vec{n}_{ik}\cdot\vec{v}_k)^2 - (\vec{n}_{ij}\cdot\vec{n}_{ik})\vec{v}_k^2 \right)$$

$$+ \frac{2(\vec{n}_{ij}-\vec{n}_{jk})\cdot\vec{v}_{ij}}{(r_{12}+r_{13}+r_{23})^2} \left(4(\vec{n}_{ij}+\vec{n}_{ik})\cdot\vec{v}_{ij} + (\vec{n}_{ik}+\vec{n}_{jk})\cdot\vec{v}_{ik} \right)$$

$$+ \frac{9(\vec{n}_{ij}\cdot\vec{v}_{ij})^2 - 9\vec{v}_{ij}^2 - 2(\vec{n}_{ij}\cdot\vec{v}_{ik})^2 + 2\vec{v}_{ik}^2}{r_{ij}(r_{ij}+r_{ik}+r_{jk})} \Bigg].$$

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Compare gauge invariant *classical* scattering amplitudes

$$L_{2PN}^{(3)}(x,v) \xrightarrow{\text{Legendre}} H_{2PN}^{(3)}(x,p) \xrightarrow{\text{Fourier}} H_{2PN}^{(3)}(q,p) \xrightarrow{\text{Born}} \mathcal{M}_3^{2PN}(q,p)$$

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$$

Agrees with our 2PM 3-body Hamiltonian up to $\mathcal{O}\left(G^2 p^4\right)$.

Check #2: Probe Limit

Is any part of the answer fixed by probe (test mass) limit?

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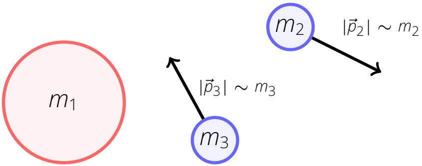

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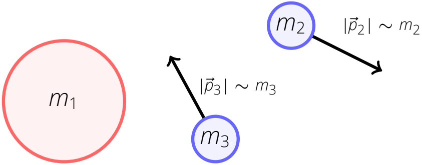

"Trivial", not sensitive to three-body interactions

$$H^{\text{3-body}}_{\text{probe}} = \left[\left(\frac{1-\frac{Gm_1}{2r_{12}}}{1+\frac{Gm_1}{2r_{12}}}\right)\left(m_2^2+\frac{|\vec{p}_2|^2}{1+\frac{Gm_1}{2r_{12}}}\right)\right]^{1/2} + \left[\left(\frac{1-\frac{Gm_1}{2r_{13}}}{1+\frac{Gm_1}{2r_{13}}}\right)\left(m_3^2+\frac{|\vec{p}_3|^2}{1+\frac{Gm_1}{2r_{13}}}\right)\right]^{1/2}$$

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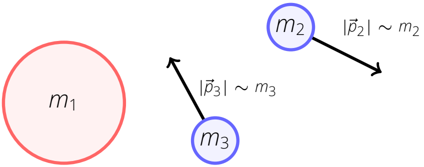

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Need multi-center black hole solutions.

Check #2: Probe Limit

Charged (extremal) black hole solution **[Majumdar 1947; Papapetrou 1947]**

$$ds^2 = -U(\vec{x})^{-2}dt^2 + U(\vec{x})^2 d\vec{x}\cdot d\vec{x}$$

$$A = U(\vec{x})^{-1}dt$$

$$U(\vec{x}) = 1 + \sum_{i=1}^{N-1} \frac{GM_i}{|\vec{x}-\vec{x}_i|}.$$

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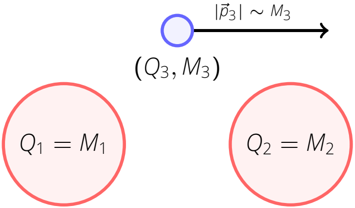

$$H_{\text{probe}}(\vec{x}_3, \vec{p}_3) \supset \frac{G^2 M_1 M_2 \left(2M_3^4 - 2Q_3\left(|\vec{p}_3|^2 + M_3^2\right)^{3/2} + 9M_3^2|\vec{p}_3|^2 + 6|\vec{p}_3|^4\right)}{\left(|\vec{p}_3|^2 + M_3^2\right)^{3/2} r_{13} r_{23}}.$$

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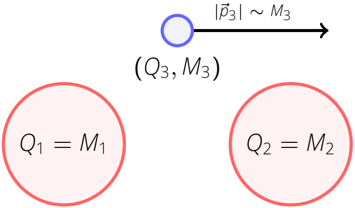

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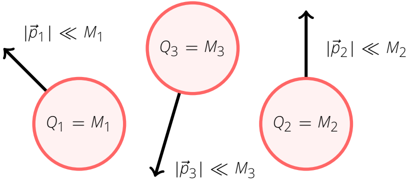
Agrees with 2PM 3-body EM Hamiltonian to LO in $M_3/M_{1,2}$ at $\mathcal{O}\left(G^2 p^\infty\right)$.

Check #3: Moduli Space Approximation



[Eardley, Ferrell 1987] effective Lagrangian for N extremal BHs at $\mathcal{O}\left(G^{\infty}v^2\right)$

$$L_{\text{MSA}} = \sum_i \frac{1}{2} M_i |\vec{v}_i|^2 + \frac{3}{8\pi} \int \mathrm{d}^3\vec{x}_0 \left[\left(1 + \sum_k \frac{GM_k}{x_{0k}}\right)^2 \sum_{i,j} \frac{GM_iM_j}{x_{0i}^3 x_{0j}^3} \left(\frac{1}{2} \left(\vec{x}_{0i} \cdot \vec{x}_{0j}\right) |\vec{v}_i - \vec{v}_j|^2 - (\vec{x}_{0i} \times \vec{x}_{0j}) \cdot (\vec{v}_i \times \vec{v}_j) \right) \right].$$

Valid in strong field regime $r_{ij} \sim GM_i$, exact to *all-loop orders*!

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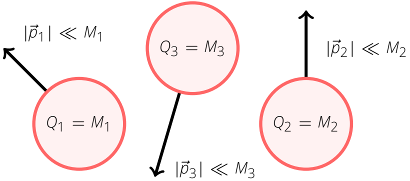

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Moduli space approximation **[Manton 1982]**:

$(0+1)$-D σ-model w/ metric on target space $=$ metric on moduli space .

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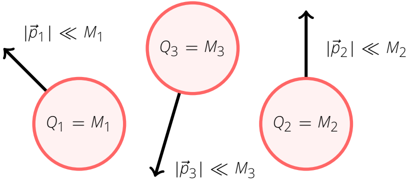

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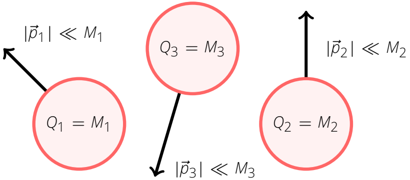
Contains only *two-*, *three-* and *four*-body interactions: consequence of supersymmetry **[Gibbons, Papadopoulos, Stelle 1997]**

Check #3: Moduli Space Approximation



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$$\supset \frac{3G^2 m_1 m_2 m_3}{4 r_{12} r_{13} r_{23}}\left[2\left(r_{12}+r_{13}-r_{23}\right)|\vec{v}_2-\vec{v}_3|^2 - \frac{r_{12}(\vec{v}_2\times\vec{v}_3)\cdot(\vec{x}_{13}\times\vec{x}_{23}) + r_{13}\left((\vec{v}_2\times\vec{v}_3)\cdot(\vec{x}_{12}\times\vec{x}_{23})\right) + r_{23}\left((\vec{v}_2\times\vec{v}_3)\cdot(\vec{x}_{13}\times\vec{x}_{12})\right)}{2(r_{12}+r_{13}+r_{23})^2}\right] + (\text{cyclic})\,.$$

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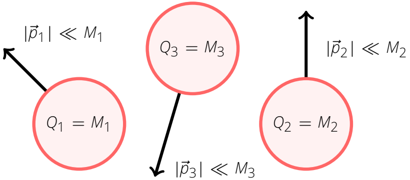

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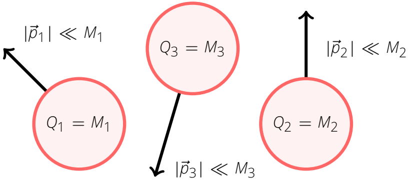

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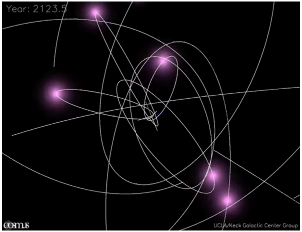
Conclusions and Future Directions



N-body classical gravitational dynamics are an exciting new direction for scattering amplitudes methods.

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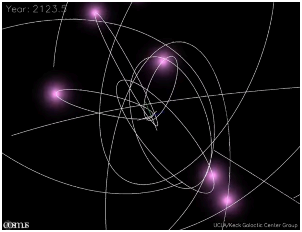

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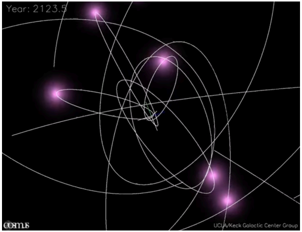

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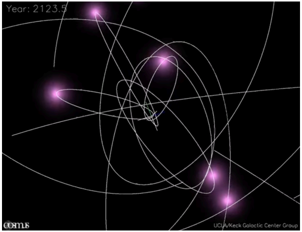

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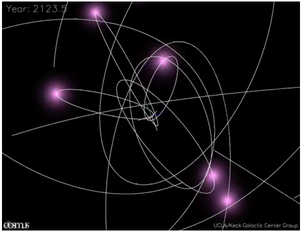

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Thank You!

UCLA