Null Wilson loop with Lagrangian insertion in $\mathcal{N} = 4$ super-Yang-Mills theory

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based on works 2202.05596

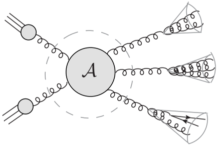
and 2204.00329 with Johannes Henn

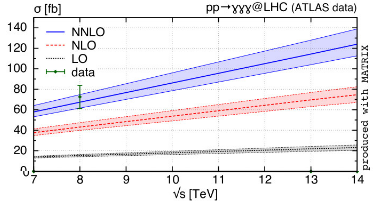
European Research Council
Established by the European Commission

10 August 2022

The era of precision collider physics

- Ever improving experimental precision @ LHC
- Next-to-next-to-leading order theoretical predictions for QCD processes are required nowadays
- Great interest in $2 \to 2$ and $2 \to 3$ processes





plot from arXiv:2010.04681

- Parton QCD amplitudes are input for cross-section (and event-shape) calculations [talk by S. Zoia]

$$\mathcal{A}_n^{\mathrm{QCD}} = \mathcal{Z}_{\mathrm{IR}} \times \mathcal{H}_n$$

(Planar) maximally supersymmetric Yang-Mills theory is a theoretical laboratory for amplitude calculations

Established for $\mathcal{N} = 4$ sYM amplitudes and applied to QCD amplitudes

- on-shell recursion relations
  [Britto, Cachazo, Feng, Witten '05]
- generalized unitarity
  [Bern, Dixon, Dunbar, Kosower '94]
- symbols and alphabets
  [Goncharov, Spradlin, Vergu, Volovich '10]
- differential equations [Henn '13]
- pure integrals and leading singularities
  [Arkani-Hamed, Cachazo, Cheung, Kaplan '09]
  [Arkani-Hamed, Bourjaily, Cachazo, Trnka '10]
- color-kinematics duality
  [Bern, Carrasco, Johansson '08]
- ...

Massless QCD amplitudes vs planar $\mathcal{N} = 4$ sYM amplitudes

finite part of $\mathcal{A}_n$	QCD	planar $\mathcal{N} = 4$ sYM
# variables	$3n - 11$	$3n - 15$

The dual-conformal symmetry of the planar $\mathcal{N} = 4$ sYM severely restricts the functional dependence of amplitudes

n	QCD	planar $\mathcal{N} = 4$ sYM
4	1	**0**
5	4	**0**
6	7	3
7	10	6

immediate phenomenological interest (n = 4, 5)

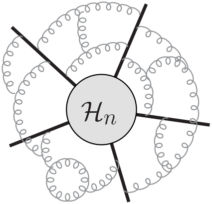
[talk by J. Henn] → n = 6

Planar four-point QCD amplitudes are highly nontrivial

Counterpart in the planar $\mathcal{N} = 4$ sYM ?

Wilson loops with Lagrangian insertion is a planar $\mathcal{N}=4$ sYM analog of the finite part of massless QCD amplitudes

$$\mathcal{H}_n^{\mathrm{QCD}} = \frac{\mathcal{A}_n^{\mathrm{QCD}}}{\mathcal{Z}_{\mathrm{IR}}}$$



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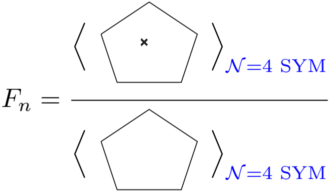
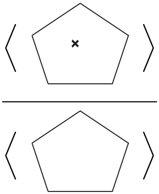


Why are Wilson loops with Lagrangian insertion in planar $\mathcal{N} = 4$ sYM interesting?

$F_n =$ 

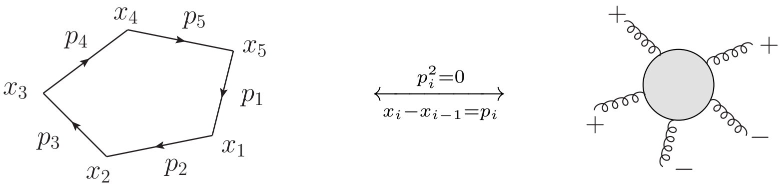
- Finite observable in $\mathcal{N} = 4$ super Yang-Mills theory [Alday, Buchbinder, Tseytlin '11]
- Displays nice properities of $\mathcal{N} = 4$ sYM amplitudes

  hidden symmetries and Yangian

  Grassmannian representation

  positivity properties and Amplituhedron geometry

Null Wilson loop / scattering amplitude duality in planar $\mathcal{N} = 4$ sYM



[Alday, Maldacena '07] [Drummond, Korchemsky, Sokatchev '07][Brandhuber, Heslop, Travaglini '07]

- Duality at the level of regularised quantities

$$\langle W_n \rangle \sim \frac{A_n^{\text{MHV}}}{A_{n,\text{tree}}^{\text{MHV}}}$$

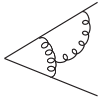
- Duality at the level of their integrands

$$L\text{-loop } 4D \text{ integrand of } \langle W_n \rangle \quad = \quad L\text{-loop } 4D \text{ integrand of } A_n^{\text{MHV}}$$

Anomalous symmetries in the regularised theory

Cusp divergences of WL exponentiate

$$\log\langle W_n\rangle = \sum_{L\geq 1} \frac{g^{2L}\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \,, \qquad g^2 \equiv \frac{N_c g^2_{\text{YM}}}{16\pi^2}$$



Anomalous dual-conformal symmetry of $\langle W_n\rangle$ [Drummond, Henn, Korchemsky, Sokatchev '07]

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

Wilson loop integrands and Lagrangian insertion

How to define the integrand of the L-loop correction of the Wilson loop?

After rescaling of the fields $A^\mu \to \frac{1}{g_{\rm YM}} A^\mu$, ...

$$\langle W_n \rangle \sim \int \mathcal{D}A^\mu \, e^{\frac{i}{g^2_{\rm YM}} \int d^d y \mathcal{L}(y)} W_n$$

differentiation in the coupling constant results in the Lagrangian insertion formula

$$\boxed{g^2_{\rm YM} \partial_{g^2_{\rm YM}} \langle W_n \rangle \sim \int d^d y \, \langle W_n \mathcal{L}(y) \rangle}$$

with $\mathcal{N} = 4$ sYM Lagrangian, $\mathcal{L} = -\frac{1}{2} \mathrm{tr} \left(F_{\alpha\beta}\right)^2 + \ldots$

Wilson loop integrands are Born-level correlators

$$\langle W_n \rangle = 1 + g^2\, W_n^{(1)} + g^4\, W_n^{(2)} + \ldots\,, \qquad g^2 \equiv \frac{N_c g_{\mathrm{YM}}^2}{16\pi^2}$$

Lagrangian insertion formula

$$g^2 \partial_{g^2} \langle W_n \rangle \sim \int d^d y\, \langle W_n \mathcal{L}(y) \rangle$$

One-loop correction

$$W_n^{(1)} \sim \int d^d y\, \underbrace{\langle W_n\, \mathcal{L}(y) \rangle_{\mathrm{Born}}}_{\text{one-loop integrand}}$$

L-loop correction

$$\boxed{W_n^{(L)} \sim \int d^d y_1 \ldots d^d y_L\, \underbrace{\langle W_n\, \mathcal{L}(y_1) \ldots \mathcal{L}(y_L) \rangle_{\mathrm{Born}}}_{L\text{-loop integrand}}}$$

The integrands are finite rational functions in four dimensions

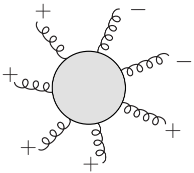
Planar amplitude integrands

The planar amplitude integrand is unambiguous in the dual-momenta variables

$$M_n^{(L)}(x_1,\ldots,x_n) \sim \int d^d y_1 \ldots d^d y_L \, \mathcal{I}_n^{(L)}(\underbrace{x_1,\ldots,x_n}_{\text{external dual momenta}} \mid \underbrace{y_1,\ldots,y_L}_{\text{loop dual momenta}})$$

$$M_n \equiv \frac{A_n^{\text{MHV}}}{A_{n,\text{tree}}^{\text{MHV}}}$$

$$M_n = 1 + g^2 M_n^{(1)} + g^4 M_n^{(2)} + \ldots$$



$4D$ integrands are very-well studied and known, in principle, at any loop order

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka '10]

Four-particle $4D$ integrands are explicitly known up to ten-loop order

[Bourjaily, Heslop, Tran '16]

Null Wilson loop / scattering amplitude duality at the integrand level

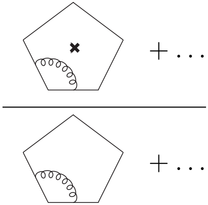
The planar $4D$ integrands are dual-conformal and chiral (parity-even + parity-odd)

$$\mathcal{I}_{\rm MHV}(x_1,\ldots,x_n|y_1,\ldots,y_L) = \langle W[x_1,\ldots,x_n]\mathcal{L}(y_1)\ldots\mathcal{L}(y_L)\rangle_{\rm Born}$$

The integrands satisfy the same set of recurrence relations

Null Wilson loop with Lagrangian insertion

$$F_n(x_1,\ldots,x_n;x_0) \equiv \frac{\langle W[x_1,\ldots,x_n]\mathcal{L}(x_0)\rangle}{\langle W[x_1,\ldots,x_n]\rangle} =$$



Finite observable in $4D$. Cusp divergences cancel out in the ratio

Due to the Lagrangian insertion trick, F_n is $\log\langle W_n\rangle$ with one of the loop integrations 'frozen'

$$g^2\partial_{g^2}\log\langle W_n\rangle \sim \int d^dx_0 \underbrace{F_n(x_1,\ldots,x_n;x_0)}_{\text{finite in 4D}} \sim \frac{1}{\epsilon^2}$$

Null Wilson loop with Lagrangian insertion

$$\Big[\log\langle W_n\rangle\Big]_{(L+1)\text{-loop}} \sim \int d^d y_1 \dots d^d y_{L+1}\, \Omega_n^{(L+1)}(x_1,\dots,x_n|y_1,\dots,y_L,y_{L+1})$$

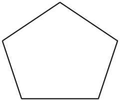
$(L+1)$-loop integrand of the logarithm: $\Omega_n^{(L+1)}$ – a rational function with numerator which suppresses cusp singularities

$$F_n^{(L)}(y_{L+1}) \sim \int d^4 y_1 \dots d^4 y_L\, \Omega_n^{(L+1)}(x_1,\dots,x_n|y_1,\dots,y_L,y_{L+1})$$

L-loop integrations in 4D are finite!

L-loop F_n is $(L+1)$-loop $\log\langle W_n\rangle$ with one of the loop integrations frozen

Null Wilson loop with Lagrangian insertion is an interesting observable with nice properties

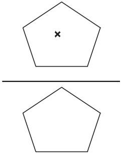
$\langle W_n \rangle =$ 

cusp divergences

transcendental function

of $3n - 15$ variables

anomalous dual-conformal symmetry

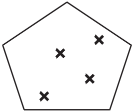
$F_n =$ 

finite in four dimensions

transcendental function

of $3n - 11$ variables

exactly dual-conformal



$\langle W_n \, \mathcal{L}(y_1) \ldots \mathcal{L}(y_L) \rangle_{\text{Born}}$

finite in four dimensions

rational function

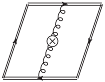
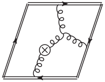
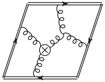
of $3n + 4L - 15$ variables

exactly dual-conformal

Structure of loop corrections

Perturbative weak-coupling expansion

$$F_n = \underbrace{g^2\, F_n^{(0)}}_{\text{Born-level}} + \underbrace{g^4\, F_n^{(1)}}_{\text{one-loop}} + \underbrace{g^6\, F_n^{(2)}}_{\text{two-loop}} + \ldots$$

...  ... ...  ... ...  ...

L-loop correction

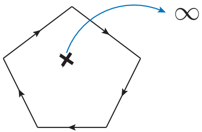
$$F_n^{(L)} = \sum_j \underbrace{R_{n,j}(x_1, \ldots, x_n; x_0)}_{\text{leading singularity – rational function}} \; \underbrace{g_{n,j}^{(L)}}_{\text{transcendental function weight } 2L} \; \underbrace{(u_1, \ldots, u_{3n-11})}_{\text{dual-conformal cross-ratios}}$$

- What is the set of all-loop n-point leading singularities?
- What is the class of the transcendental functions for n-point observable?

Wilson Loop with Lagrangian insertion in the "particle" frame

Cancel out dual-conformal weight $+4$ at the Lagrangian insertion point,

$$f_n(x_1,\ldots,x_n) \equiv \lim_{x_0\to\infty} (x_0^2)^4\, F_n(x_1,\ldots,x_n;x_0)$$



The frame $x_0 = \infty$ breaks the dual-conformal symmetry

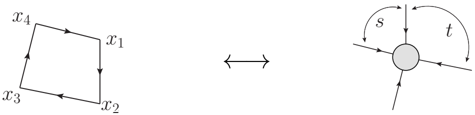
$$f_n(x_1,\ldots,x_n) \sim f_n(p_1,\ldots,p_n)$$

- Finite function in four dimensions
- Kinematics of n-particle scattering in a massless QFT

Perturbative data for the Wilson Loop with Lagrangian insertion

4-cusp	one- and two-loop	[Alday, Heslop, Sikorowski '12] [Alday, Henn, Sikorowski '13]
4-cusp	three-loop	[Henn, Korchemsky, Mistlberger '19]
4-cusp	strong coupling	[Alday, Buchbinder, Tseytlin '11]
4-cusp	all-loop 'tree' part in negative geometries expansion	[Arkani-Hamed, Henn, Trnka '21]
5-cusp	one- and two-loop	[DC, Henn '22]
n-cusp	one-loop	[DC, Henn '22]

Four-particle observable



$$f_4^{(L)} = x_{13}^2 x_{24}^2 \, g^{(L)}\left(\frac{x_{13}^2}{x_{24}^2}\right) = \underbrace{s\,t}_{\text{leading sing.}} \; \underbrace{g^{(L)}}_{\text{HPL}}\left(\frac{t}{s}\right)$$

- One leading singularity $x_{13}^2 x_{24}^2$
- L-loop corrections are Harmonic Polylogarithms (HPL) of weight $2L$

$$g^{(0)}(z) = -1\,, \qquad g^{(1)}(z) = \log^2(z) + \pi^2\,, \qquad \ldots$$

Five-particle observable $(L = 0, 1, 2)$

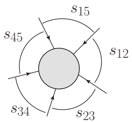
$$f_5^{(L)} = f_5^{(0)}\, g_0^{(L)}(\mathbf{u}) + \sum_{i=1}^{5} r_i\, g_i^{(L)}(\mathbf{u})$$

- Six leading singularities [DC, Henn '22]

$$r_0 \equiv f_5^{(0)}\,, \quad r_1\,, \quad r_2\,, \quad r_3\,, \quad r_4\,, \quad r_5$$

- Loop corrections $g_i^{(L)}$ are transcendental functions (pentagon functions) of four-variables

$$\mathbf{u} = \left\{ \frac{s_{12}}{s_{15}}\,, \frac{s_{23}}{s_{15}}\,, \frac{s_{34}}{s_{15}}\,, \frac{s_{45}}{s_{15}} \right\}\,, \qquad s_{ij} \equiv (p_i + p_j)^2$$



- Kinematics of a five-particle scattering amplitude in a massless QFT
- Nontrivial parity properties

$$i\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

The leading singularities are conformally invariant in momentum space

Six leading singularities normalized by the Parke-Taylor factor, $\mathrm{PT}_5\, r_i$, have a very simple form

$$\frac{[51]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} + \frac{[34]^2}{\langle 51\rangle\langle 12\rangle\langle 25\rangle} - \frac{[13]^2}{\langle 24\rangle\langle 45\rangle\langle 52\rangle}\,,$$

$$\frac{[34]^2}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\,,\; \frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,,\; \frac{[51]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle}\,,\; \frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle}\,,\; \frac{[23]^2}{\langle 45\rangle\langle 51\rangle\langle 14\rangle}$$

Remarkable conformal symmetry (in momentum space)

$$\mathbb{K}_{\alpha\dot\alpha} = \sum_{i=1}^{5} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde\lambda_i^{\dot\alpha}}\,, \qquad \mathbb{K}_{\alpha\dot\alpha}\Big(\underbrace{\mathrm{PT}_5}_{\frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle 51\rangle}}\; r_i\Big) = 0$$

Dual-conformal symmetry of $F_5(x_0)$ and conformal symmetry of f_5

Loop corrections of the five-particle observable: Planar pentagon functions

- pentagon functions are classified at weight $w \leq 4$ (two-loop order)

[Gehrmann, Henn,Lo Presti '18][DC, Sotnikov '20]

- polylogarithmic iterated integral representation

$$\int d\log(W_{i_1})\ldots d\log(W_{i_w})$$

- 26-letter planar pentagon alphabet

W_1	$2\,p_1\cdot p_2$	$+(4)$
W_6	$2\,p_4\cdot(p_3+p_5)$	$+(4)$
W_{11}	$2\,p_3\cdot(p_4+p_5)$	$+(4)$
W_{16}	$2\,p_1\cdot p_3$	$+(4)$
W_{21}	$2\,p_3\cdot(p_1+p_4)$	$+(4)$
W_{26}	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}{\mathrm{tr}[(1+\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}$	$+(4)$
W_{31}	$i\epsilon(p_1,p_2,p_3,p_4)$	

- efficient routines for numerical evaluations (as required for NNLO QCD)

Definite sign of the loop corrections

Positivity of the integrated loop corrections?

Observed for $f_4^{(L)} = s\,t \cdot g^{(L)}(t/s)$ [Arkani-Hamed, Henn, Trnka '21]

Five-particle positivity is highly nontrivial! [DC, Henn '22]

$$(-1)^{L+1} f_5^{(L)} > 0 \qquad \text{at} \quad L = 0, 1, 2$$

in the one-loop Amplituhedron region of the five-particle scattering

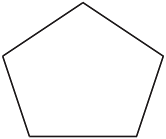
$$s_{12} < 0\,,\; s_{23} < 0\,, \ldots, s_{15} < 0\,, \quad i\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma > 0$$

The sign of six terms varies inside the Amplituhedron region!

$$f_5^{(L)} = \underbrace{r_0}_{\pm}\,\underbrace{g_0^{(L)}(\mathbf{u})}_{\pm} + \ldots + \underbrace{r_5}_{\pm}\,\underbrace{g_5^{(L)}(\mathbf{u})}_{\pm}$$

At two loop order $L = 2$, the definite sign of the sum appears as a miracle

Duality with all-plus amplitude in pure Yang-Mills



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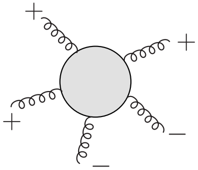
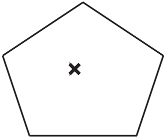


$\mathcal{N} = 4$ super-Yang-Mills $\qquad$ $\mathcal{N} = 4$ super-Yang-Mills



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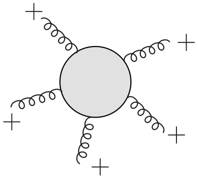
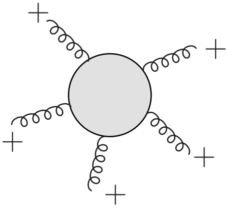


$\mathcal{N} = 4$ super-Yang-Mills $\qquad$ pure Yang-Mills

Planar all-plus amplitude in pure Yang-Mills

$$\mathcal{A}_n^{\rm YM} = g_{\rm YM}^{n-2} \sum_{L\geq 1} g^{2L} \sum_{\sigma\in S_n/Z_n} \mathrm{tr}\left(T^{a_{\sigma_1}} \ldots T^{a_{\sigma_n}}\right) A^{(L)}_{{\rm YM},n}(\sigma_1^+,\ldots,\sigma_n^+) + \mathcal{O}\left(N_c^{-1}\right)$$



tree-level $A^{(0)}_{{\rm YM},n} = 0$

one-loop $A^{(1)}_{{\rm YM},n}$ finite rational function

two-loop $A^{(2)}_{{\rm YM},n}$ highest transcendental weight two

three-loop $A^{(3)}_{{\rm YM},n}$ highest transcendental weight four

Planar all-plus amplitude in pure Yang-Mills

$$A_n^{\text{YM}} = \underbrace{\mathcal{Z}_{\text{IR}}^{\text{YM}}}_{\text{IR poles}} \underbrace{g^2 A^{(1)}_{\text{YM,n}}}_{\text{Born level}} \underbrace{\mathcal{H}_n^{\text{YM}}}_{\text{finite part}} + \mathcal{O}\left(N_c^{-1}\right)$$

Available perturbative data:

one-loop	n-particle	[Bern, Chalmers, Dixon, Kosower '93] [Henn, Power, Zoia '19]
two-loop	five-particle	[Gehrmann, Henn, Lo Presti '15] [Badger, DC, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '18]
two-loop	n-particle	[Dunbar, Jehu, Perkins '16][Dunbar, Godwin, Jehu, Perkins '17] [Dunbar, Perkins, Strong '20] [Kosower, Pögel '22]
three-loop	four-particle	[Jin, Luo '19] [Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21]

Duality planar all-plus amplitude / Wilson loop with Lagrangian insertion

$$\langle W_n[x_1, \ldots, x_n]\, \mathcal{L}(x_0 = \infty) \rangle \;\sim\; A^{\text{all-plus}}_{\text{YM},n}(p_1, \ldots, p_n)$$

- Duality at the lowest perturbative order

$$\text{PT}_n\, f_n^{(0)} = A^{(1)}_{\text{YM},n}$$

- Duality at the loop level (in 4D and at the highest transcendentality level)

$$\log \mathcal{H}_n^{\text{MHV}} + \log \left(\frac{f_n}{g^2 f_n^{(0)}} \right) \sim \log \mathcal{H}_n^{\text{YM}} + \mathcal{O}(\epsilon)$$

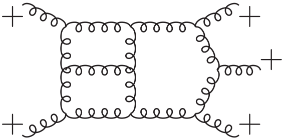
Duality planar all-plus amplitude / Wilson loop with Lagrangian insertion

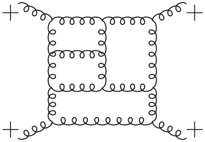
L-loop Lagrangian insertion $\sim$ $(L+1)$-loop all-plus YM amplitude

✓ Agreement with available perturbative data

✓ Predictions for all-plus YM amplitude (planar and maximal transcendentality)

- three-loop five-particle
- four-loop four-particle





n-particle leading singularities

Conjecture: "Kermits" of one-loop MHV amplitude are $(n-1)(n-2)^2(n-3)/12$ leading singularities $r_{n,j}$ of the n-particle f_n

$$f_n^{(L)} = \sum_j r_{n,j}\, g_{n,j}^{(L)}(\mathbf{u})$$

Agrees with Born-level $f_n^{(0)}$ and one-loop $f_n^{(1)}$

Conformal symmetry (in momentum space)

$$\mathbb{K}_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^\alpha \partial\tilde{\lambda}_i^{\dot{\alpha}}}\,, \qquad \mathbb{K}_{\alpha\dot{\alpha}}\Big(\underbrace{\mathrm{PT}_n}_{\frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}} \; r_n \Big) = 0$$

Yangian-like relations for the leading singularities

Yangian
r_n and $R_n(x_0)$

↙ conformal symmetry r_n

↘ dual-conformal symmetry $R_n(x_0)$

Conclusions

Wilson loop with Lagrangian insertion has similarities with finite parts of massless scattering amplitudes in QCD

Several remarkable properties:

- Positivity in the Amplituhedron region
- Duality with all-plus amplitudes in pure Yang-Mills
- Conformal (and Yangian) symmetry of the leading singularities