Hamiltonian Mechanics, Nambu Mechanics, and Generalized Phase Space

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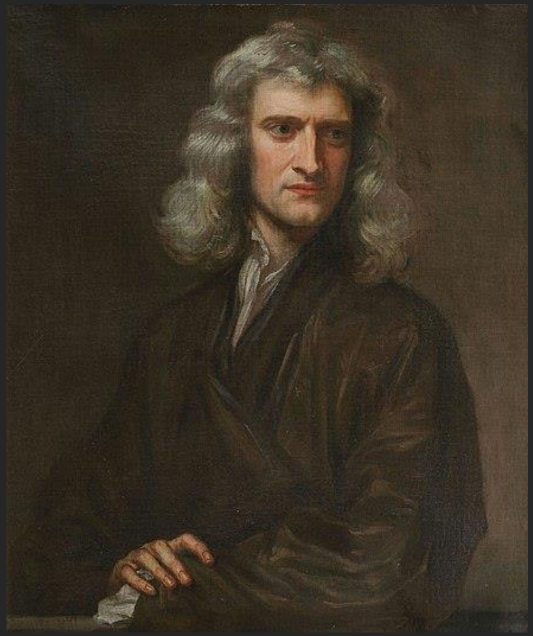
Newtonian Mechanics and Hamiltonian Mechanics



Sir Isaac Newton (1643-1727)

$$\boldsymbol{F} = m\boldsymbol{a}.$$

Sir William Rowan Hamilton (1805-1865)

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}.$$

- The correspondence between the two formulations can be made explicit: for $H = \frac{\boldsymbol{p}^2}{2m} + V$ both give $\dot{\boldsymbol{p}} = -\nabla V$.

*Pictures from Wikipedia

In an **ideal mechanical system**, there is no entropy growth mechanism (dissipation).

- Ideal mechanical systems usually arise as **Hamiltonian systems**.
- In the ideal limit, Newtonian mechanics and related theories (fluid mechanics, plasma physics, etc.) can be written in Hamiltonian form.

Hamiltonian mechanics can account for non-Newtonian physics as well. For example,

- Quantum mechanics (Schrödinger equation)
- General relativity (ADM formalism)

- Hamiltonian mechanics is a statement regarding the existence of phase space, while the properties of matter are encapsulated into the Hamiltonian function (energy) of the system.

The Schrödinger equation of quantum mechanics describes the deterministic evolution of the probability amplitude Ψ,

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V \right) \Psi.$$

Via the Madelung representation $\Psi = \sqrt{\rho}\, \exp(i\, \theta/\hbar)$,

$$\frac{\partial \rho}{\partial t} = -\frac{1}{m} \nabla \cdot (\rho \nabla \theta), \qquad \frac{\partial \theta}{\partial t} = \frac{\hbar^2}{2m} \left(-\frac{|\nabla \rho|^2}{4\rho^2} - \frac{|\nabla \theta|^2}{\hbar^2} + \frac{\Delta \rho}{2\rho} \right) - \mathrm{V}.$$

This system has the **canonical Hamiltonian form**

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \theta}, \qquad \frac{\partial \theta}{\partial t} = -\frac{\delta H}{\delta \rho}, \qquad H = \int_{\mathbb{R}^3} \rho \left(\frac{|\nabla \theta|^2}{2m} + V + \frac{\hbar^2}{8m} |\nabla \log \rho|^2 \right) \mathrm{dV}.$$

Magnetohydrodynamics is a fluid theory describing electrically conducting fluids:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v}),$$
$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) - \nabla\left(h + \frac{\boldsymbol{v}^2}{2}\right) + \frac{1}{\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B},$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}).$$

This system can be written in the **noncanonical Hamiltonian form**

$$\frac{\partial u}{\partial t} = \mathcal{J}\frac{\delta H}{\delta u}, \qquad u = (\rho, \boldsymbol{v}, \boldsymbol{B})^T,$$

where the Hamiltonian H and the Poisson operator $\mathcal{J}$ are given by:

$$H = \int_\Omega \left[\rho\left(\frac{\boldsymbol{v}^2}{2} + U(\rho)\right) + \frac{\boldsymbol{B}^2}{2}\right] \mathrm{dV}, \qquad \mathcal{J} = \begin{bmatrix} 0 & -\nabla \cdot & 0 \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{v}) \times & -\rho^{-1}\boldsymbol{B} \times (\nabla \times) \\ 0 & -\nabla \times (\rho^{-1}\boldsymbol{B} \times) & 0 \end{bmatrix}.$$

There are mechanical systems that cannot be written in Hamiltonian form:

- **Dissipative systems:** entropy grows. An example is the diffusion equation with Neumann boundary conditions,

$$\frac{\partial f}{\partial t} = D\Delta f, \qquad \frac{dS}{dt} = D\int_\Omega f|\nabla \log f|^2 \mathrm{dV} \geq 0, \qquad S = -\int_\Omega f \log f \, \mathrm{dV}.$$

- **Nonholonomic systems**: systems with non-integrable constraints that do not foliate the phase space into symplectic submanifolds. An example is $\boldsymbol{E} \times \boldsymbol{B}$ dynamics in a non-integrable magnetic field $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} \neq 0$,

$$m\ddot{\boldsymbol{x}} = q(\dot{\boldsymbol{x}} \times \boldsymbol{B} - \nabla\phi), \qquad m = 0 \rightarrow$$

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{B} \times \nabla\phi}{B^2}.$$

Note that the energy ϕ is constant, $\dot{\boldsymbol{x}} \cdot \nabla\phi = 0$, while the non-integrable constraint is given by

$$\dot{\boldsymbol{x}} \cdot \boldsymbol{B} = 0.$$

Ideal mechanical systems usually arise as **Hamiltonian systems**.

When all degrees of freedom are considered, the equations of motion take a **canonical Hamiltonian form**.

Hamilton's canonical equations for a Hamiltonian $H(p^1, \dots, p^m, q^1, \dots, q^m)$ are

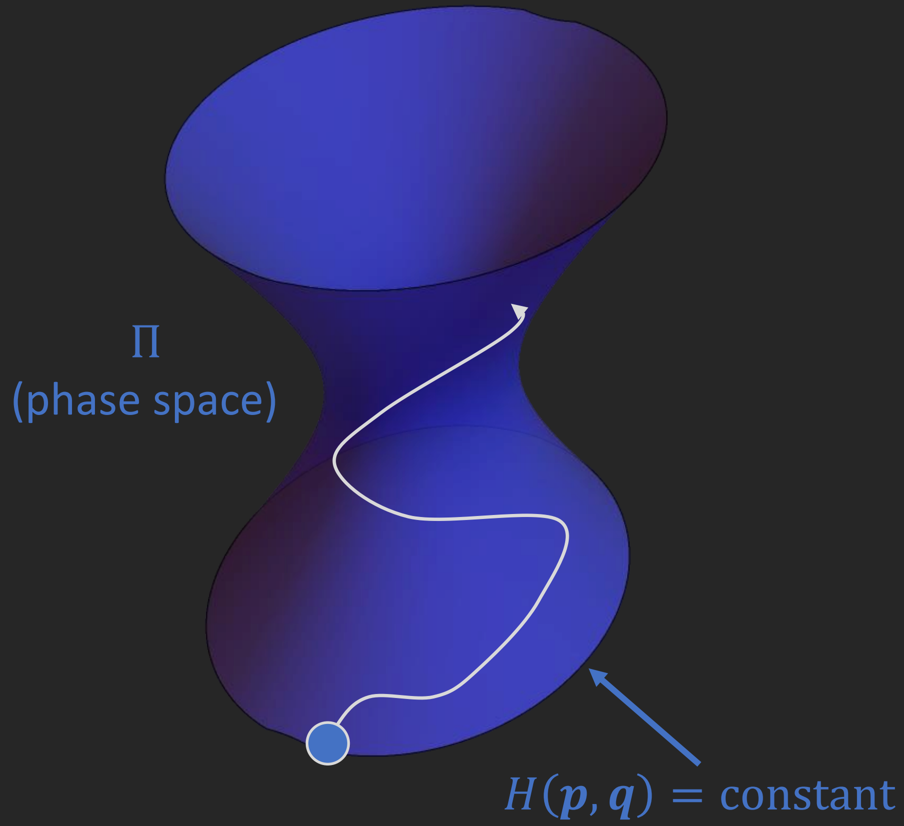
$$\dot{p}^i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p^i}, \qquad i = 1, \dots, m. \tag{1}$$

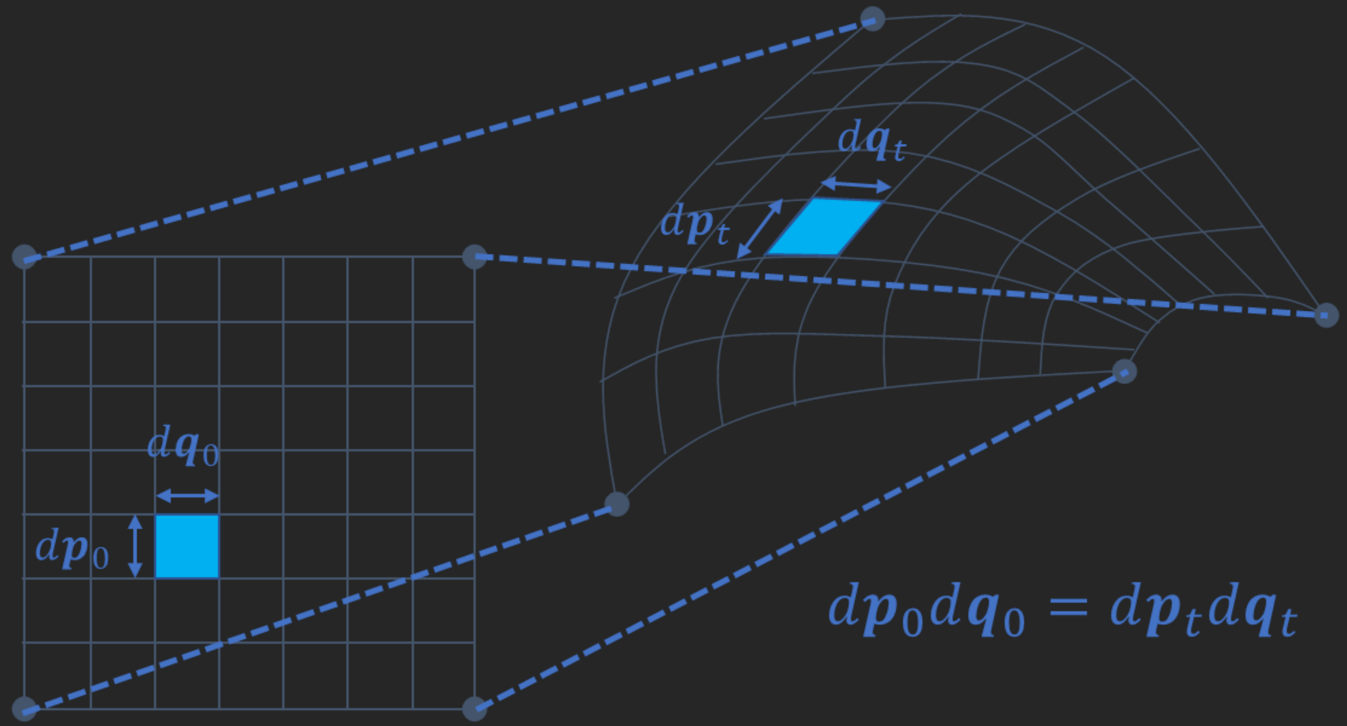
$\boldsymbol{F} = -\nabla V$

$K = \frac{\boldsymbol{p}^2}{2m}$

- H represents the energy of the system. From (1), one sees that $\dot{H} = 0$.
- The coordinate pairs (p^i, q^i), $i = 1, \dots, m$, are called **canonical pairs**, with $n = 2m$ the dimension of the system.

- Conservation of energy H.
- Conservation of phase space volume (**Liouville theorem**) $d\Pi = dp^1 \wedge \cdots \wedge dp^m \wedge dq^1 \wedge \cdots \wedge dq^m$.

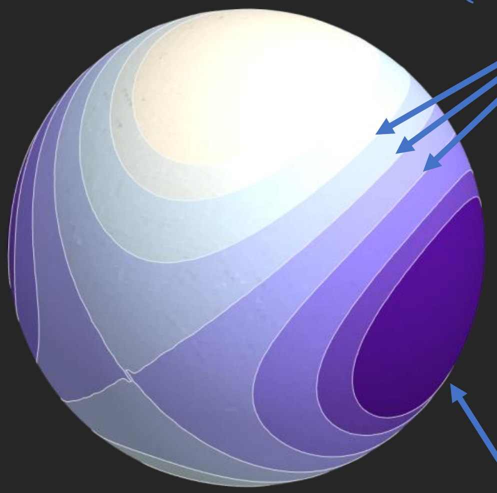




Hamilton's canonical equations can be generalized to the so called **noncanonical Hamiltonian form**.

Noncanonical Hamiltonian systems arise when redundant degrees of freedom are discarded.

Example: Euler's rotation equation for a rigid body.



$H(\boldsymbol{x}) = \text{constant}$

$C(\boldsymbol{x}) = \text{constant}$

$$\boldsymbol{x} \in \mathbb{R}^3 \quad \text{(angular momentum)},$$

$$\dot{\boldsymbol{x}} = \boldsymbol{w} \times \nabla H, \qquad \boldsymbol{w} = \nabla \frac{\boldsymbol{x}^2}{2}, \qquad H = \frac{1}{2}\left(\frac{x^2}{I_x} + \frac{y^2}{I_y} + \frac{z^2}{I_z}\right) \tag{2}$$

Poisson operator $\mathcal{J} = \boldsymbol{w} \times$

Casimir invariant $C = \boldsymbol{x}^2/2$

- In the noncanonical setting, canonical variables (p^i, q^i), $i = 1, \dots, m$, are 'hidden', and new conservation laws independent of H appear.

In an n-dimensional noncanonical Hamiltonian system, the equations of motion are written as:

$$\dot{x}^i = \mathcal{J}^{ij} H_j, \qquad i = 1, \dots, n. \tag{3}$$

Here, $\mathcal{J}^{ij}$ is an n-dimensional second-order **skew-symmetric** contravariant tensor satisfying the **Jacobi identity**:

$$\mathcal{J}^{ij} = -\mathcal{J}^{ji}, \qquad \mathcal{J}^{im}\mathcal{J}^{jk}_m + \mathcal{J}^{jm}\mathcal{J}^{ki}_m + \mathcal{J}^{km}\mathcal{J}^{ij}_m = 0, \qquad i, j, k = 1, \dots, n. \tag{4}$$

The tensor $\mathcal{J}^{ij}$ is called **Poisson operator**. It can be used to define the **Poisson bracket** $\{\cdot,\cdot\}$:

$$\{F, G\} = F_i \mathcal{J}^{ij} G_j \quad \rightarrow \quad \frac{dF}{dt} = \{F, H\}. \tag{5}$$

Let $\mathcal{X}$ be a vector space over a field K. A **Poisson bracket** on $\mathcal{X}$ is a binary operation

$$\{\cdot,\cdot\} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}, \tag{6}$$

$\mathcal{X} = C^\infty(\Omega), \quad \Omega \subset \mathbb{R}^n,$
$K = \mathbb{R}$

that satisfies the following properties for all $F, G, H \in \mathcal{X}$ and $a, b \in K$:

1) Bilinearity

$$\{aF + bG, H\} = a\{F, H\} + b\{G, H\}, \qquad \{H, aF + bG\} = a\{H, F\} + b\{H, G\},$$

2) Alternativity

$$\{F, F\} = 0,$$

$\dot{H} = \{H, H\} = 0$

3) Skew-symmetry

$$\{F, G\} = -\{G, F\}, \tag{7}$$

4) Leibniz rule

$$\{FG, H\} = F\{G, H\} + \{F, H\}G,$$

$(\dot{FG}) = \{FG, H\} = F\dot{G} + \dot{F}G$

5) Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

Hamiltonian (phase space) structure

The **Jacobi identity** is equivalent to the existence of a **closed** 2-form of even rank $2m = n - s$, the **symplectic 2-form** ω. Here, s is the dimension of the kernel of the Poisson bracket.

Due to the Lie-Darboux theorem [1-3], the closure of the 2-form ω implies that the phase space is **locally** spanned by $2m = n - s$ **canonically conjugated variables** (p^i, q^i), $i = 1, \dots, m$, and $s = n - 2m$ **Casimir invariants** C_i, $i = 1, \dots, s$, which fill the center (**kernel**) of the Poisson bracket:

$$\frac{dC^i}{dt} = \{C^i, H\} = 0, \qquad \forall H \in C^\infty(\Omega), \qquad i = 1, \dots, s. \tag{8}$$

The noncanonical equations of motion therefore take the **local canonical form**

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \qquad \frac{dC^j}{dt} = 0, \qquad i = 1, \dots, m, \qquad j = 1, \dots, s. \tag{9}$$

[1] M. de Léon, *Methods of Differential Geometry in Analytical Mechanics*, Elsevier, New York, pp. 250–253 (1989). [2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, second ed., Springer, New York, pp. 230–232 (1989). [3] R. Littlejohn, *Singular Poisson tensors*, in: M. Tabor, Y. Treve (Eds.), Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, in: American Institute of Physics Conference Proceedings, 88, American Institute of Physics, New York, pp. 47–66 (1982).

In 1973, Y. Nambu proposed a '*possible generalization of classical Hamiltonian dynamics to a 3-dimensional phase space*': the classical canonical pair (p, q) is replaced by a **canonical triplet** (p, q, r), while the number of generating functions is increased to **two Hamiltonians**, (G, H) [4].

Rigid body

$$(p, q, r) = (x, y, z), \qquad G = C = \frac{\boldsymbol{x}^2}{2}, \qquad H = \frac{1}{2}\left(\frac{x^2}{I_x} + \frac{y^2}{I_y} + \frac{z^2}{I_z}\right)$$

For the $n = 3$ case, Hamilton's canonical equations are replaced by Nambu's canonical equations:

$$\frac{dp}{dt} = \frac{\partial G}{\partial q}\frac{\partial H}{\partial r} - \frac{\partial G}{\partial r}\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial G}{\partial r}\frac{\partial H}{\partial p} - \frac{\partial G}{\partial p}\frac{\partial H}{\partial r}, \qquad \frac{dr}{dt} = \frac{\partial G}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial G}{\partial q}\frac{\partial H}{\partial p}. \tag{10}$$

Setting $(x^1, x^2, x^3) = (p, q, r)$, system (10) can be written through a ternary operation (**Nambu bracket**) $\{\circ,\circ,\circ\}: C^\infty(\Omega) \times C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega)$ as below:

$$\frac{dx^i}{dt} = \{x^i, G, H\} = \epsilon^{ijk} G_j H_k, \qquad i = 1,2,3, \qquad \to \qquad \frac{dF}{dt} = \{F, G, H\} = \epsilon^{ijk} F_i G_j H_k. \tag{11}$$

[4] Y. Nambu, *Generalized Hamiltonian Dynamics*, Phys. Rev. D **7**, 8 (1973).

The generalization of the Nambu bracket (11) to an algebraic framework analogous to the Poisson bracket of classical Hamiltonian mechanics has proven difficult because the generalization of the Poisson bracket axioms (7) to the Nambu bracket is nontrivial [5].

Several authors [5, 6] have proposed the following set of **axioms** for the **Nambu bracket**: trilinearity, skew-symmetry, Leibniz rule, and fundamental identity:

1) Trilinearity

$$\{aF_1 + bF_2, F_3, F_4\} = a\{F_1, F_3, F_4\} + b\{F_2, F_3, F_4\},$$

2) Skew–symmetry

$$\{F_1, F_2, F_3\} = \epsilon^{ijk}\{F_i, F_j, F_k\}, \qquad \text{(not summed)},$$

$$\dot{G} = \{G, G, H\} = 0, \quad \dot{H} = \{H, G, H\} = 0$$

3) Leibniz rule

$$\{F_1F_2, F_3, F_4\} = F_1\{F_2, F_3, F_4\} + F_2\{F_1, F_3, F_4\},$$

$$(F_1 \dot{F_2}) = F_1\dot{F}_2 + \dot{F}_1 F_2$$

4) Fundamental identity

$$\{\{F_1, F_2, F_3\}, F_4, F_5\} = \{\{F_1, F_4, F_5\}, F_2, F_3\} + \{F_1, \{F_2, F_4, F_5\}, F_3\} + \{F_1, F_2, \{F_3, F_4, F_5\}\}. \tag{12}$$

$\forall F_1, F_2, F_3, F_4, F_5 \in C^\infty(\Omega)$ and $a, b \in \mathbb{R}$.

[5] P. M. Ho and Y. Matsuo, *The Nambu bracket and M-theory*, Prog. Theor. Exp. Phys. **2016**, 06A104 (2016).
[6] L. Takhtajan, *On Foundation of the Generalized Nambu Mechanics*, Commun. Math. Phys. **160**, pp. 295-315 (1994).

- The **fundamental identity**, which replaces the **Jacobi identity** for the Poisson bracket, implies **distribution of derivatives** (a property satisfied by Poisson brackets), that is, given two Hamiltonians F_4 and F_5, one has

$$\frac{d}{dt}\{F_1, F_2, F_3\} = \left\{\frac{dF_1}{dt}, F_2, F_3\right\} + \left\{F_1, \frac{dF_2}{dt}, F_3\right\} + \left\{F_1, F_2, \frac{dF_3}{dt}\right\}. \tag{13}$$

- The fundamental identity leads to the property that **the bracket** $\{\circ,\circ\}_G = \{\circ, G, \circ\}$ defined by fixing the second entry with a given generating function G **assigns a Poisson algebra**. This can be verified by observing that the fundamental identity reduces to the Jacobi identity when $F_1 = F_4 = G$.

- However, the fundamental identity also implies that **constant skew-symmetric 3-tensors** $\mathcal{J}^{ijk} \in \mathbb{R}$ **do not define a Nambu bracket** in general, i.e. $\{F, G, H\} = \mathcal{J}^{ijk} F_i G_j H_k$ does not satisfy the axioms (12) even if $\mathcal{J}^{ijk}$ has constant entries.

- This situation points to the fact that **the fundamental identity is more stringent than the Jacobi identity** required for a Poisson bracket (on this point, see [6-7]).

[6] L. Takhtajan, *On Foundation of the Generalized Nambu Mechanics*, Commun. Math. Phys. 160, pp. 295-315 (1994).
[7] R. Chatterjee and L. Takhtajan, *Aspects of classical and quantum Nambu mechanics*, Lett. Math. Phys. 37, pp. 475-482 (1996).

Consider a 1-dimensional system of 2 quantum oscillators [8] with Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \xi_1 + \xi_2) + \lambda q_1 \xi_2. \quad (14)$$

The phase space variables $\boldsymbol{x} = (p_1, q_1, \xi_1, p_2, q_2, \xi_2)$ correspond to the expectation values

$$p_i = \langle \hat{p}_i \rangle, \qquad q_i = \langle \hat{q}_i \rangle, \qquad \xi_\mathrm{i} = \langle \hat{q}_i^2 \rangle, \qquad i = 1,2. \quad (15)$$

The system is Hamiltonian with Poisson operator $\mathcal{J}$ given by

$$\mathcal{J} = \begin{bmatrix} 0 & -1 & -2q_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2q_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2q_2 & 0 & 0 \end{bmatrix}, \qquad \dot{\boldsymbol{x}} = \mathcal{J}(dH) = \begin{bmatrix} -q_1 - \lambda\xi_2 \\ p_1 \\ 2q_1p_1 \\ -q_2 - 2\lambda q_1 q_2 \\ p_2 \\ 2q_2p_2 \end{bmatrix}. \quad (16)$$

[8] A. Horikoshi, *Nambu dynamics and its noncanonical Hamiltonian representation in many degrees of freedom systems*, Prog. Theor. Exp. Phys. **2021**, ptab075 (2021).

Hamiltonian Mechanics vs Nambu Mechanics – Example

The system is endowed with 2 additional constants of motion:

$$G_1 = \xi_1 - q_1^2, \qquad G_2 = \xi_2 - q_2^2, \qquad G = G_1 + G_2. \tag{17}$$

The equations of motion can be expressed through a Nambu bracket

$$\{F, G, H\} = \epsilon_1^{ijk} F_i G_j H_k + \epsilon_2^{ijk} F_i G_j H_k, \tag{18}$$

as below

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{\xi}_1 \\ \dot{p}_2 \\ \dot{q}_2 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \{p_1, G, H\} \\ \{q_1, G, H\} \\ \{\xi_1, G, H\} \\ \{p_2, G, H\} \\ \{q_2, G, H\} \\ \{\xi_2, G, H\} \end{bmatrix} = \begin{bmatrix} H_{\xi_1} G_{q_1} - H_{q_2} G_{\xi_1} \\ H_{p_1} G_{\xi_1} - H_{\xi_1} G_{p_1} \\ H_{q_1} G_{p_1} - H_{p_1} G_{q_1} \\ H_{\xi_2} G_{q_2} - H_{q_2} G_{\xi_2} \\ H_{p_2} G_{\xi_2} - H_{\xi_2} G_{p_2} \\ H_{q_2} G_{p_2} - H_{p_2} G_{q_2} \end{bmatrix} \quad \leftrightarrow \quad \dot{x}_m^i = \epsilon_m^{ijk} G_j H_k, \qquad m = 1,2. \tag{19}$$

- The Nambu bracket (18) does not satisfy the fundamental identity despite having constant entries and the system being Hamiltonian.

Jacobi Identity – Geometrical Meaning

The Jacobi identity of classical Hamiltonian mechanics has a simple geometrical interpretation: if $\mathcal{J}^{ij}$ is invertible with inverse ω_{ij}, the noncanonical equations of motion can be written as

$$\dot{x}^i = \mathcal{J}^{ij} H_j \quad \rightarrow \quad \omega_{ij}\dot{x}^j = H_i. \tag{20}$$

Defining the **symplectic 2-form** $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$, one can show that

$$\begin{matrix} \mathcal{J}^{im}\mathcal{J}^{jk}_m + \mathcal{J}^{jm}\mathcal{J}^{ki}_m + \mathcal{J}^{km}\mathcal{J}^{ij}_m = 0, \\ i,j,k = 1,\dots,n. \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \frac{\partial \omega_{kj}}{\partial x^i} + \frac{\partial \omega_{ik}}{\partial x^j} + \frac{\partial \omega_{ji}}{\partial x^k} = 0 \\ i,j,k = 1,\dots,n. \end{matrix} \quad \leftrightarrow \quad d\omega = 0. \tag{21}$$

Jacobi identity

Closure

The Jacobi identity of generalized Hamiltonian mechanics should express the **closure of a symplectic 3-form** w:

$$d\omega = 0 \quad \Rightarrow \quad dw = 0, \qquad w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k. \tag{22}$$

Generalized Hamiltonian Mechanics – A Possible Approach

Classical Hamiltonian mechanics can be regarded as a special case ($n = 2$) of dynamical theories written as

$$\mathcal{L}_X \omega^n = 0, \qquad d\omega^n = 0, \tag{23}$$

where ω^n is an n-form. We suggest that Nambu mechanics corresponds to $n = 3$. On a contractible domain,

$$i_X \omega^n = -d\lambda^{n-2}. \tag{24}$$

Remark 1: *Let $\omega \in \Lambda^2 T^*\Omega$ denote a closed 2-form. Suppose that the vector field $X \in T\Omega$ satisfies*

$$i_X \omega = -dH, \qquad i_X dG = 0,$$

where dG and dH are linearly independent exact 1-forms. Then, the (Nambu or symplectic) 3-form

$$w = \omega \wedge dG,$$

has the following properties

$$i_X w = -dH \wedge dG, \qquad dw = 0.$$

Converse question: given w and X such that $i_X w = -dH \wedge dG$ and $i_X dG = 0$, can one find ω with $d\omega = 0$?

Generalized Hamiltonian Mechanics – Properties

1. Given 2 Hamiltonians $G, H \in C^\infty(\Omega)$ and a skew-symmetric third order contravariant tensor $\mathcal{J}^{ijk} \in C^\infty(\Omega)$, the equations of motion are given by

$$\dot{x}^i = \mathcal{J}^{ijk} G_j H_k = \sum_{j<k} \mathcal{J}^{ijk}\left(G_j H_k - G_k H_j\right), \qquad i = 1, \dots, n. \tag{25}$$

2. The tensor $\mathcal{J}^{ijk}$ (**generalized Poisson operator**) satisfies a generalized Jacobi identity expressing the closure of a smooth 3-form w (**symplectic 3-form**), $dw = 0$.

3. The symplectic 3-form w is Lie-invariant, $\mathcal{L}_{\dot{x}} w = 0$.

4. When w has rank n, there exists local coordinates $(y^1, \dots, y^n)$ and a constant skew-symmetric tensor B^{ijk} such that (17) takes the **generalized canonical form**

$$\dot{y}^i = \sum_{j<k} B^{ijk}\left(\frac{\partial G}{\partial y^j}\frac{\partial H}{\partial y^k} - \frac{\partial G}{\partial y^k}\frac{\partial H}{\partial y^j}\right), \qquad i = 1, \dots, n. \tag{26}$$

Nambu's canonical equation when $n = 3$

5. The local coordinates $(y^1, \dots, y^n)$ define an invariant (Liouville) measure $d\Xi = dy^1 \wedge \dots \wedge dy^n$ such that $\mathcal{L}_{\dot{x}} d\Xi = 0$.

Theory Comparison– A Schematic View

Generating functions	H	G, H
Operator	$\mathcal{J}^{ij}$	$\mathcal{J}^{ijk}$
Bracket	$\{F, G\} = F_i \mathcal{J}^{ij} G_j$	$\{F, G, H\} = \mathcal{J}^{ijk} F_i G_j H_k$
Symplectic form	$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$	$w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k$
Jacobi identity	$d\omega = 0$	$dw = 0$
Lie-Darboux theorem	$\omega = \sum_{i=1}^{m} dp^i \wedge dq^i$	$w = A_{ijk} dy^i \wedge dy^j \wedge dy^k, \quad A_{ijk} \in \mathbb{R}$
Canonical equations	$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}$	$\frac{dy^i}{dt} = \sum_{j<k} B^{ijk} \left(\frac{\partial G}{\partial y^j} \frac{\partial H}{\partial y^k} - \frac{\partial G}{\partial y^k} \frac{\partial H}{\partial y^j} \right),$ $B^{ijk} \in \mathbb{R}$
Liouville measure	$d\Pi = dp^1 \wedge \cdots \wedge dp^m \wedge dq^1 \wedge \cdots \wedge dq^m$	$d\Xi = dy^1 \wedge \cdots \wedge dy^n$
Special functions (Casimir invariants)	$\mathcal{J}^{ij} C_j = 0$	$\mathcal{J}^{ijk} C_k = 0, \quad \mathcal{J}^{ijk} C_j D_k = 0$

Symplectic 3-Form – Inverse of a 3-Tensor

The core of the theory lies in the assumption that $\dot{x}$ defines a generalized Hamiltonian system if there exist a smooth 3-form w and generating functions G, H with the following properties

$$i_{\dot{x}} w = -dH \wedge dG, \qquad dw = 0. \tag{27}$$

$\to \mathcal{L}_{\dot{x}} w = 0$

Explicitly, (27) can be written as

$$\dot{x}^i w_{ijk} = H_k G_j - H_j G_k, \qquad \frac{\partial w_{ijk}}{\partial x^\ell} + \frac{\partial w_{i\ell j}}{\partial x^k} + \frac{\partial w_{ik\ell}}{\partial x^j} + \frac{\partial w_{j\ell k}}{\partial x^i} = 0, \qquad i, j, k, \ell = 1, \dots, n. \tag{28}$$

We say that a skew-symmetric third order contravariant tensor $\mathcal{J}^{jk\ell}$ is the **inverse** of w_{ijk} if

$\sum_{j<k} \epsilon_{ijk} \epsilon^{jk\ell} = \delta_i^\ell$

$$\sum_{j<k} w_{ijk} \mathcal{J}^{jk\ell} = \delta_i^\ell, \qquad i, \ell = 1, \dots, n \quad \to \quad \dot{x}^\ell = \mathcal{J}^{\ell jk} G_j H_k, \qquad \ell = 1, \dots, n. \tag{29}$$

The notion of invertibility (29) for w_{ijk} is equivalent to the **existence of a right-inverse for the $n \times n^2$ matrix** $w_{i(jk)}$. The right-inverse $\mathcal{J}^{(jk)\ell}$ exists when $w_{i(jk)}$ has **rank** n. In such case $w_{i(jk)}$ is the left-inverse of $\mathcal{J}^{(jk)\ell}$.

Inverse of a 3-Tensor – Examples

3-tensors as w_{ijk} can be thought of as matrices having rows, columns, and 'depth'. To a tensor w_{ijk} we can assign a unique conventional matrix $w_{i(jk)}$ having n rows and n^2 columns. For example, if $w_{ijk} = \epsilon_{ijk}$,

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}_1, \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}_2, \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_3 \to \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \quad (30)$$

rank $= 3$
$\to \mathcal{J}^{jk\ell} = \epsilon^{jk\ell}$

An example in higher dimensions ($n = 6$) is the following. The 3-form

$$w = dx^1 \wedge dx^2 \wedge dx^3 + dx^4 \wedge dx^5 \wedge dx^6 + dx^1 \wedge dx^2 \wedge dx^4, \quad (31)$$

has rank 6, and therefore possesses an inverse $\mathcal{J}$. The inverse is

$$\mathcal{J} = -\partial_1 \wedge \partial_2 \wedge \partial_3 - \partial_4 \wedge \partial_5 \wedge \partial_6 + \partial_6 \wedge \partial_3 \wedge \partial_5. \quad (32)$$

Symplectic 3-Form – Closure as Generalized Jacobi Identity

Using the inverse $\mathcal{J}^{jk\ell}$ it is possible to derive a necessary condition for closure $dw = 0$ to hold:

$$0 = \sum_{i<j<k} w_{ijk} \left(\mathcal{J}^{\beta k\ell} \mathcal{J}_{\ell}^{\alpha ij} + \mathcal{J}^{\alpha i\ell} \mathcal{J}_{\ell}^{\beta jk} + \mathcal{J}^{\alpha j\ell} \mathcal{J}_{\ell}^{\beta ki} + \mathcal{J}^{\beta j\ell} \mathcal{J}_{\ell}^{\alpha ki} + \mathcal{J}^{\alpha k\ell} \mathcal{J}_{\ell}^{\beta ij} + \mathcal{J}^{\beta i\ell} \mathcal{J}_{\ell}^{\alpha jk} \right). \tag{33}$$

This expression is analogous to the Jacobi identity $\mathcal{J}^{im}\mathcal{J}_m^{jk} + \mathcal{J}^{jm}\mathcal{J}_m^{ki} + \mathcal{J}^{km}\mathcal{J}_m^{ij} = 0$ for the Poisson operator in classical Hamiltonian mechanics.

- Any invertible skew-symmetric third order tensor with constant entries, $\mathcal{J}^{jk\ell} \in \mathbb{R}$, automatically satisfies (33), and $dw = 0$ as well.
- The generalization of Hamiltonian mechanics following from the present construction is weaker than that resulting from enforcing the fundamental identity.

Hamiltonian Mechanics vs Nambu Mechanics – Example

Consider again the previous system of quantum oscillators. Perform the change of variables $(p_1, q_1, \xi_1, p_2, q_2, \xi_2) \to (p_1, q_1, G_1, p_2, q_2, G_2)$. The Hamiltonian can be written as

$$\widetilde{H} = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \lambda q_1(G_2 + q_2^2). \tag{34}$$

The equations of motion become

$$\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{G}_1 \\ \dot{p}_2 \\ \dot{q}_2 \\ \dot{G}_2 \end{bmatrix} = \begin{bmatrix} -q_1 - \lambda(G_2 + q_2^2) \\ p_1 \\ 0 \\ -q_2 - 2\lambda q_1 q_2 \\ p_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\widetilde{H}_{q_1} \\ \widetilde{H}_{p_1} \\ 0 \\ -\widetilde{H}_{q_2} \\ \widetilde{H}_{p_2} \\ 0 \end{bmatrix}. \tag{35}$$

This system, which fails the fundamental identity, is a generalized Hamiltonian system with symplectic 3-form

$$w = \omega \wedge dG_2 = (dp_1 \wedge dq_1 + dp_2 \wedge dq_1) \wedge dG_2. \tag{36}$$

It remains to prove the Lie-Darboux and Liouville theorems for the generalized theory.

Theorem. *Let $w \in \Lambda^3 T\Omega$ be a smooth closed 3-form on a smooth manifold Ω of dimension n. Let w_{ijk}, $i, j, k = 1, \dots, n$ denote the components of w with respect to a coordinate system $(x^1, \dots, x^n)$ on Ω,*

$$w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k .$$

Suppose that the $n \times n^2$ matrix $w_{i(jk)}$ has rank n. Then, for every point $\boldsymbol{x}_0 \in \Omega$ there exist a neighborhood $U \subset \Omega$ of $\boldsymbol{x}_0$ and a local smooth coordinate system $(y^1, \dots, y^n)$ such that

$$w = \sum_{i<j<k} A_{ijk} dy^i \wedge dy^j \wedge dy^k \quad \text{in } U,$$

where $A_{ijk} = w_{ijk}(\boldsymbol{x}_0)$, $i, j, k = 1, \dots, n$, is a skew-symmetric third order covariant tensor with constant entries.

Proof. We follow the steps of the classical proof of the Lie-Darboux theorem based on Moser's method [1,9]. Let w_0 denote the constant form on $\mathbb{R}^n$,

$$w_0 = \sum_{i<j<k} A_{ijk} dy^i \wedge dy^j \wedge dy^k \,,$$

with A_{ijk}, $i, j, k = 1, \dots, n$, real constants. Consider a family of vector fields $X_t \in T\Omega$, $0 \le t \le 1$, defined in a neighborhood U of a point $\boldsymbol{x}_0 \in \Omega$ that generates a one-parameter group of diffeomorphisms g_t as follows,

$$\frac{d}{dt} g_t(\boldsymbol{x}_0) = X_t\big(g_t(\boldsymbol{x}_0)\big), \qquad g_0(\boldsymbol{x}_0) = \boldsymbol{x}_0.$$

Next, define the family of 3-forms

$$w_t = w_0 + t(w - w_0).$$

We wish to obtain X_t, and thus g_t, so that (g_t^* is the pullback of w_t by g_t)

Closure $dw_t = 0$

$$g_t^* w_t = w_0 \quad \rightarrow \quad \frac{d}{dt} g_t^* w_t = g_t^* \left(\frac{dw_t}{dt} + d i_{X_t} w_t + i_{X_t} dw_t \right) = g_t^* \left(\frac{dw_t}{dt} + d i_{X_t} w_t \right) = 0.$$

By the Poincaré lemma, in a sufficiently small neighborhood W of $\boldsymbol{x}_0$, the closed differential form dw_t/dt is exact.

[1] M. de Léon, Methods of Differential Geometry in Analytical Mechanics, Elsevier, New York, pp. 250–253 (1989).
[9] J. Moser, On the volume elements on a manifold, Trans. Am. Mat. Soc. **120**, pp. 286-294 (1965).

Hence, there is a 2-form $\sigma_t = \sum_{j<k} \sigma_{tjk} dx^j \wedge dx^k$ such that

$$\frac{dw_t}{dt} = d\sigma_t \quad \text{in } W.$$

Therefore, the equation $\frac{d}{dt} g_t^* w_t = g_t^* \left(\frac{dw_t}{dt} + d i_{X_t} w_t \right) = 0$ can be solved by finding a vector field X_t satisfying

$$\sigma_t = -i_{X_t} w_t \quad \rightarrow \quad \sigma_{tjk} = -X_t^i w_{tijk}, \qquad j,k = 1,\dots,n.$$

By hypothesis, the $n \times n^2$ matrix $w_{i(jk)}$ has rank n. Similarly, setting $A_{ijk} = w_{ijk}(\boldsymbol{x}_0)$, the $n \times n^2$ matrix $A_{i(jk)}$ has rank n. Furthermore, at $\boldsymbol{x}_0$ we may assume $w(\boldsymbol{x}_0) = w_0(\boldsymbol{x}_0)$ since the matrices w_{ijk} and A_{ijk} coincide there. Then, for $0 \le \mathrm{t} \le 1$,

$$w_t(\boldsymbol{x}_0) = w_0(\boldsymbol{x}_0).$$

This implies that the $n \times n^2$ matrix $w_{ti(jk)}(\boldsymbol{x}_0)$ has rank n at $\boldsymbol{x}_0$. By continuity of the tensor w_{tijk} it follows that there exists a neighborhood V of $\boldsymbol{x}_0$ where the rank of $w_{ti(jk)}$ is n. Define $U = W \cap V$. Then, the matrix $w_{ti(jk)}$ has an inverse $\mathcal{J}_t^{(jk)\ell}$ giving X_t in U as

$$X_t^\ell = -\mathcal{J}_t^{jk\ell} \sigma_{tjk}, \qquad \ell = 1,\dots,n. \quad \blacksquare$$

The previous theorem implies that a generalized Hamiltonian system has an **invariant (Liouville) measure**

$$d\Xi = dy^1 \wedge \cdots \wedge dy^n \quad \text{in } \ U. \tag{37}$$

To see that $d\Xi$ is invariant, notice that, setting $B^{jk\ell} = \mathcal{J}^{jk\ell}(\boldsymbol{x}_0)$, one has

$$i_{\dot{x}} w = -dH \wedge dG \ \rightarrow \ A_{ijk}\frac{dy^i}{dt} = \frac{\partial G}{\partial y^j}\frac{\partial H}{\partial y^k} - \frac{\partial G}{\partial y^k}\frac{\partial H}{\partial y^j} \ \rightarrow \ \frac{dy^\ell}{dt} = B^{\ell jk}\frac{\partial G}{\partial y^j}\frac{\partial H}{\partial y^k}, \qquad j,k,\ell = 1,\dots,n. \tag{38}$$

It follows that

$$\mathcal{L}_{\dot{x}} d\Xi = \frac{\partial}{\partial y^i}\left(\frac{dy^i}{dt}\right) d\Xi = B^{ijk}\left(\frac{\partial^2 G}{\partial y^i \partial y^j}\frac{\partial H}{\partial y^k} + \frac{\partial G}{\partial y^k}\frac{\partial^2 H}{\partial y^i \partial y^j}\right) d\Xi = 0. \tag{39}$$

Both the symplectic form w and the Liouville measure $d\Xi$ are preserved as in the classical formulation. However, there is a difference with respect to the existence of canonical variables.

In the classical proof of the Lie-Darboux theorem the skew-symmetry of the tensor ω_{ij} associated with the symplectic 2-form ω is sufficient to ensure that there exists a linear change of basis transforming the skew-symmetric matrix $\omega_{ij}(\boldsymbol{x}_0)$ into block diagonal form at any $\boldsymbol{x}_0 \in \Omega$,

$$Q^T \omega(\boldsymbol{x}_0) Q = \begin{bmatrix} 0 & 1 & \cdots \\ -1 & 0 & 0 \\ \vdots & 0 & \ddots \end{bmatrix}, \qquad QQ^T = I. \tag{40}$$

An analogous result is not available for third order tensors like $w_{ijk}(\boldsymbol{x}_0)$: one cannot guarantee the local invertibility of the tensor $\tilde{w}_{tijk}$ associated with the 3-form $\tilde{w}_t = \tilde{w}_0 + t(w - \tilde{w}_0)$, with

$$\tilde{w}_0 = \sum_{i=1}^{m} dp^i \wedge dq^i \wedge dr^i \,. \tag{41}$$

Local canonical triplets (p^i, q^i, r^i), $i = 1, \dots, m$, are not expected to be available in general.

Nevertheless, whenever $n = 3m$ with m an integer, canonical triplets locally exist in the neighborhood of all points $\boldsymbol{x}_0 \in \Omega$ such that $w_{ijk}(\boldsymbol{x}_0)$ can be transformed by a linear change of basis into the generalized Levi-Civita symbol E_{ijk},

$$E_{ijk} = \begin{cases} \epsilon^{ijk} & \text{if } \sigma(i,j,k) = (\ell, m+\ell, 2m+\ell), \quad \ell = 1, \dots, m \\ 0 & \text{otherwise} \end{cases}. \tag{42}$$

- We have formulated a generalization of classical Hamiltonian mechanics to a 3-dimensional phase space.
- The theory relies on a symplectic 3-form w and a pair of Hamiltonians G, H.
- The Jacobi identity is identified with the closure condition $dw = 0$ written in terms of the inverse $\mathcal{J}^{ijk}$.
- The closure condition is weaker than the fundamental identity: constant 3-tensors define generalized Poisson operators.
- The closure of w ensures that there exist local coordinates $(y^1, \ldots, y^n)$ such that the components of w are constants, and the volume form $d\Xi = dy^1 \wedge \cdots \wedge dy^n$ is invariant.

Concluding Remarks – Open Questions

- Identify sufficient condition for existence of canonical triplets (p^i, q^i, r^i), $i = 1, \dots, m$.
- Identify conditions under which a generalized Hamiltonian system has a classical Hamiltonian structure and vice versa.
- Identify systems that are Hamiltonian in the generalized sense, but that do not possess a classical Hamiltonian structure.
- Prove uniqueness and skew-symmetry of inverse matrix $\mathcal{J}^{(jk)\ell}$ when $w_{i(jk)}$ has rank n.
- Does the theory perform better with respect to bracket quantization?

Thank you for your attention!