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Amplitudes, Geometry & Soft Theorems

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w/ Cheung, Helset [2111.03045,2202.06972] + Derda [WIP]

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Other talks today…

Amplitudes → Geometry

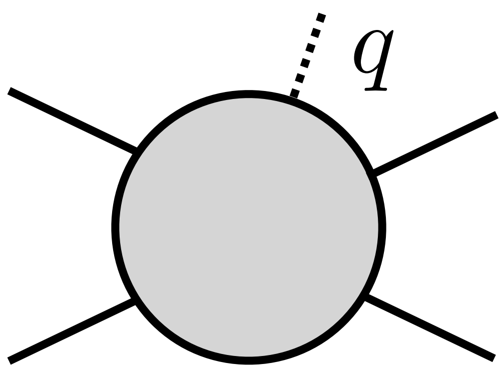
My talk today…

Geometry → Amplitudes

See also talk by Nathaniel Craig and references therein!

Soft theorems

- The behavior of scattering amplitudes when the momentum of a particle is small is often universal



$$\lim_{q \to 0} A_{n+1} = \mathcal{S} A_n$$

- Earliest example: Soft photon theorem [Low; Burnett, Kroll; Weinberg]

"leading" "subleading"

$$\lim_{q \to 0} A_{n+1} = \sum_a \frac{1}{q \cdot p_a} \left[\epsilon \cdot p_a + \epsilon \cdot J_a \cdot q \right] A_n$$

similar for soft gluons, gravitons

- Adler zero: soft pion in χPT

$$\lim_{q_\pi \to 0} A_{n+1} = 0$$

Motivation 1: two perspectives

Symmetry

"Soft theorems are a consequence of symmetry"

e.g. gauge (photons, gravitons), non-linearly realized (pions)

Effective field theory

"Soft theorems are EFT of long-wavelenght modes"

e.g. factorization, ultrasoft decoupling in SCET

Which one is it?

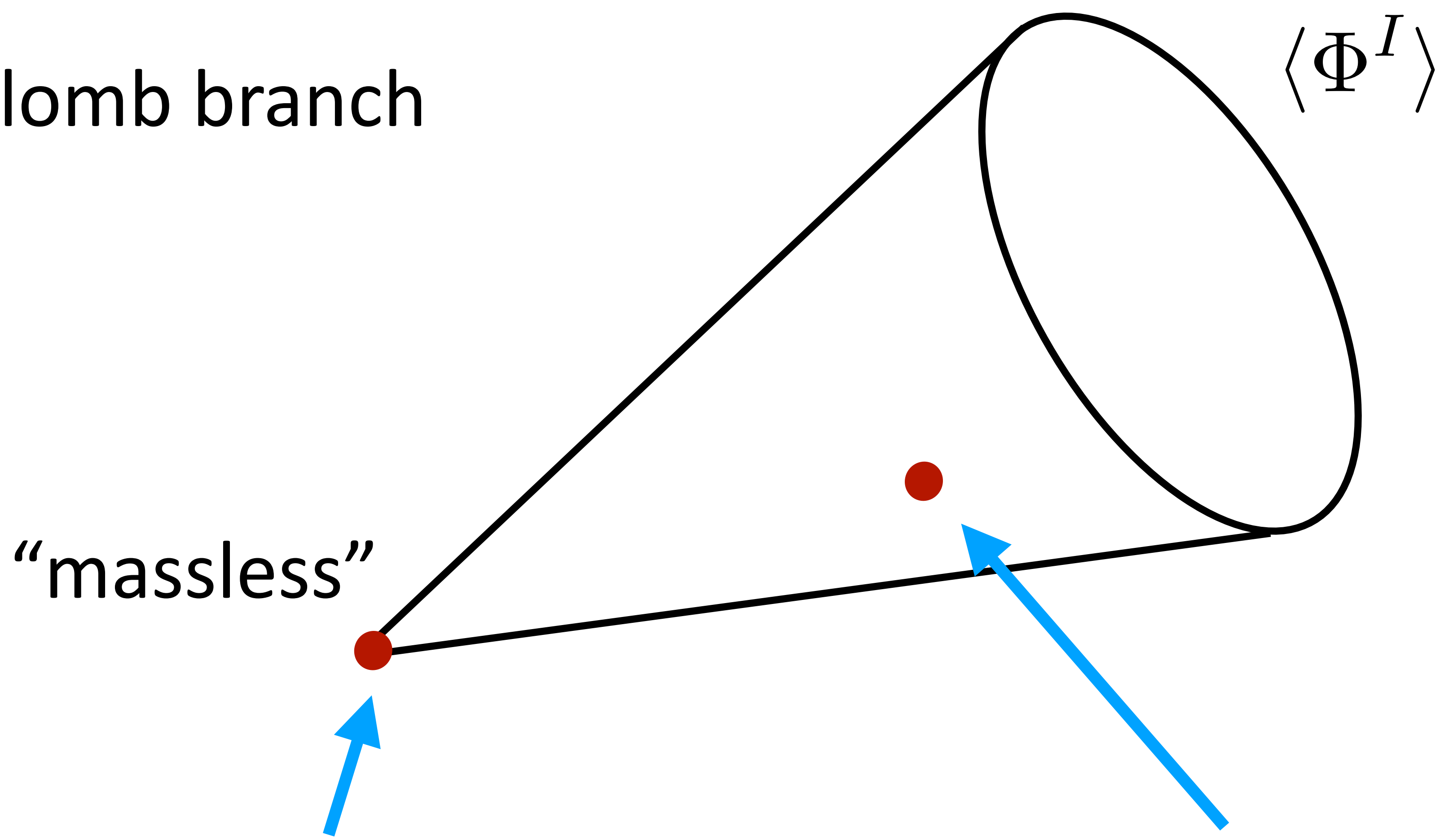
Third perspective: soft theorems related to asymptotic symmetries, memory, CCFT (will not discuss here)

See talks by Sabrina, Monica, Mark, Natalie!

Motivation 2: moduli space

Many of our favorite theories have moduli spaces of vacua, parameterized by v.e.v of scalars

e.g., N=4 Coulomb branch



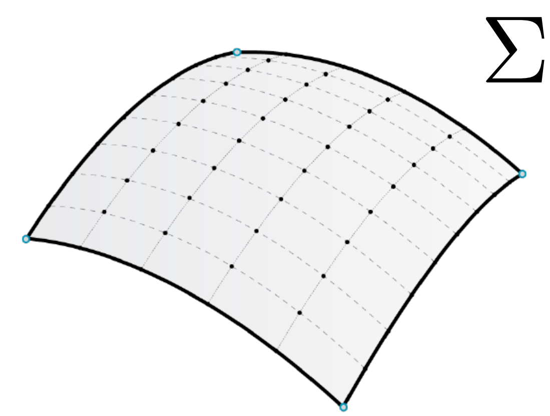
Are amplitudes <u>here</u> related to amplitudes <u>here</u>?

Outline

- Amplitudes & geometry
- Geometric soft theorems

Amplitudes & Geometry

Geometry of fields



- Scalar field take values in a target space manifold
- Lagrangian can be organized by derivative order

$$\frac{1}{2}g_{IJ}(\Phi)\partial_\mu\Phi^I\partial^\mu\Phi^J - V(\Phi) + \frac{1}{4}\lambda_{IJKL}(\Phi)\partial_\mu\Phi^I\partial^\mu\Phi^J\partial_\nu\Phi^K\partial^\nu\Phi^L + \cdots\,,$$

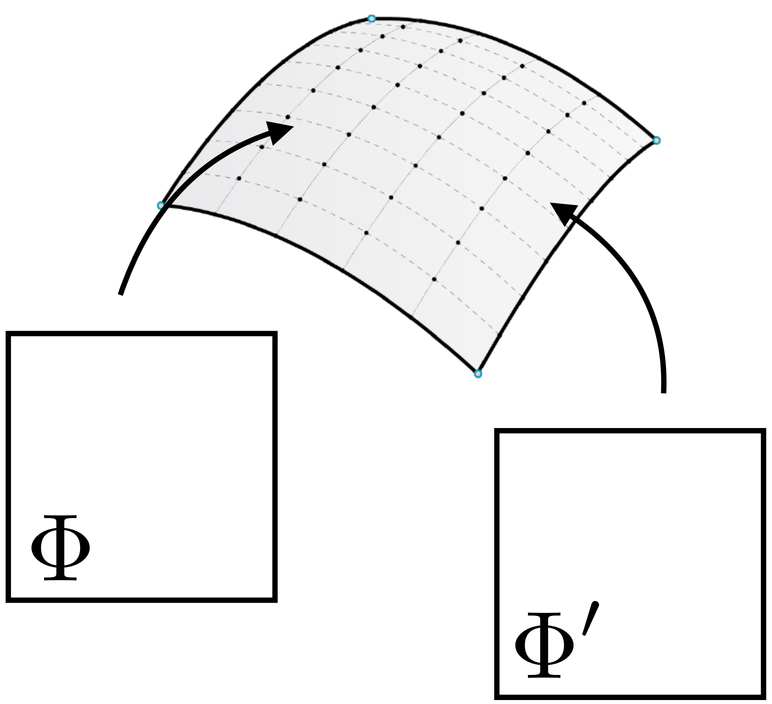
- Field redefinitions = changes of coordinates $\Phi^I = \Phi^I(\Phi')$

$$\partial_\mu\Phi^I \quad\to\quad \frac{\partial\Phi'^I}{\partial\Phi^J}\partial_\mu\Phi^J$$

- Most couplings are tensors in field space

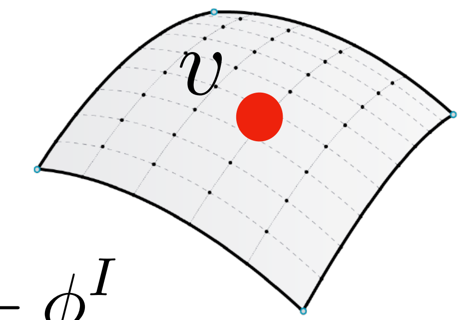
  e.g. two-derivative = metric

$$g_{IJ}(\Phi) \quad\to\quad \frac{\partial\Phi^K}{\partial\Phi'^I}\frac{\partial\Phi^L}{\partial\Phi'^J}g_{KL}(\Phi')$$



See Nathaniel's talk!

Geometry of amplitudes



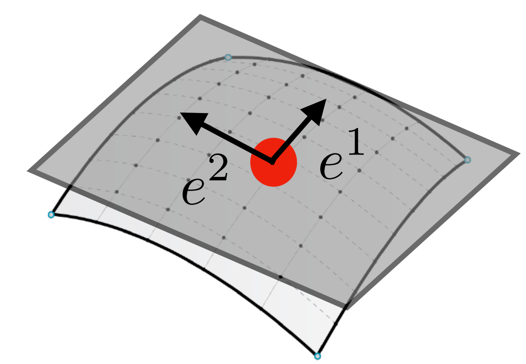
- Amplitudes defined by expanding around VEV $\Phi^I = v^I + \phi^I$

- Do not depend on field basis $\phi \to \phi + \epsilon f(\phi)$

$$S(\phi) \to S(\phi) + \frac{\delta S}{\delta \phi} \epsilon f(\phi) + \cdots$$

equations of motion

$$\propto p^2 - m^2 + \cdots$$



but on a choice of frame $\langle p^i | \phi^J(x) | 0 \rangle = e^{iJ}(v) e^{ip\cdot x}$

- Must be a function of geometric invariants! e.g. curvature of Σ

  [Volkov; Dixon, Kaplunovsky, Louis]

$$R^{ijkl}(v) \qquad \nabla^m R^{ijkl}(v)$$

Example

- Two-derivative theory $\frac{1}{2} g_{IJ}(\Phi)\partial_\mu \Phi^I \partial^\mu \Phi^J$

$$
\begin{aligned}
A_4^{i_1 i_2 i_3 i_4} &= R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}, \\
A_5^{i_1 i_2 i_3 i_4 i_5} &= \nabla^{i_3} R^{i_1 i_4 i_2 i_5} s_{45} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} s_{35} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} s_{25} \\
&\quad + \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4} (s_{24} + s_{45}), \\
A_6^{i_1 i_2 i_3 i_4 i_5 i_6} &= -\tfrac{1}{72} \left(R^{i_1 i_3 i_2 j} s_{12} + R^{i_1 i_2 i_3 j} s_{13} \right) \frac{1}{s_{123}} \left(R_j{}^{i_6 i_5 i_4} s_{46} + R_j{}^{i_5 i_6 i_4} s_{45} \right) \\
&\quad + \tfrac{1}{108} \left(R^{i_1 i_3 i_2 j} (s_{12} - \tfrac{1}{6} s_{123}) + R^{i_1 i_2 i_3 j} (s_{13} - \tfrac{1}{6} s_{123}) \right) \left(R_j{}^{i_6 i_5 i_4} + R_j{}^{i_5 i_6 i_4} \right) \\
&\quad + \tfrac{1}{90} R^{i_1 i_6 i_5 j} R_j{}^{i_2 i_3 i_4} s_{13} + \tfrac{1}{80} \nabla^{i_6} \nabla^{i_5} R^{i_1 i_2 i_3 i_4} s_{13} + \text{perm}\,.
\end{aligned}
$$

- Function of geometric invariants evaluated at the point/VEV v

- Valid for any two derivative theory, manifest FR invariance!

  e.g. G/H NLSM, obscures symmetry

$$R^{ijkl} \sim f^{ij}{}_m f^{klm}$$

$$(\nabla)^n R^{ijkl} = 0$$

Field/kinematic dependent geometry

- Background instead of VEV $\quad v \rightarrow \bar{\phi}(p) \;\text{ or }\; \bar{\phi}(x)$

  c.f. Tim's talk $\quad \partial_v \rightarrow \dfrac{\delta}{\delta\bar{\phi}(p)}$

- Lagrangian is a generating function $\quad \frac{1}{2} g_{IJ}(p_1, p_2; \Phi) p_1 \cdot p_2 \Phi^I(p_1) \Phi^J(p_2)$

- Geometry now depends on kinematics and background field

$$g_{ij} \rightarrow g_{ij}(p_1, p_2, \bar{\phi})$$

$$\partial_k g_{ij} \rightarrow \frac{\delta}{\delta\bar{\phi}^k(p_3)} g_{ij}(p_1, p_2, \bar{\phi}) = \partial_k g_{ij}(p_1, p_2 | p_3) + f(\bar{\phi})$$

$$\partial_{i_3} \cdots \partial_{i_n} g_{i_1 i_2} = g(p_1, p_2 | p_3, \cdots, p_n) = \frac{1}{p_1 \cdot p_2} \times n - \text{point Feynman vertex}$$

Geometric invariants

- FR covariant off-shell $R \sim \partial\Gamma + \Gamma\Gamma \sim \partial^2 g + \partial g \partial g$

  Even under derivative FR! $\Phi^I = \Phi^I(\Phi', \partial\Phi', \cdots)$

- Two-derivative theory: this reproduces usual invariants

- Potential $R(p_1, p_2, p_3, p_4) = V_{(3)}^2 \left(\frac{1}{t^2} - \frac{1}{u^2}\right) + \frac{V_{(4)}}{3}\left(\frac{1}{t} - \frac{1}{u}\right) + f(\bar{\phi})$

- Dirac-Born-Infeld $R(p_1, p_2, p_3, p_4) = -\frac{\lambda_4}{4}(t-u)$

- Special Galileon $R(p_1, p_2, p_3, p_4) = \frac{\lambda_4}{12}(t^2 - u^2)$

- More generally $R(p_1, p_2, p_3, p_4) \sim t^\rho - u^\rho$

Checks: symmetries, Bianchi identities, ..

$$R(p_1, p_2, p_3, p_4) + R(p_2, p_3, p_1, p_4) + R(p_3, p_1, p_2, p_4) = 0$$

Amplitudes

- Just insert new invariants!

$$A_4^{i_1 i_2 i_3 i_4} \quad = \quad R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}$$

e.g. potential theory, DBI

$$A(p_1, p_2, p_3, p_4) = -V_{(3)}^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) - V_{(4)} \quad \checkmark$$

$$A(p_1, p_2, p_3, p_4) = \frac{\lambda_4}{4}(s^2 + t^2 + u^2) \quad \checkmark$$

- Manifests FR invariance at the expense of locality

- Note: Amplitude only a function of geometric invariants on-shell, but invariants are FR invariant off-shell

Theories with spin

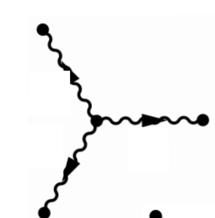
- It obviously generalizes to other bosonic theories (e.g. Yang-Mills)

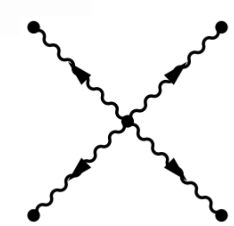
$$g_{ab}^{\mu\nu}(p_1,p_2)\delta(p_{12}) = -\,\delta_{ab}\eta^{\mu\nu}\delta(p_{12}) + \int_{p_3} g_{ab|c}^{\mu\nu|\rho}(p_1,p_2|p_3)\delta(p_{123})A_\rho^c(p_3) + \cdots$$

in Feynman gauge

$$g_{ab|c}^{\mu\nu|\rho}(p_1,p_2|p_3) = -\frac{if_{abc}}{p_1\cdot p_2}(p_3^\mu\eta^{\nu\rho} - p_3^\nu\eta^{\mu\rho})$$

$$g_{ab|cd}^{\mu\nu|\rho\sigma}(p_1,p_2|p_3,p_4) = \frac{f_{abe}f_{cd}{}^e}{2p_1\cdot p_2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$$





- Geometric expansion still works. Manifests FR invariance at the expense of locality & gauge invariance

$$A_4^{i_1i_2i_3i_4} \quad = \quad R^{i_1i_3i_2i_4}s_{34} + R^{i_1i_2i_3i_4}s_{24}$$

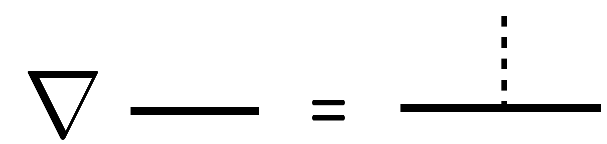
Geometric soft theorems

[Cheung, Helset, **JPM**]

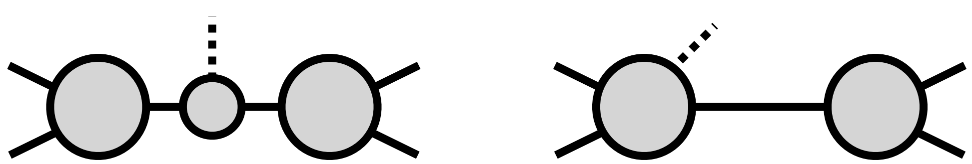
New soft scalar theorem

- Intuition:

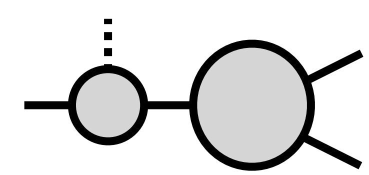
$$\lim_{q\to 0} A_{n+1} \sim \left(\nabla + \frac{\nabla m^2}{p^2 - m^2} \right) A_n$$



“Derivative w.r.t VEV”



“on-shell connection”



- Dotting the i’s etc

leading

$$\lim_{q\to 0} A_{n+1}^{i_1\cdots i_n i} = \quad \nabla^i A_n^{i_1\cdots i_n} + \sum_{a=1}^{n} \frac{\nabla^i V^{i_a}{}_{j_a}}{(p_a+q)^2 - m_{j_a}^2} \left(1 + q^\mu \frac{\partial}{\partial p_a^\mu} \right) A_n^{i_1\cdots j_a \cdots i_n}$$

subleading

Examples

- Two-derivative theory

$$
\begin{aligned}
A_5^{i_1 i_2 i_3 i_4 i_5} \quad &= \quad \nabla^{i_3} R^{i_1 i_4 i_2 i_5} s_{45} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} s_{35} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} s_{25} \\
&\quad + \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4} (s_{24} + s_{45}) \\
&= \nabla^{i_5} (R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24})
\end{aligned}
$$

- Quartic theory

$$
\begin{aligned}
A_{4,\lambda}^{i_1 i_2 i_3 i_4} \quad &= \quad \tfrac{1}{2} s_{12} s_{34} \lambda^{i_1 i_2 i_3 i_4} + \tfrac{1}{2} s_{13} s_{24} \lambda^{i_1 i_3 i_2 i_4} + \tfrac{1}{2} s_{23} s_{14} \lambda^{i_2 i_3 i_1 i_4} \,, \\
A_{5,\lambda}^{i_1 i_2 i_3 i_4 i_5} \quad &= \quad \tfrac{1}{2} s_{12} s_{34} \nabla^{i_5} \lambda^{i_1 i_2 i_3 i_4} + \tfrac{1}{2} s_{13} s_{24} \nabla^{i_5} \lambda^{i_1 i_3 i_2 i_4} + \tfrac{1}{2} s_{23} s_{14} \nabla^{i_5} \lambda^{i_2 i_3 i_1 i_4} \\
&+ \tfrac{1}{2} s_{23} s_{45} \nabla^{i_1} \lambda^{i_2 i_3 i_4 i_5} + \tfrac{1}{2} s_{24} s_{35} \nabla^{i_1} \lambda^{i_2 i_4 i_3 i_5} + \tfrac{1}{2} s_{34} s_{25} \nabla^{i_1} \lambda^{i_3 i_4 i_2 i_5} \\
&+ \tfrac{1}{2} s_{13} s_{45} \nabla^{i_2} \lambda^{i_1 i_3 i_4 i_5} + \tfrac{1}{2} s_{14} s_{35} \nabla^{i_2} \lambda^{i_1 i_4 i_3 i_5} + \tfrac{1}{2} s_{34} s_{15} \nabla^{i_2} \lambda^{i_3 i_4 i_1 i_5} \\
&+ \tfrac{1}{2} s_{12} s_{45} \nabla^{i_3} \lambda^{i_1 i_2 i_4 i_5} + \tfrac{1}{2} s_{14} s_{25} \nabla^{i_3} \lambda^{i_1 i_4 i_2 i_5} + \tfrac{1}{2} s_{24} s_{15} \nabla^{i_3} \lambda^{i_2 i_4 i_1 i_5} \\
&+ \tfrac{1}{2} s_{12} s_{35} \nabla^{i_4} \lambda^{i_1 i_2 i_3 i_5} + \tfrac{1}{2} s_{13} s_{25} \nabla^{i_4} \lambda^{i_1 i_3 i_2 i_5} + \tfrac{1}{2} s_{23} s_{15} \nabla^{i_4} \lambda^{i_2 i_3 i_1 i_5}
\end{aligned}
$$

Examples

- Two-derivative theory

$$
\begin{aligned}
A_5^{i_1 i_2 i_3 i_4 i_5} &= \nabla^{i_3} R^{i_1 i_4 i_2 i_5} \cancelto{0}{s_{45}} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} \cancelto{0}{s_{35}} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} \cancelto{0}{s_{25}} \\
&\quad + \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4} (s_{24} + \cancelto{0}{s_{45}}) \\
&= \nabla^{i_5} (R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24})
\end{aligned}
$$

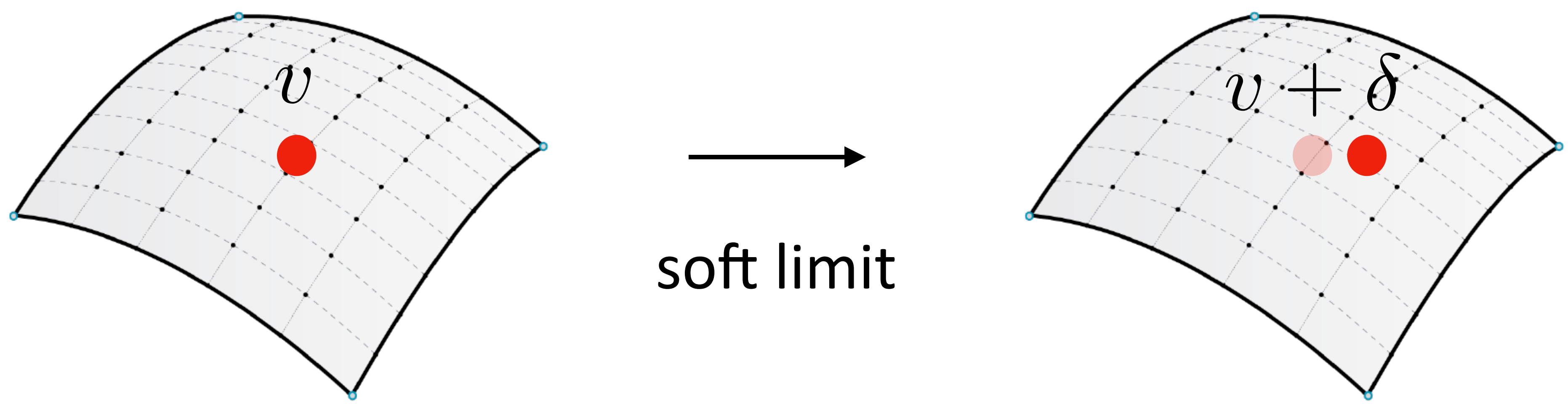
- Quartic theory

$$
\begin{aligned}
A_{4,\lambda}^{i_1 i_2 i_3 i_4} &= \tfrac{1}{2} s_{12} s_{34} \lambda^{i_1 i_2 i_3 i_4} + \tfrac{1}{2} s_{13} s_{24} \lambda^{i_1 i_3 i_2 i_4} + \tfrac{1}{2} s_{23} s_{14} \lambda^{i_2 i_3 i_1 i_4}\,, \\
A_{5,\lambda}^{i_1 i_2 i_3 i_4 i_5} &= \tfrac{1}{2} s_{12} s_{34} \nabla^{i_5} \lambda^{i_1 i_2 i_3 i_4} + \tfrac{1}{2} s_{13} s_{24} \nabla^{i_5} \lambda^{i_1 i_3 i_2 i_4} + \tfrac{1}{2} s_{23} s_{14} \nabla^{i_5} \lambda^{i_2 i_3 i_1 i_4} \\
&\quad \cancelto{0}{+ \tfrac{1}{2} s_{23} s_{45} \nabla^{i_1} \lambda^{i_2 i_3 i_4 i_5} + \tfrac{1}{2} s_{24} s_{35} \nabla^{i_1} \lambda^{i_2 i_4 i_3 i_5} + \tfrac{1}{2} s_{34} s_{25} \nabla^{i_1} \lambda^{i_3 i_4 i_2 i_5}} \\
&\quad \cancelto{0}{+ \tfrac{1}{2} s_{13} s_{45} \nabla^{i_2} \lambda^{i_1 i_3 i_4 i_5} + \tfrac{1}{2} s_{14} s_{35} \nabla^{i_2} \lambda^{i_1 i_4 i_3 i_5} + \tfrac{1}{2} s_{34} s_{15} \nabla^{i_2} \lambda^{i_3 i_4 i_1 i_5}} \\
&\quad \cancelto{0}{+ \tfrac{1}{2} s_{12} s_{45} \nabla^{i_3} \lambda^{i_1 i_2 i_4 i_5} + \tfrac{1}{2} s_{14} s_{25} \nabla^{i_3} \lambda^{i_1 i_4 i_2 i_5} + \tfrac{1}{2} s_{24} s_{15} \nabla^{i_3} \lambda^{i_2 i_4 i_1 i_5}} \\
&\quad \cancelto{0}{+ \tfrac{1}{2} s_{12} s_{35} \nabla^{i_4} \lambda^{i_1 i_2 i_3 i_5} + \tfrac{1}{2} s_{13} s_{25} \nabla^{i_4} \lambda^{i_1 i_3 i_2 i_5} + \tfrac{1}{2} s_{23} s_{15} \nabla^{i_4} \lambda^{i_2 i_3 i_1 i_5}}
\end{aligned}
$$

- Also works beyond contact terms i.e. factorization

Some comments

- Precise encoding of the intuition: “Soft scalar = shift of VEV”



- Valid for **any** massless scalar

- Challenges common lore = soft scalar theorems *iff* symmetry

- No symmetry required! soft theorems move us around space of vacua instead.

More comments

- Non-perturbative in massless theory, tree-level in theory with masses
- Geometry encodes both leading and subleading soft theorems
- Generalizes and extends soft theorems for Goldstone bosons (Adler zero, non-symmetric G/H, soft dilaton, …)
- Has been interpreted in CCFT (vacua = exactly marginal deformations, special soft operator)

  [Kapec, Law, Narayanan]

Double soft measures curvature

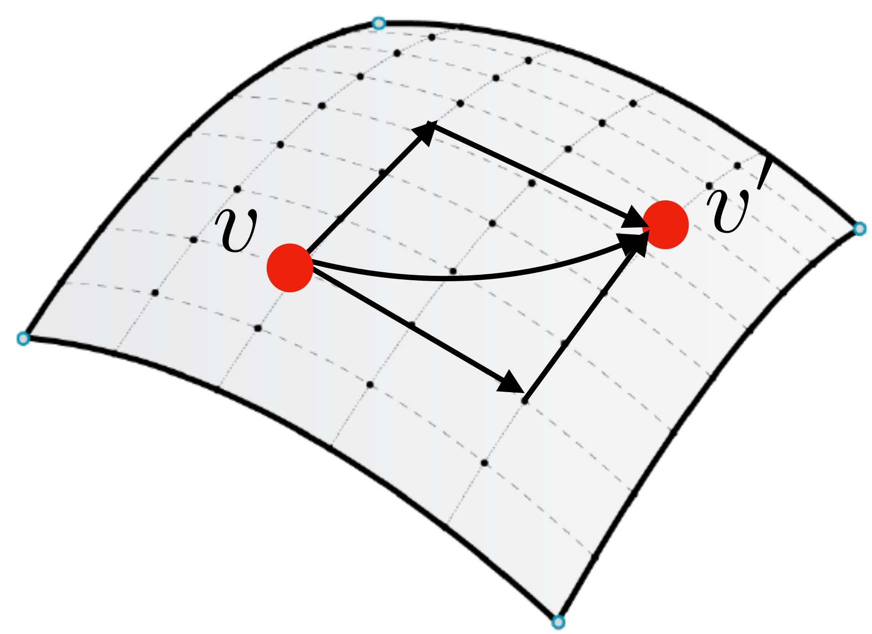
- Consecutive double soft

$$\left[\lim_{q_a\to 0},\lim_{q_b\to 0}\right] A_{n+2}^{i_1\cdots i_n i_a i_b} = [\nabla^{i_a},\nabla^{i_b}]A_n^{i_1\cdots i_n} = \sum_{c\neq a,b} R^{i_a i_b i_c}{}_{j_c} A_n^{i_1\cdots j_c\cdots i_n}$$

- Simultaneous double soft

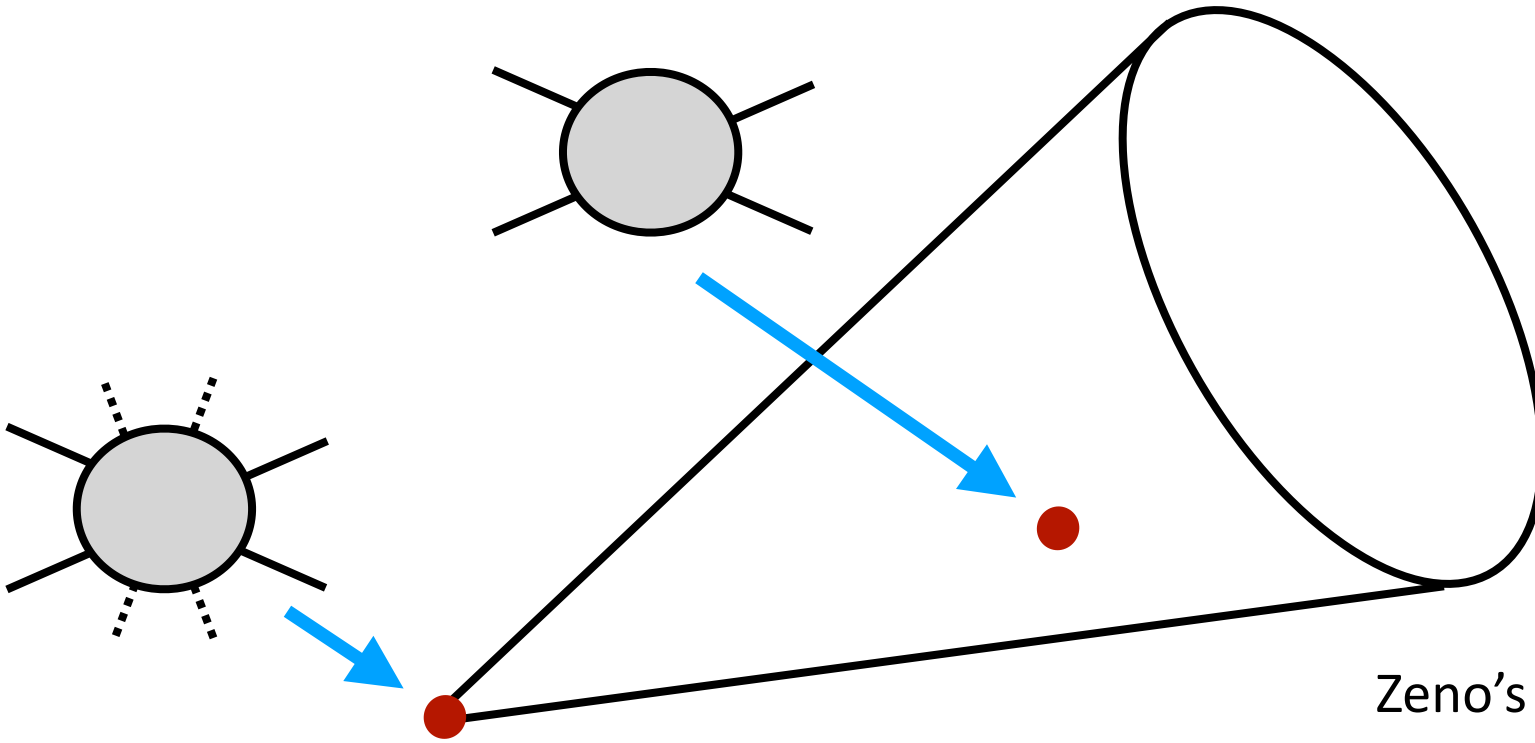
$$\lim_{q_a,q_b\to 0} A_{n+2}^{i_1\cdots i_n i_a i_b} = \frac{1}{2}\sum_{c\neq a,b}\frac{s_{ac}-s_{bc}}{s_{ac}+s_{bc}} R^{i_a i_b i_c}{}_{j_c} A_n^{i_1\cdots j_c\cdots i_n} + \nabla^{(i_a}\nabla^{i_b)}A_n^{i_1\cdots i_n}$$

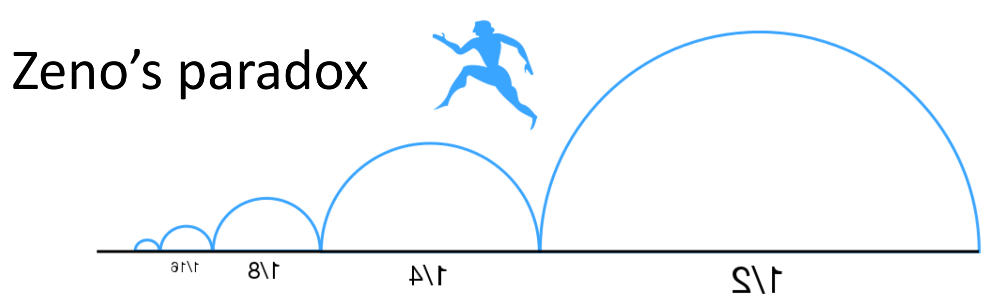
- Difference: which path in field space



Exploring moduli space

- Geometric (multi)soft theorem lets us move around moduli space



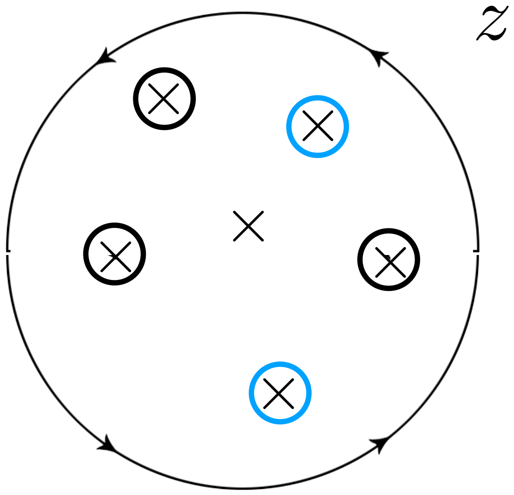


- One know example: W scattering in N=4

  [Craig, Elvang, Kiermaier, Slatyer; Kiermaier]

- Must be explore further, make systematic

Geometric soft recursion



- Recursion with soft-subtractions

  [Cheung, Kampf, Novotny, Shen, Trnka; Luo, Wen]

$$A_n(0) = \frac{1}{2\pi i}\oint \frac{dz}{z}\frac{A_n(z)}{F_{n,m}(z)} = -\sum_\alpha \mathrm{Res}_{z=z_\alpha^\pm}\left(\frac{A_n(z)}{zF_{n,m}(z)}\right)$$

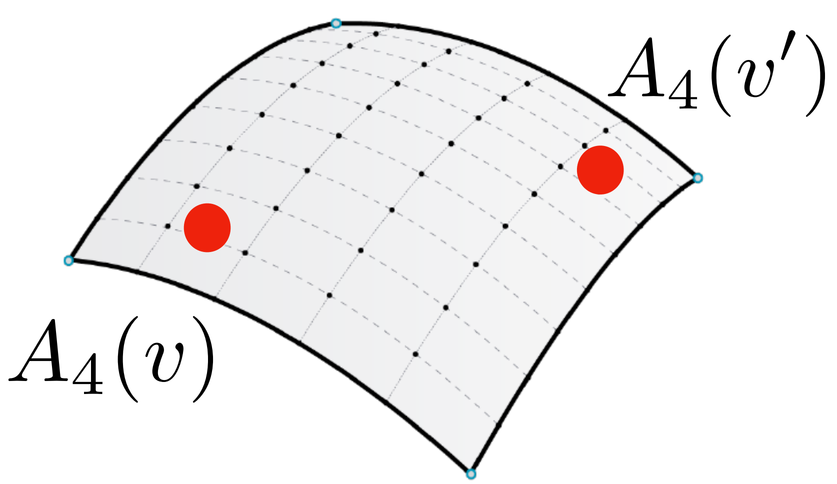
$$F_{n,m}(z) = \prod_{a=1}^{n}(1-c_a z)^m$$

- Soft theorem enables on-shell recursion

  [Cheung, Helset, **JPM**]

$$A_n(0) = \sum_\alpha \frac{A_L(z_\alpha^+)A_R(z_\alpha^+)}{(1-z_\alpha^+/z_\alpha^-)F_{n,1}(z_\alpha^+)} + (z_\alpha^+ \leftrightarrow z_\alpha^-) + \sum_a \frac{\nabla_{i_a}A_{n-1}(1/c_a)}{\Pi_{b\neq a}(1-c_b/c_a)}$$

- Seed of the recursion is four-point amplitude at every VEV

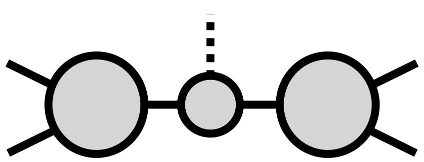


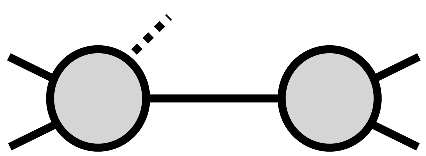
A general soft theorem?

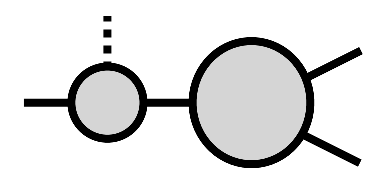
Generalize the intuition:

Does this work?

$$\lim_{q\to 0} A_{n+1} \sim \left(\nabla + \bar{\Gamma}\right) A_n$$







"Soft particle = derivative w.r.t. const. background"

3pt-vertex "on-shell connection"

(3pt function as connection also appeared recently in [Cohen, Craig, Lu, Sutherland]) See Nathaniel's talk!

Soft scalar coupled to matter

- Leading coupling to fermions $\Omega_{PQI}(\Phi)\bar{\psi}^P\gamma^\mu\psi^Q\partial_\mu\Phi^I$
- Soft theorem works!

$$\lim_{q\to 0} A_{n+1} = \bar{\nabla} A_n = \left(\nabla + \bar{\Omega}\right) A_n$$

torsion!

- E.g. dipole coupling $D_{PQA}(\Phi)(\bar{\psi}^P\sigma^{\mu\nu}\psi^Q)W^A_{\mu\nu}$

$$\mathcal{A}_3^{pqa} \sim \langle 13\rangle\langle 23\rangle D^{pqa},$$
$$\mathcal{A}_4^{pqai_4} \sim \langle 13\rangle\langle 23\rangle \bar{\nabla}^{i_i} D^{pqa},$$

- Minimal coupling to vectors gives version of Goldstone boson equivalence theorem.

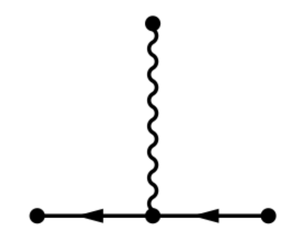
Soft photon theorem

- Scalar-photon metric $g^{\rho}_{\phi\bar{\phi}}(p_1, p_2|p_3) = \dfrac{q}{p_1 \cdot p_2}(p_1 - p_2)^{\rho}$
- General soft theorem

$$\lim_{q \to 0} A_{n+1} \sim \left(\nabla + \bar{\Gamma}\right) A_n$$

$$\epsilon^{\mu} \frac{\delta}{\delta \bar{A}^{\mu}} \overset{D = \partial + q\bar{A}}{=} -\sum_a q_a \epsilon \cdot \frac{\partial}{\partial p_a}$$

Three-point vertex



$$q_a \frac{\epsilon \cdot p_a}{q \cdot p_a}\left(1 + p_a \cdot \frac{\partial}{\partial q}\right)$$

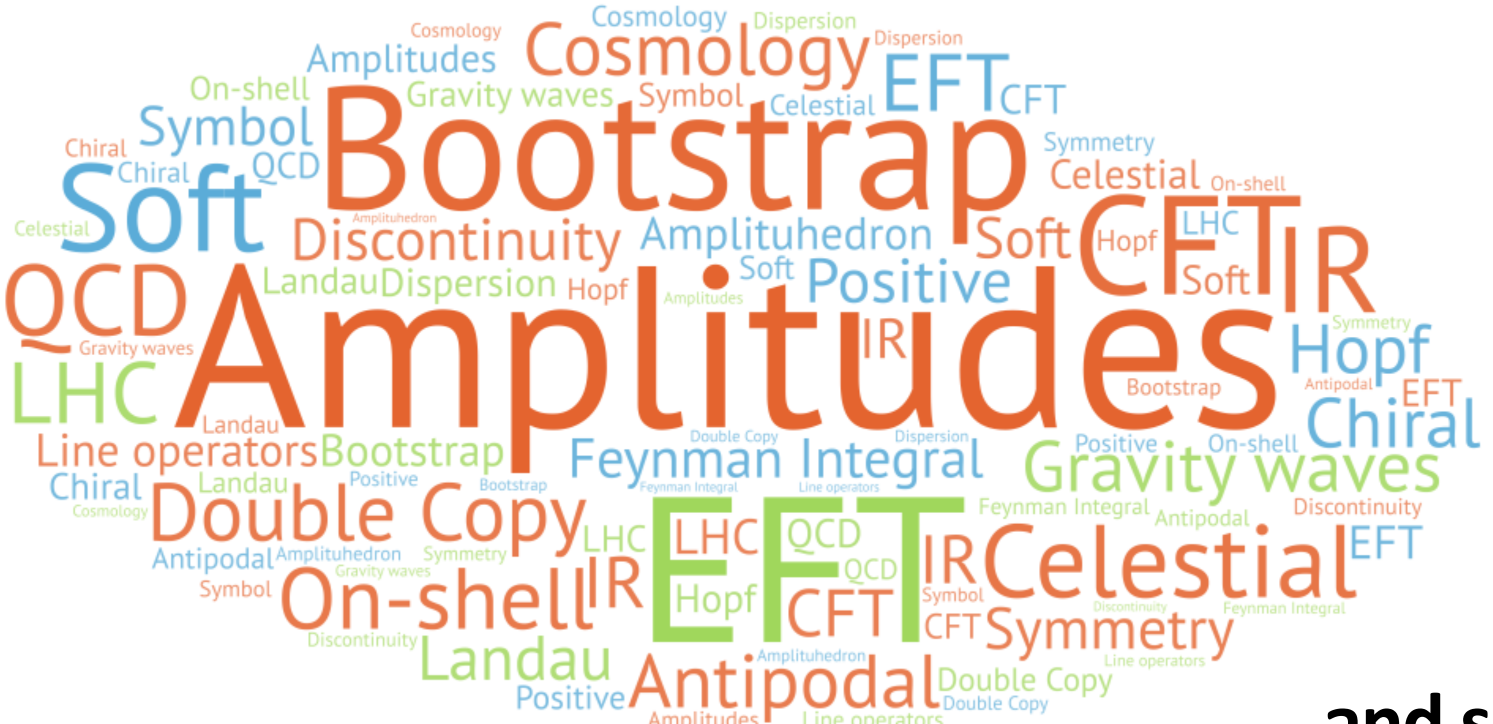
$$\sum_a \frac{1}{q \cdot p_a}\left[\epsilon \cdot p_a + \epsilon \cdot J_a \cdot q\right]$$

Gravitons and gluons WIP

Conclusions

- Geometry of field space is a useful principle which
  - …can be extended beyond simple changes of coordinates to general field redefinitions
  - …can be used to organize bosonic amplitudes in a geometric way that exposes field-redefinition invariance
- Geometric perspective gives new and general soft theorems
- Important question remains: What is the systematic way to move a finite distance in field space? Massive from massless?
- Tantalizing hints of general organizing principle for general (leading and sub-leading) soft theorems. Can we find new ones?

Thank you!



... and see you at CERN!

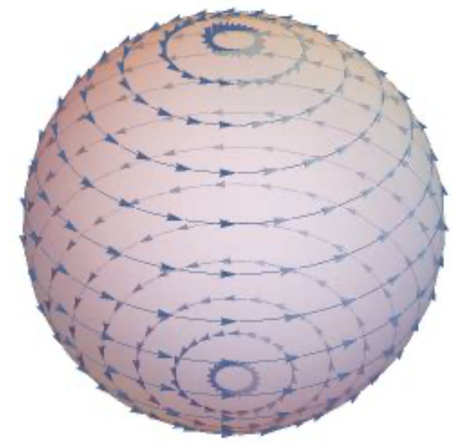
What is special about symmetry?

Symmetry $\cancel{\longleftrightarrow}$ soft theorem

Symmetry $\rightarrow$ “multiplicative” soft theorem

$$\lim_{q\to 0} A_{n+1} = \mathcal{S} A_n$$

Geometry of symmetry



- Symmetry = Killing vector

$$\Phi^I \quad \to \quad \Phi^I + \mathcal{K}^I(\Phi)$$

$$g_{IJ}(\Phi) \quad \to \quad g_{IJ}(\Phi) + \mathcal{L}_{\mathcal{K}} g_{IJ}(\Phi)$$

- Ward identity

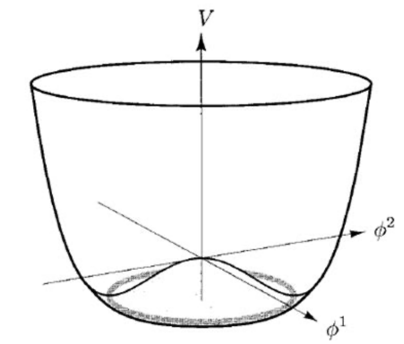
$$\mathcal{L}_{\mathcal{K}} A_n^{i_1\cdots i_n} = \mathcal{K}_i \nabla^i A_n^{i_1\cdots i_n} - \sum_{a=1}^{n} \nabla_{j_a} \mathcal{K}^{i_a} A_n^{i_1\cdots j_a \cdots i_n} = 0$$

- Makes soft theorem multiplicative

$$\lim_{q\to 0} \mathcal{K}_i A_{n+1}^{i_1\cdots i_n i} = \sum_{a=1}^{n} \nabla_{j_a} \mathcal{K}^{i_a} A_n^{i_1\cdots j_a \cdots i_n}$$

Example: non-symmetric NBG

[Kampf, Novotny, Shifman, Trnka; Cheung, Helset, **JPM**]



- Spontaneously broken global symmetry

- Coset space G/H

$$[\mathcal{T}_a, \mathcal{T}_b] = f_{ab}{}^{c}\mathcal{T}_c\,,$$
$$[\mathcal{T}_a, \mathcal{X}_i] = f_{ai}{}^{j}\mathcal{X}_j\,,$$
$$[\mathcal{X}_i, \mathcal{X}_j] = f_{ij}{}^{a}\mathcal{T}_a + f_{ij}{}^{k}\mathcal{X}_k$$

- Soft theorem

$$\lim_{q\to 0} A_{n+1}^{i_1\ldots i_n i} = -\frac{1}{2}\sum_{a=1}^{n} f_{j_a}{}^{i_a i} A_n^{i_1\cdots j_a\cdots i_n}$$

$\nabla\mathcal{X}$

- Symmetric coset ($\mathcal{X} \to -\mathcal{X}$) = Adler zero

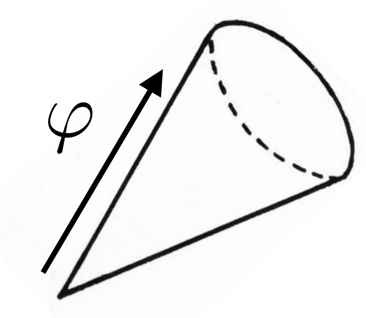
$$\lim_{q\to 0} A_{n+1}^{i_1\ldots i_n i} = 0$$

Example: soft dilaton

[Callan]

- Spontaneously broken spacetime symmetry - scale invariance

- Dilaton (NGB) = flat direction in field space $\nabla_\varphi = \partial_\varphi$



- Dimensional analysis $$\left(f_\varphi \partial_{\langle\varphi\rangle} + \sum_{a=1}^{n} p_a^\mu \frac{\partial}{\partial p_a^\mu} \right) A = (D - n\Delta) A$$

- Soft dilaton theorem

$$\lim_{q\to 0} A_{n+1} = \partial_{\langle\varphi\rangle} A_n = \frac{1}{f_\varphi} \left(D - n\Delta - \sum_{a=1}^{n} p_a^\mu \frac{\partial}{\partial p_a^\mu} \right) A_n$$

also works with masses!