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Phase transitions and critical phenomenon in QCD

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Indico page : https://indico.cern.ch/event/1103564/

Outline

- Symmetries of the QCD Lagrangian
- Z_3 symmetry of a purely gluonic system and the Polyakov loop
- Chiral condensate as an order parameter
- Universality class of the chiral phase transition
- The Columbia plot
- The sign problem in QCD and the realistic *walk arounds*.

Phase transitions

- Ising model and Landau theory
- Fluctuations and critical exponents
- Approaches to the critical point
- Connections to the experimental programmes

Critical phenomenon

Gauge invariance of the QCD Lagrangian

The QCD action is given by an integral over the spacetime of the QCD Lagrangian

$$S_{QCD} = \int d^4x \left[\frac{1}{4g^2} G^k_{\mu\nu} G^{\mu\nu}_k + \bar{\psi}\gamma^\mu D_\mu \psi + \bar{\psi} m \psi \right]$$

Kinetic term for the gluons

Kinetic term for quarks

Mass term for quarks

The gauge group for the theory is $SU(3)$: Unitary 3x3 matrices with determinant equal to unity.
An element of the group : $\Omega = exp(i\omega^k t_k)$ with $\Omega^+ = \Omega^{-1}$; $det[\Omega] = 1$. t_k are Hermitian 3x3 matrices with $k = 1,2,..,8$.

$\psi^{\alpha a f}$ are fermionic fields with three indices

Direct indices : $\alpha = 1,2,3,4$ **:** The Dirac gamma matrices acts on α

Colour Indices: $a = 1,2,3$ **:** The gauge transformation & covariant derivative acts on a

Flavour index: $f = u, d, s$ **: 6x6** mass matrices acts on f

Gauge invariance of the QCD Lagrangian

Covariant derivative: $(D_\mu)_{ab} = \delta_{ab} - i(G_\mu)_{ab}$ G_μ is the gluonic field, a 3x3 matrix

In order to have SU(3) symmetry, we want the fermionic fields to transform as :

$$\psi'(x) = \Omega(x)\psi(x) \qquad \bar{\psi}'(x) = \bar{\psi}'(x)\Omega^+(x)$$

$$D'_\mu\psi'(x) = \Omega D_\mu\psi(x) = \Omega D_\mu[\Omega^+\psi'] \rightarrow D'_\mu = \Omega D_\mu\Omega^+$$

This tells us how the gauge field transforms

$$\partial_\mu - iG'_\mu = \Omega[\partial_\mu - iG_\mu]\Omega^+ = \Omega\partial_\mu\Omega^+ + \Omega\Omega^+\partial_\mu - i\Omega G_\mu\Omega^+$$

Transformation of the gauge fields then can be written as

$$G'_\mu = i\Omega\partial_\mu\Omega^+ + \Omega G_\mu\Omega^+$$

Finite temperature field theory

Grand canonical Partition function: $Z = Tre^{-\beta[\hat{H}-\mu\hat{N}]}$

$\hat{H} \equiv$ Hamiltonian; $\hat{N} \equiv$ Operator for the conserved charge; $\mu \equiv$ Chemical potential

We can rewrite the Z making use of the QCD action: The integral over time becomes imaginary which becomes the temperature of the system.

Grand canonical Partition function: $Z = \int_{b.c} D\psi D\bar{\psi} DG_\mu exp\left[-\int_0^\beta d\tau \int d^3x L_{QCD}\right]$ Where $\beta = \frac{1}{T}$

Crucial for Order parameter definitions

Boundary conditions are anti periodic for ψ & $\bar{\psi}$: $\psi(\vec{x}, x_4+\beta) = -\psi(\vec{x}, x_4)$

Boundary conditions are periodic for G_μ : $G_\mu(\vec{x}, x_4+\beta) = G_\mu(\vec{x}, x_4)$

Combine: Boundary conditions and gauge transformation properties

Z_3 symmetry of a purely gluonic system

For the gauge fields we need

1. $G_\mu(\overrightarrow{x}, x_4 + \beta) = G_\mu(\overrightarrow{x}, x_4)$

2. $G'_\mu = \Omega(G_\mu + i\partial_\mu)\Omega^+$

The simplest choice of Omegas would be : $\Omega(\overrightarrow{x}, x_4 + \beta) = \Omega(\overrightarrow{x}, x_4)$

Can we define a transformation like this? : $\Omega(\overrightarrow{x}, x_4 + \beta) = h\Omega(\overrightarrow{x}, x_4)$

If we apply this to the gauge transformation property for the field: Can we get an h which multiplies the gauge transformation and still gives me a periodic boundary condition for G_μ

$$G'_\mu(\overrightarrow{x}, x_4 + \beta) = \Omega(\overrightarrow{x}, x_4 + \beta)[G_\mu(\overrightarrow{x}, x_4 + \beta) + i\partial_\mu]\Omega^+(\overrightarrow{x}, x_4 + \beta)$$

$$= h\Omega(\overrightarrow{x}, x_4)[G_\mu(\overrightarrow{x}, x_4) + i\partial_\mu]\Omega^+(\overrightarrow{x}, x_4)h^+$$

$$G'_\mu(\overrightarrow{x}, x_4) = hG'_\mu h^+$$

This is possible when h commutes with G_μ

Z_3 symmetry of a purely gluonic system

We need a matrix h which commutes with all the matrices in the SU(3) group.

$Z(3) \equiv$ Centre of the SU(3) group

If we define h as follows:

$h \in Z(3)$ $\quad h = z\mathbf{1}$ $\quad$ where $\quad z = exp\left[\frac{2\pi i n}{3}\right]; n = 1,2,3.$

We get a set of transformations which obey the boundary conditions and the transformation properties of the field.

$Z(3)$ symmetry: We found out an additional symmetry for a purely gluonic system which is not present at the level of the Lagrangian. When we have quarks, this symmetry is explicitly broken.

It allows us to define a rigorous phase transition between confinement and deconfinement phases in the limit of infinitely heavy quarks

Polyakov loop as an order parameter

Polyakov loop is defined as

$$\Phi(\overrightarrow{x}) = \frac{1}{3}Tr\left[Pexp\left(i\int_0^\beta dx_4 G_4(\overrightarrow{x}, x_4)\right)\right]$$

$P \equiv Path\ Ordering$

which transforms non trivialy under the Z(3) Symmetry

Expectation value of the Polyakov loop can be considered as an order parameter for a purely gluonic field.

$\langle \Phi(\overrightarrow{x}) \rangle = 0 \longrightarrow$ The vacuum is symmetric under the Z(3)

$\langle \Phi(\overrightarrow{x}) \rangle \neq 0 \longrightarrow$ The vacuum will not be symmetric under the Z(3)

Polyakov loop as an order parameter

Transformation of the Polyakov loop

$$\Phi'(\vec{x}) = \frac{1}{3} Tr \left[Pexp \left(i \int_0^\beta dx_4 G_4'(\vec{x}, x_4) \right) \right]$$

$$= \frac{1}{3} Tr \left[\Omega Pexp \left(i \int_0^\beta dx_4 G_4(\vec{x}, x_4) \right) \Omega^+(\vec{x}, 0) \right]$$

$$= \frac{1}{3} Tr \left[Z\Omega(\vec{x}, 0) Pexp \left(i \int_0^\beta dx_4 G_4(\vec{x}, x_4) \right) \Omega^+(\vec{x}, 0) \right]$$

$$= Z\Phi(\vec{x})$$

Polyakov loop transforms nontrivially under the Z(3) Symmetry

Physical meaning of the Polyakov loop

If we add a very heavy quark into the system, it satisfies the static Dirac equation given by:

$$[\partial_\tau - igG_4 + M]\psi(\vec{r},\tau) = 0$$

Separate the variables and integrate

$$\frac{\partial_\tau \psi(\vec{r},\tau)}{\psi(\vec{r},\tau)} = -M + gG_4(\vec{r},\tau)$$

$$\ln \psi(\vec{r},\tau) = -M + ig\int_0^\tau G_4(\vec{r},\tau') + \ln \psi(\vec{r},0)$$

Something which *looks* similar to Φ

Exponentiate and get the solution

$$\psi(\overrightarrow{x},\tau) = exp^{-M\tau} T\left[\boxed{exp\left(ig\int_0^\tau d\tau G_4(\vec{r},\tau)\right)}\right]\psi(\overrightarrow{x},0)$$

Physical meaning of the Polyakov loop

Consider the free energy of the very heavy static quark we put in to the gluonic system

$$e^{-\beta F} = \frac{1}{N_c}\sum_{i,n} < n|\psi_i(\vec{r})e^{-\beta H}\psi_i^+(\vec{r})|n >$$

$i \rightarrow |n > \equiv$ *Possible gluonic states*

Imaginary time evolution operator: $e^{\beta H}\psi_i(\vec{r})e^{-\beta H} = \psi_i(\vec{r},\beta)$

Use it here!

$$e^{-\beta F} = \frac{1}{N_c}\sum_{i,n} < n|e^{-\beta H}\psi_i(\vec{r},\beta)\psi_i^+(\vec{r},0)|n >$$

Solution of the static Dirac equation

$$= \frac{1}{N_c}\sum_{i,n} e^{-\beta E_n} < n|\psi_i(\vec{r},\beta)\psi_i^+(\vec{r},0)|n >$$

$$= e^{-M\beta}\sum_{n} e^{-\beta E_n} < n|\frac{1}{N_c}\sum_{i,j} T\left[exp\left(ig\int_0^{\tau} d\tau G_4(\vec{r},\tau)\right)\right]_{i,j} \overbrace{\psi_j(\vec{r},0)\psi_i^+(\vec{x},0)}^{\delta_{ij}}|n >$$

$$= e^{-M\beta}\sum_{n} e^{-\beta E_n} < n|\frac{1}{N_c}Tr\, T\left[exp\left(ig\int_0^{\tau} d\tau G_4(\vec{r},\tau)\right)\right]|n > = e^{-M\beta}\sum_{n} e^{-\beta E_n}\boxed{< n|\Phi(\vec{r})|n >}$$

$$\boxed{e^{-\beta(F-F_0-M)} = < \Phi(\vec{r}) >}$$

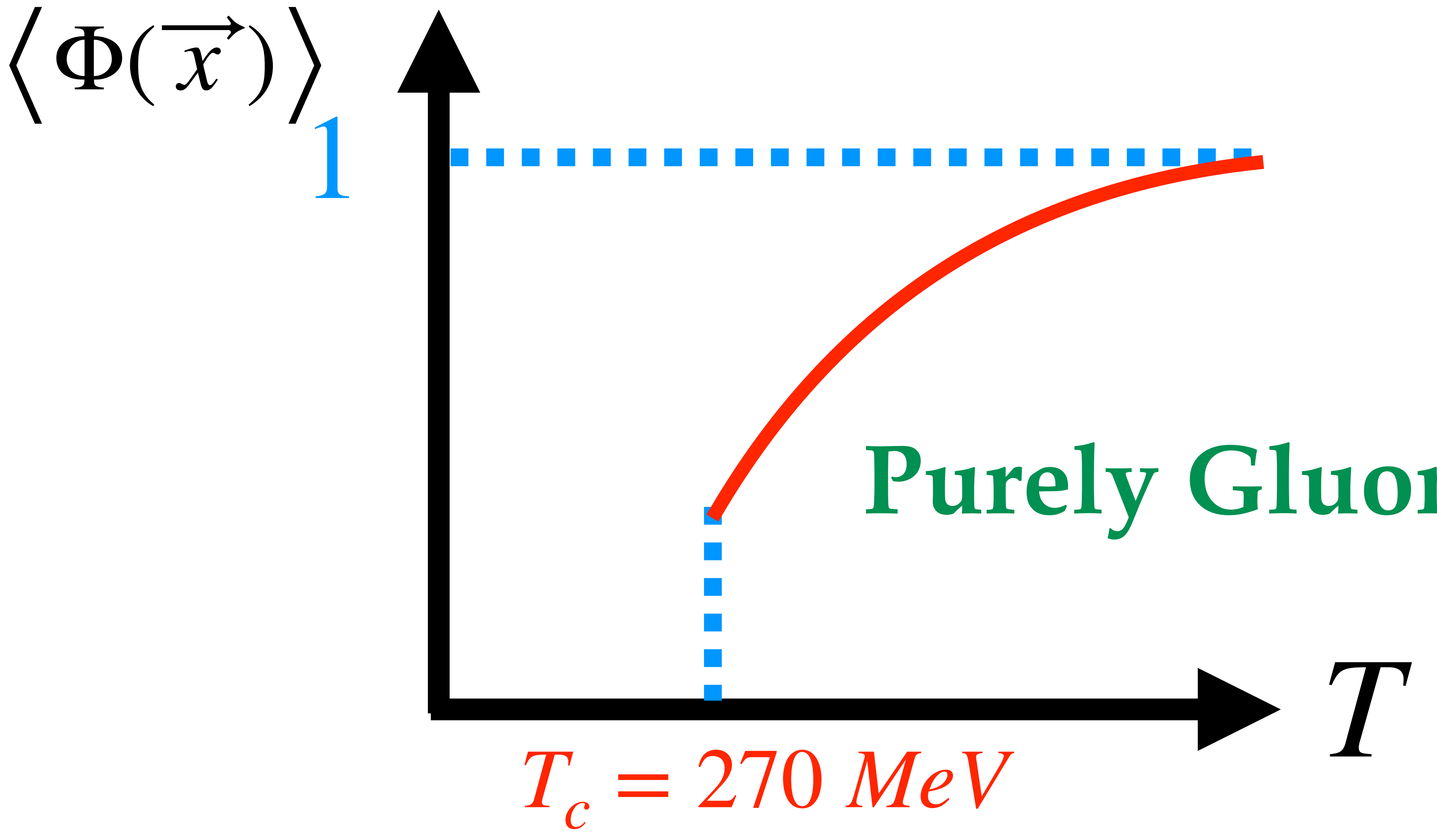
Polyakov loop as an order parameter

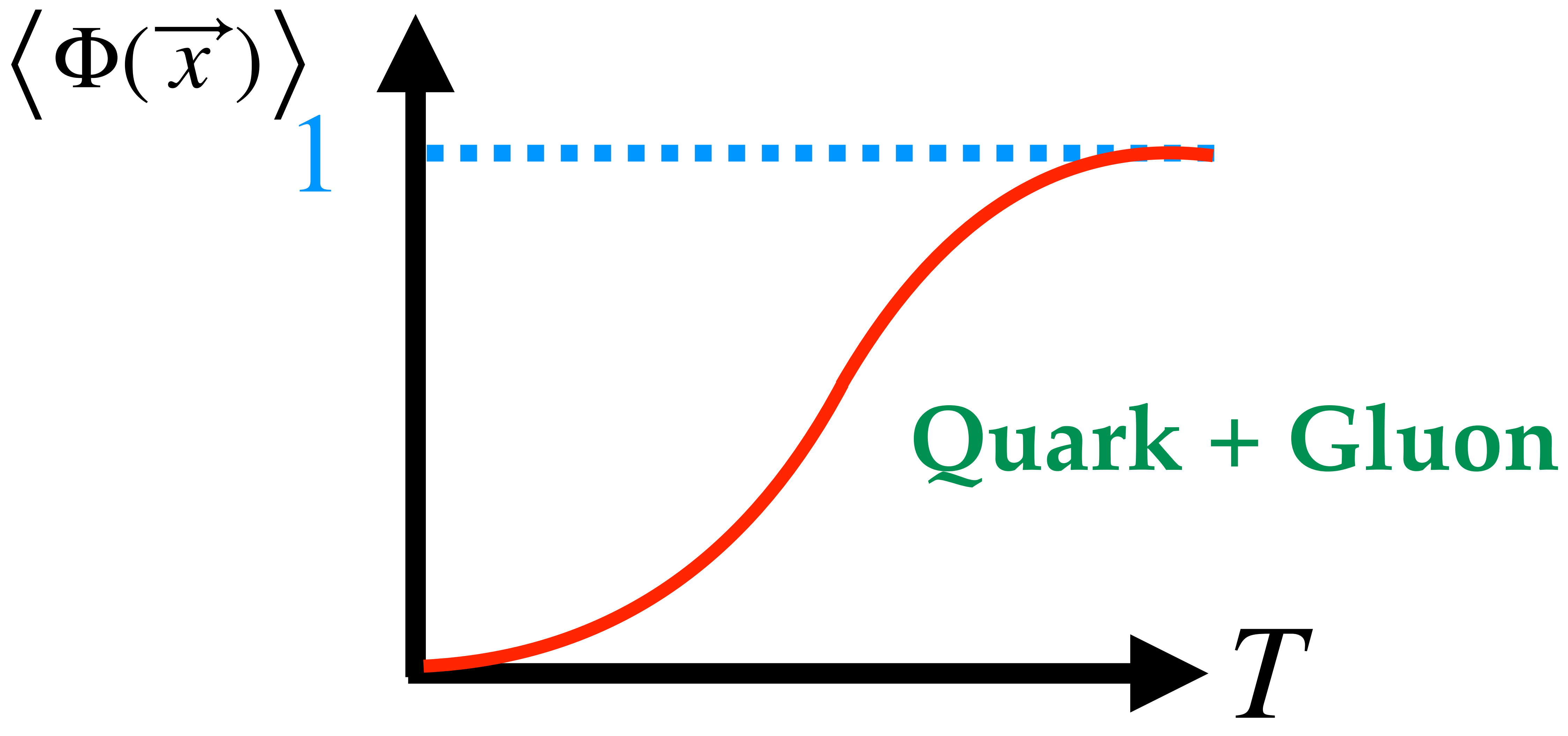
Free energy of the static quark in a thermal bath of gluons is the expectation value of the Polyakov loop!

$\langle \Phi(\vec{x}) \rangle = 0$ → Z(3) Symmetry → $F = \infty$ → **Confinement**

$\langle \Phi(\vec{x}) \rangle \neq 0$ → Z(3) Symmetry Broken → $F \neq \infty$ → **Deconfinement**

Expectation value of the Polyakov loop is an order parameter for the confinement phase transition of a purely gluonic system





Symmetries of the massive QCD Lagrangian

Let's make the system a bit complicated by putting quarks with masses very close to zero

The Lagrangian of the QCD is given by: $\mathscr{L} = \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + i\bar{\psi}\not{D}\psi - \bar{\psi}m\psi$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$\{\gamma_5, \gamma^\mu\} = 0$

where $\not{D} = \gamma^\mu D_\mu; N_f = 2$

It has the following properties under the transformations

$U(1)_v : \ \psi \to e^{i\alpha}\psi$, $\alpha \in R$ $\Longrightarrow$ $\psi\not{D}\psi$ & $\bar{\psi}m\psi \to invariant$

$U(1)_a : \ \psi \to e^{-i\alpha\gamma_5}\psi$ $\Longrightarrow$ $\bar{\psi}\not{D}\psi \to invariant$

$SU(2)_v : \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a}\psi$ $\Longrightarrow$ $\psi\not{D}\psi$ & $\bar{\psi}m\psi \to invariant$

$SU(2)_a : \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a\gamma_5}\psi$ $\Longrightarrow$ $\bar{\psi}\not{D}\psi \to invariant$

$\boxed{\textbf{Invariant group: } SU(2)_V \times U(1)_V}$

Symmetries of the massless QCD Lagrangian: Theory

In the limit where the $m_q = 0$, the symmetry group is bigger

$U(1)_v : \ \psi \to e^{i\alpha}\psi$ $\alpha \in R$ ⟶ $\psi \not{D} \psi \ \& \ \bar{\psi} m \psi \to invariant$ ⟶ $j_\mu = \bar{\psi}\gamma_\mu\psi = Baryon\ Number$

$U(1)_a : \ \psi \to e^{-i\alpha\gamma_5}\psi$ ⟶ $\bar{\psi} \not{D} \psi \to invariant$ ⟶ $\partial_\mu j_{5\mu} \neq 0 \to Anomalous$

$SU(2)_v : \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a}\psi$ ⟶ $\psi \not{D} \psi \ \& \ \bar{\psi} m \psi \to invariant$ ⟶ $j_\mu^k = \bar{\psi}\gamma_\mu\tau^k\psi \quad Q^k \equiv Isospin$

$SU(2)_a : \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a\gamma_5}\psi$ ⟶ $\bar{\psi} \not{D} \psi \to invariant$ ⟶ $j_{5\mu}^k = \bar{\psi}\gamma_\mu\gamma_5\tau^k\psi$

In the limit of massless quarks, the group of transformations under which the SU(2) Lagrangian is invariant is : $SU(2)_v \times SU(2)_a \times U(1)_v$

This is called chiral symmetry of the QCD Lagrangian

Symmetries of massless the QCD Lagrangian: Experiments

$U(1)_v: \ \psi \to e^{i\alpha}\psi$ → *Baryon Number* → Exp: Conserved

$SU(2)_v: \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a}\psi$ → $j_\mu^k = \bar{\psi}\gamma_\mu\tau^k\psi$ → Exp: Good Symmetry $m_u \neq m_d$

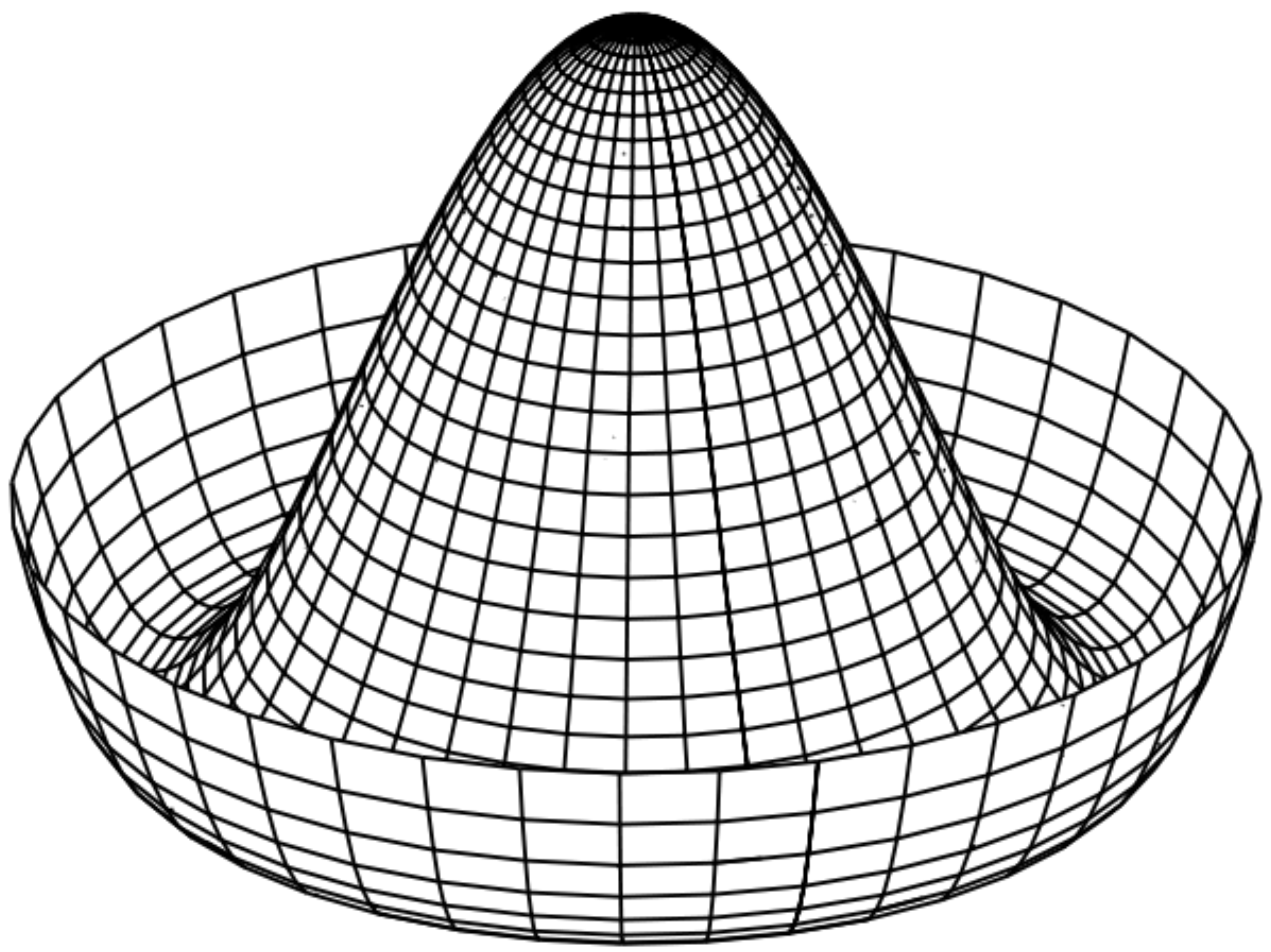
$U(1)_a: \ \psi \to e^{-i\alpha\gamma_5}\psi$ → $\partial_\mu j_{5\mu} \neq 0$ → Anomalously Broken!

$SU(2)_a: \ \psi \to e^{-i\frac{\tau_a}{2}\theta_a\gamma_5}\psi$ → $j_{5\mu}^k = \bar{\psi}\gamma_\mu\gamma_5\tau^k\psi$ → Exp: Broken!

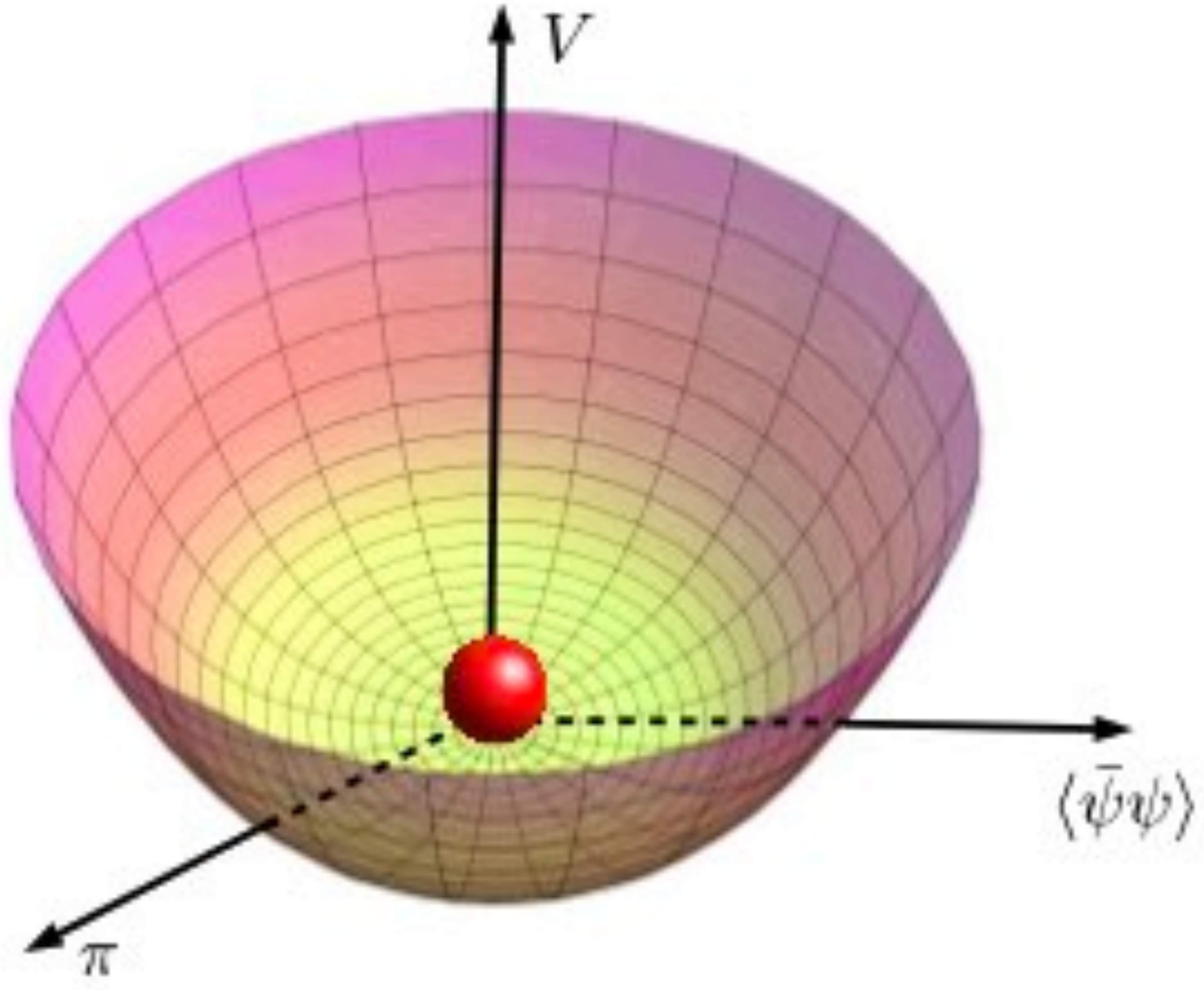
$\rho: J^\pi = 1^-; m_\rho = 770 MeV$ $\qquad$ $a_1: J^\pi = 1^+; m_{a_1} = 1.23 GeV$

Chiral symmetry is observed to have spontaneously broken in the real world that we live in. The pions ($\pi^{0,\pm}$) are identified as the corresponding Goldstone bosons of the symmetry breaking.

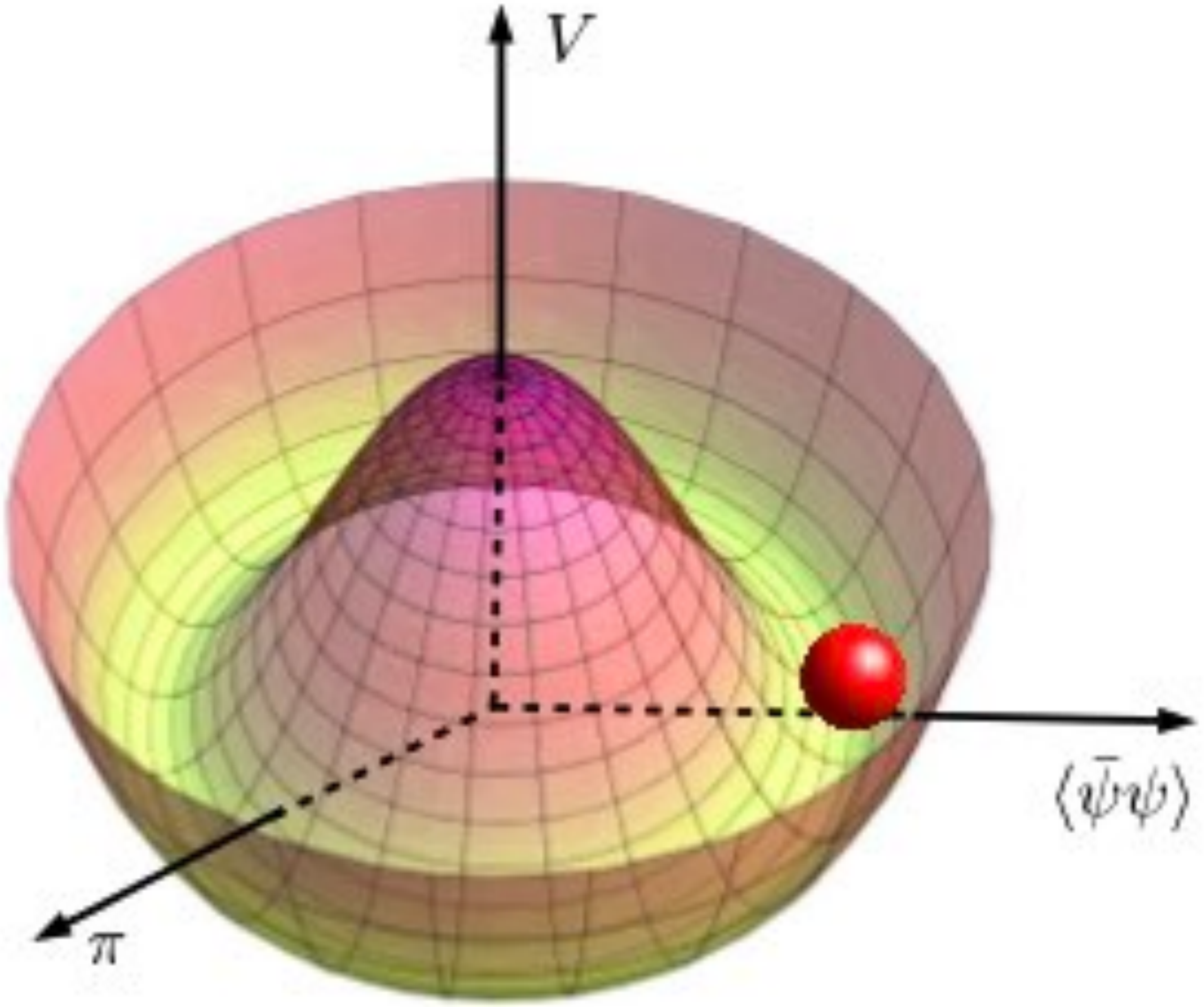
Symmetries of massless the QCD Lagrangian: Mexican hat potential



Mexican hat



Vacuum is symmetric under rotation



Vacuum is not symmetric under rotation

Chiral symmetry is observed to have spontaneously broken in the real world that we live in. The pions ($\pi^{0,\pm}$) are identified as the corresponding Goldstone bosons of the symmetry breaking.

Chiral condensate as an order parameter

The quantity which break the $SU(2)_a$ symmetry is called the chiral condensate

$$\textit{Chiral Condensate} \equiv \langle \bar{\psi}\psi \rangle \rightarrow \textit{Order Parameter of Chiral Phase transition}$$

$$\boxed{\text{Chiral symmetry is broken when } \langle \bar{\psi}\psi \rangle \neq 0}$$

One can calculate chiral condensate from the QCD partition function as follows:

$$Z = \int DG_\mu D\bar{\psi}\psi e^{-S_G} e^{\bar{\psi}M\psi} \qquad \textbf{where} \qquad M = i\not{D} - m$$

Then,

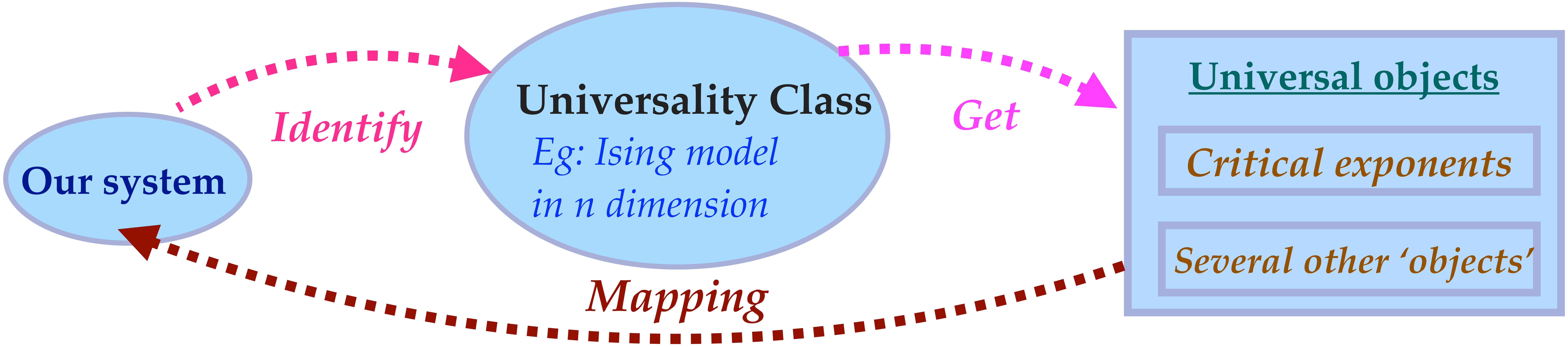
$$\frac{\partial \ln Z}{\partial m_f} = \frac{\partial \ln Z}{\partial M}\frac{\partial M}{\partial m_f} = \frac{1}{z}\frac{\partial \ln Z}{\partial M}\frac{\partial M}{\partial m_f} = \frac{1}{z}\int DG_\mu D\bar{\psi} D\psi \bar{\psi}_f \psi_f e^{-S_G} e^{\int d\tau dx^3 \bar{\psi}M\psi} \sim \langle \bar{\psi}\psi \rangle_f$$

$$\Rightarrow \quad \langle \bar{\psi}\psi \rangle_f = \frac{T}{V}\frac{\partial \ln Z}{\partial m_f}$$

Robert D. Pisarski and Frank Wilczek: Made arguments based on *Universality classes* about the behaviour of $< \bar{\psi}\psi >$ in the vicinity of a chiral phase transition

Universality classes

- Different substances have critical points with qualitative similarities, there are quantitative similarities between them.
- There are systems in which the thermodynamic variables are T and P, but T and M. One can map one transition to another. So it is possible to study the critical phenomenon using them.
- Most of the concepts can be understood in a given context and then translate into the variables for the given system (density, T, P). Essential physics remains the same.



Suggestion for the moment: Consider U.C as a group of models which has certain universal features

$\mathcal{O}(4)$ universality class of the chiral phase transition

Argument: **Chiral phase transition the in chiral limit at $\mu_B = 0$ is a second order which the same as Universality class $\mathcal{O}(4)$ spin model.**

How will the chiral condensate behave if it is in the $\mathcal{O}(4)$ universality class (U.C)?

In the vicinity of phase transition, the free energy can be written as:

$$F = \frac{T}{V} \ln Z = F_{singular}(t, h) + F_{regular}(T, m_l, m_s, \mu)$$

Specific to the U.C & relevant to the χ P.T **Specific to the theory & not relevant to the χ P.T**

Terms: Explicitly breaking symmetry : $h = \dfrac{1}{h_0} \dfrac{m_l}{m_s}$

Terms: Thermal variables which do not break chiral symmetry: $t = \dfrac{1}{t_0} \left[\dfrac{T - T_c^0}{T_c^0} + \kappa_g \left(\dfrac{\mu g}{T} \right)^2 \right]$

Terms: $\kappa_g \equiv$ How T_c varies with μ_B **Terms:** t_0 & $h_0 \rightarrow$ Normalisation Factors

$\mathcal{O}(4)$ universality class of the chiral phase transition

From the argument of universality: $F_{singular}(t,h) = h_0 h^{\frac{1}{\delta}} f_s(z)$ where $z = \dfrac{t}{h^{\frac{1}{\beta\delta}}}$

Define the order parameter as:

$$M(t,h) = \frac{\partial F_{sing}}{\partial H} = -\frac{1}{h_0}\frac{\partial F}{\partial h} = \frac{\partial}{\partial h}[h^{1+\frac{1}{\delta}} f_s(z)] = -(1+\frac{1}{\delta})h^{\frac{1}{\delta}} f_s(z)\frac{\partial f_s}{\partial z}\frac{\partial z}{\partial h}$$ where $H = h_0 h$

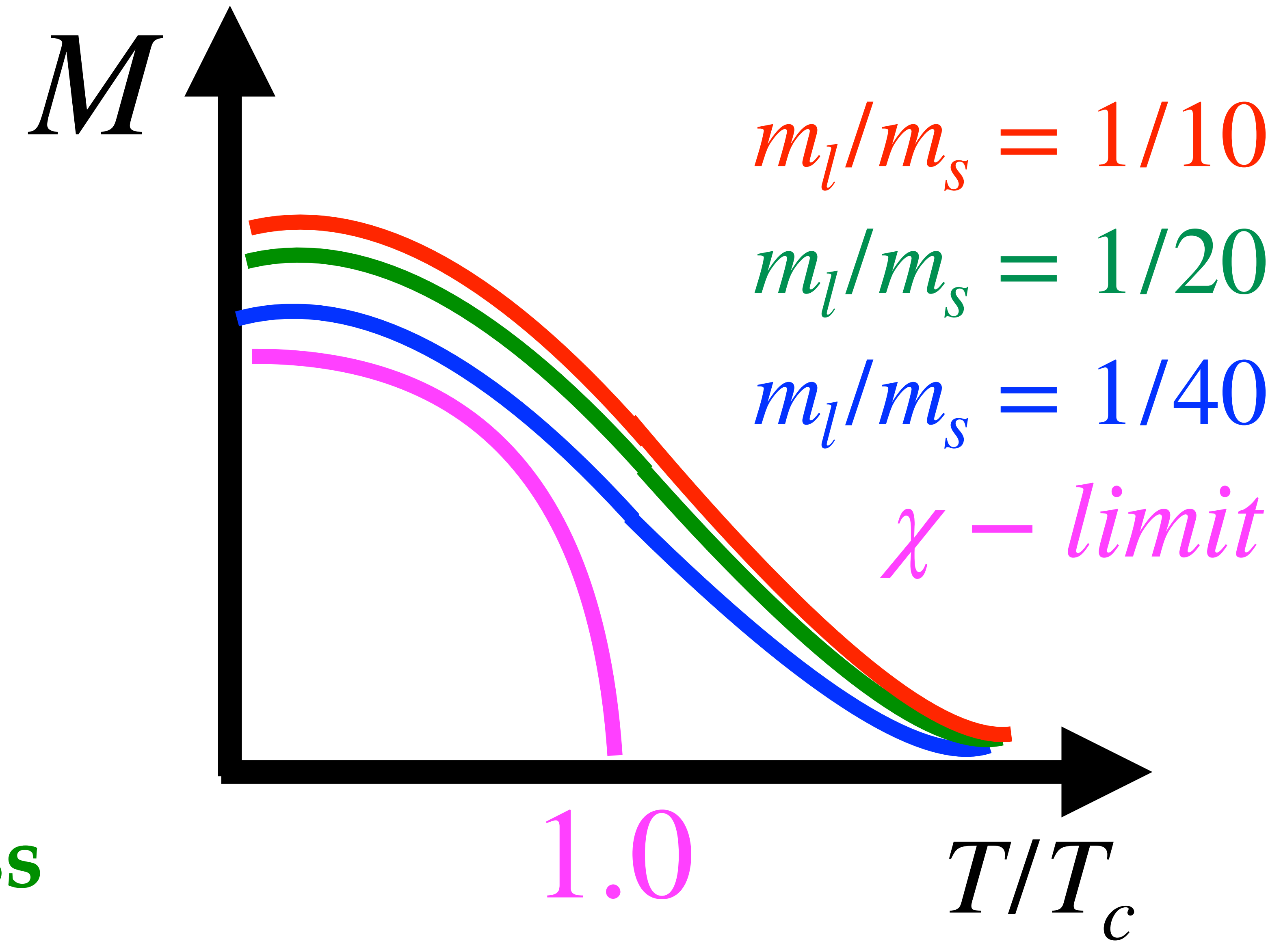
We have $$\frac{\partial z}{\partial h} = \frac{t}{h^{\frac{1}{\beta\delta}}+1}\left(\frac{-1}{\beta\delta}\right) = \frac{-z}{\beta\delta}\frac{1}{h}$$

So $M(t,h)$ would become

$$M(t,h) = -(1+\frac{1}{\delta})h^{\frac{1}{\delta}} f_s(z) + \frac{h^{\frac{1}{\delta}} z}{\beta\delta}\frac{\partial f_s}{\partial z} = h^{\frac{1}{\delta}}\underbrace{\left[-(1+\frac{1}{\delta})f_s(z) + \frac{z}{\beta\delta}\frac{df_s}{dz}\right]}_{F_G(z)}$$

$$M(t,h) = h^{\frac{1}{\delta}} F_G(z)$$

$F_G(z)$: **Specific to the universality class**



Compare with LQCD: $\mathcal{O}(4)$ is indeed the universality class and the phase transition is 2nd order in the chiral limit

Susceptibilities of the chiral condensate: Real QCD

Looking at some quantity which is is divergent at the vicinity of a phase transition is more convenient. We define the susceptibilities as follows:

$$\chi_t \sim \frac{\partial M}{\partial t} \sim \frac{\partial^2 f_s}{\partial t \partial h}$$

$$\chi_h \sim \frac{\partial M}{\partial h} \sim \frac{\partial^2 f_s}{\partial^2 h}$$

Diverges in the vicinity of the critical point

It can be used to spot the phase transition and critical point

The behaviour of these susceptibilities are governed by the following two quantities in the vicinity of the critical point:

$$f'_G = \frac{\partial f_G}{\partial z}$$

$$f_\chi = \frac{1}{\delta}\left(f_G(z) - \frac{z}{\beta} f'_G(z)\right)$$

These are specific to the Universality class and hence the susceptibilities can be calculated for the QCD system.

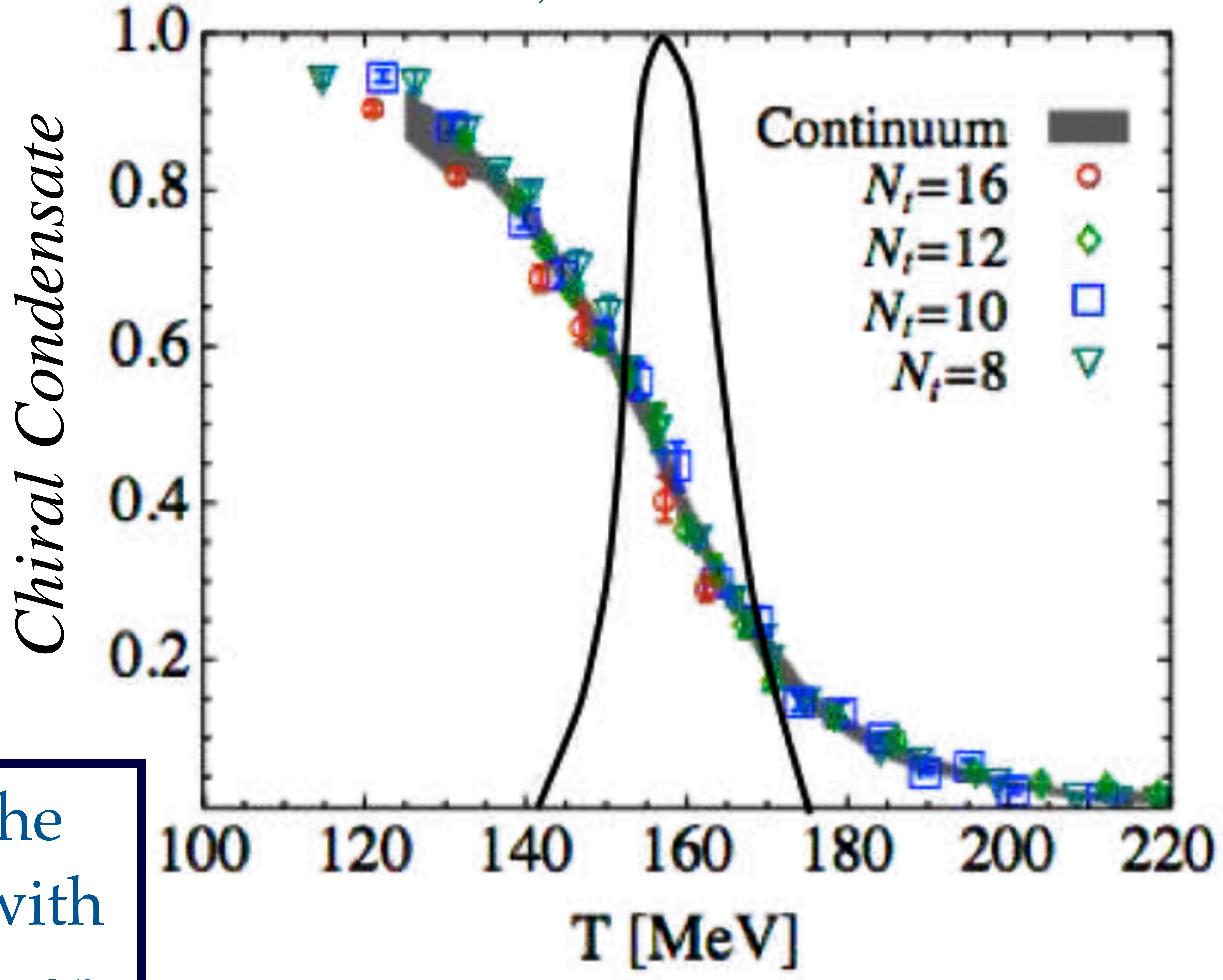
What do they tell us about the real QCD system with physical quark masses?

Susceptibilities of the chiral condensate: Real QCD

In real QGP with physical quark masses we expect for the $\chi_{t,h}$ simulations:

Analytic crossover:

As we increase the physical size of the lattice, the simulated chiral susceptibilities would fall on top of each other:



This is what is observed from LQCD: So the QCD phase transition in the real universe with physical quark masses is an analytic crossover

Nature 443:675-678,2006: Borsanyi et al., JHEP (2010) Bazavov et al., PRD (2012)

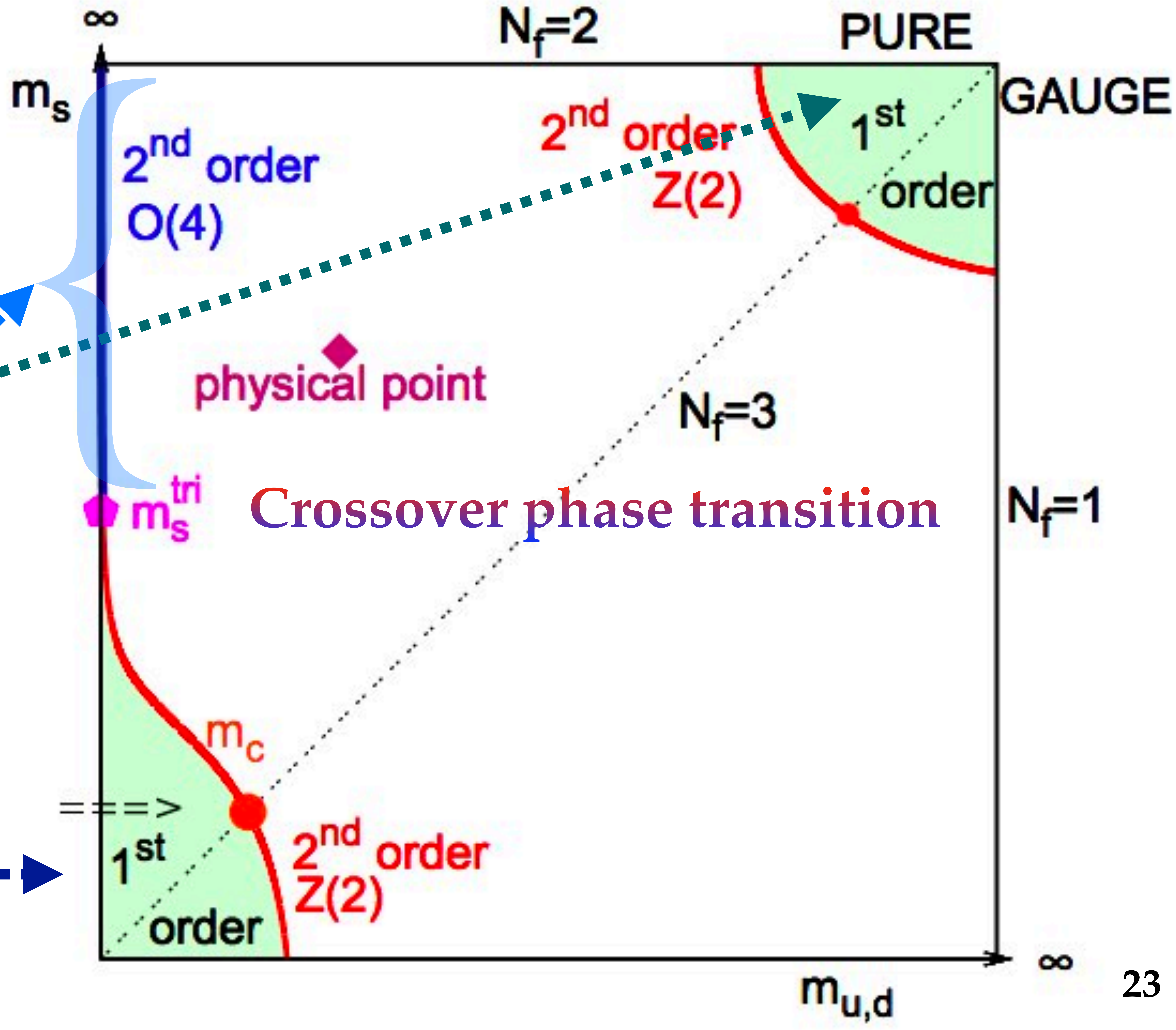
The Columbia plot at zero chemical potential

The Columbia plot summarises what we can expect from QCD when we have various values and combinations of the quark masses.

O.P is the Polyakov loop
P.T is first order

O.P is the Chiral condensate
P.T is 2nd order ($U.C = \mathcal{O}(4)$)

Effective theory for the C.C contains a cubic term and it gives a 1st order P.T



Finite chemical potential and the sign problem of QCD

The chemical potential enters in to S similar to the 4th component of the gauge field.

$$S_{QCD} = \int_0^\beta \int d\tau d^3x \bar{\psi} \left[\gamma_\mu D^\mu + \mu\gamma_4 + m\right] \psi \equiv \int_0^\beta \int d\tau d^3x \bar{\psi} \left[M\right] \psi$$

$M \equiv$ **Fermionic determinant** (***crucial in LQCD in simulations***)

When $\mu = 0 : (\gamma_5 M)^+ = \gamma_5 M$

$det[M^+] = det[\gamma_5 M \gamma_5] = det[M] \in R$

$\gamma^{4+} = \gamma^4;\ \gamma^{\mu+}\gamma_5 = \gamma^\mu\gamma_5 = \gamma_5\gamma^\mu;\ -\gamma^{4+}\gamma_5 iG_4 = \gamma^5 iG_4\gamma^4$

When $\mu \neq 0 : (\gamma_5 M)^+ = \gamma_5 M$

$M^+(\mu) = \gamma_5 M(-\mu)^*\gamma_5$

When $\mu \in R$

$det[M(\mu)]^* = det[M(-\mu)] \in C$

Problematic!

When $\mu \in Im$

$det[M(\mu)]^* = det[M(\mu)] \in R$

Still, it is problematic!

This is the *sign problem in QCD* with finite chemical potentials

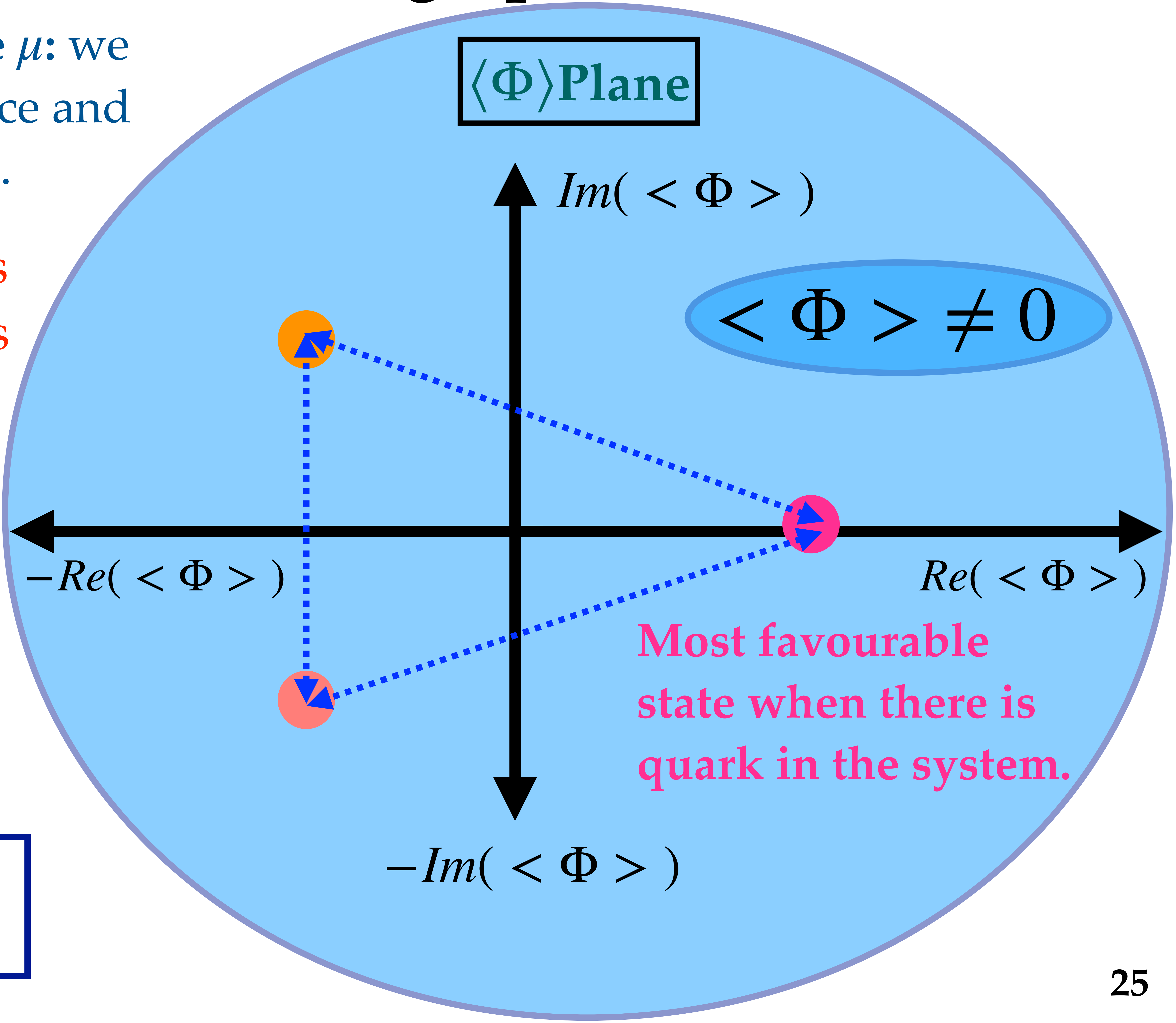
Finite chemical potential and the sign problem of QCD

If we can simulate to arbitrarily large μ: we can estimate the quantities on the lattice and analytically map it to the real μ values.

Problem with imaginary μ: There is a periodicity in the action that limits the range of explorable μ.

For a gluonic system: $< \Phi > \neq 0$, the Z(3) symmetry is broken. In this discrete situation, the three values the ground state can take are $z\mathbf{1}$ where $z = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$.

Lets see what happens when we have quarks with finite imaginary μ



Finite chemical potential and the sign problem of QCD

The *link variable* on the lattice for the gauge field is : $U_\mu = e^{-iG_\mu}$

The Polyakov loop on the lattice can be written as : $\Phi = \frac{1}{N} Tr \Pi U_4(\tau, \vec{x})$

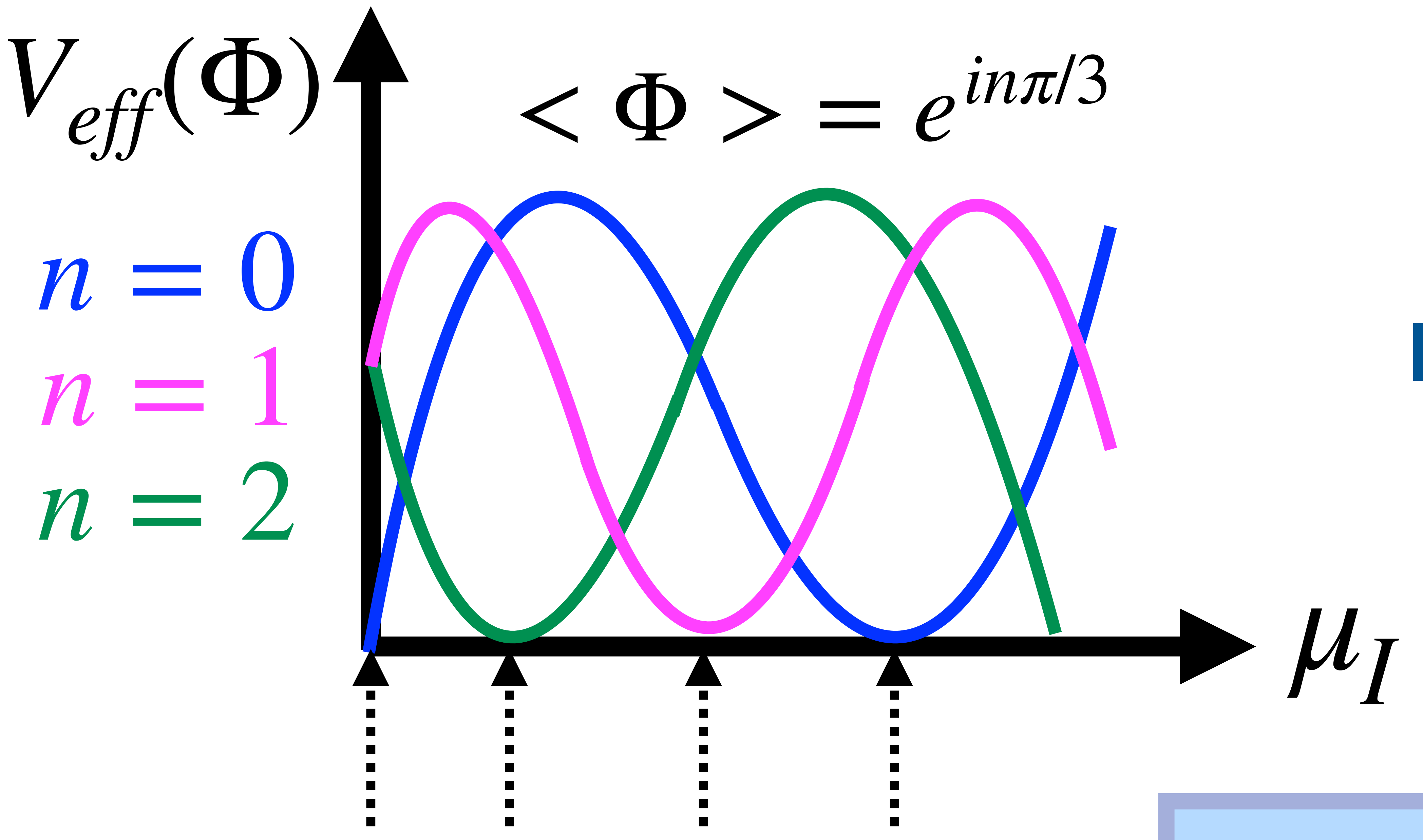
Z(3) transforms the U_4 on the lattice as: $U_4 \to z^k U_4$

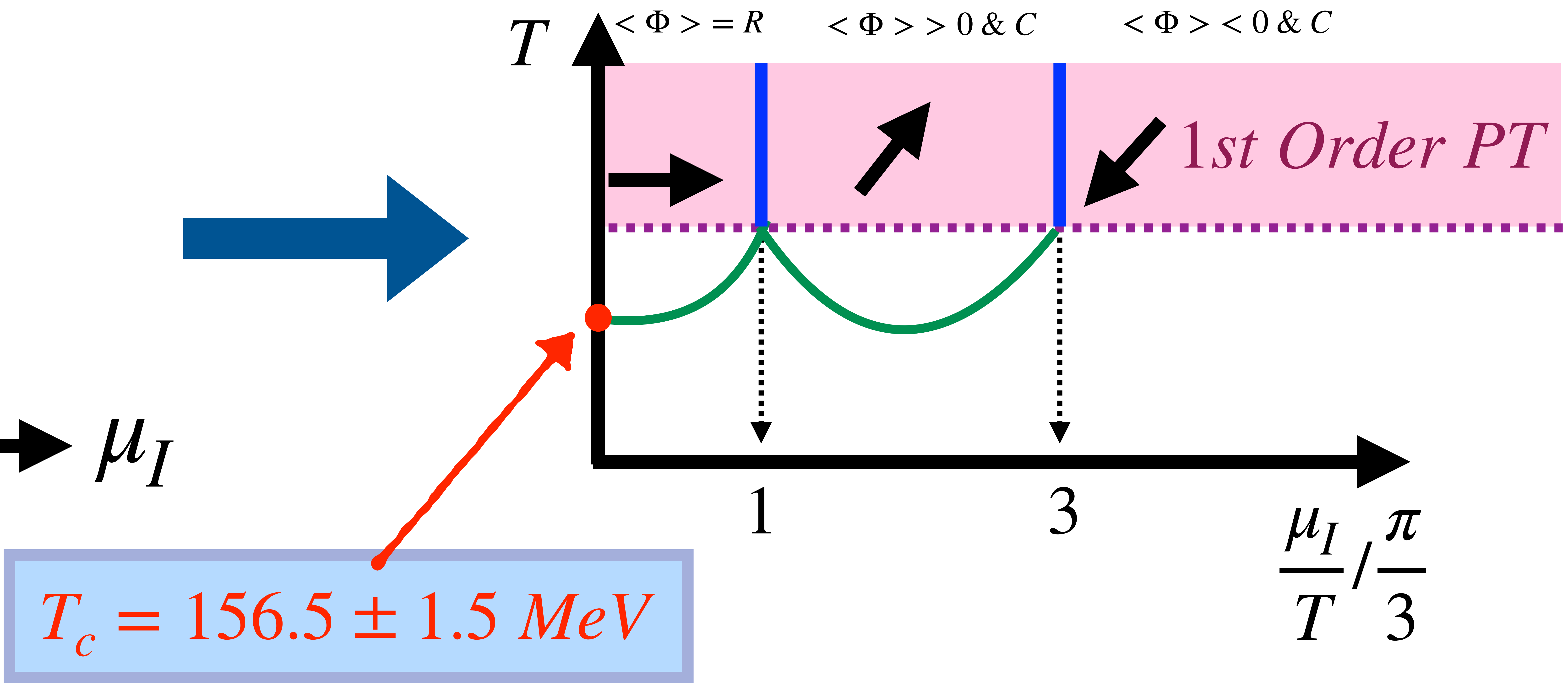
Chemical potentials induce transformations on lattice as follows: $U_4 \to e^{i\mu_I/T} U_4$

There is an interplay from the phase coming from Z(3) symmetry and μ_B due to the phase structure of transformations! **For which of the values of μ will it happen ?**

Finite chemical potential and the sign problem of QCD

As the μ_I increases, the real ground state oscillates. From the effective theory for the Polyakov loop, we can define an effective potential which reflect this scenario.





In the phase diagram at finite imaginary μ: We can simulate the quantities up to $\frac{\mu_I}{T} = \frac{\pi}{3}$.

When we increase μ_I further, the physics repeats itself. Hence we cannot extend the LQCD simulations meaningfully towards larger values of imaginary μ.

Finite chemical potential and the sign problem of QCD

Another way to do the finite μ_B estimations is by the Taylor series expansion.

One can write the pressure to the temperature ratio as

$$\frac{P(T,\mu_B)}{T^4} = \sum_0^{\infty} c_n(T)\left(\frac{\mu_B}{T}\right)^4$$

And calculate the coefficients at $\frac{\mu_B}{T} = 0$ to put into the series

$$c_n(T) = \frac{1}{n!}\frac{\partial^n (p/T)^4}{\partial(\mu_B/T)^n}\bigg|_{\frac{\mu_B}{T}=0}$$

In order to go to higher μ_B values, we need to calculate higher order derivatives which becomes noisy. **To calculate one order higher, it takes about two years!**.
As far as I know we have values up to c_6 and some estimates about c_8.

Within the range of the explored μ_B, the higher order corrections are **getting smaller**. It suggest that the **series would converge** unlike in the vicinity of a critical point. So there is **no hint** about the existence of a critical point from Taylor series approach.

Outline

- Symmetries of the QCD Lagrangian
- Z_3 symmetry of a purely gluonic system and the Polyakov loop
- Chiral condensate as an order parameter
- Universality class of the chiral phase transition
- The Columbia plot
- The sign problem in QCD and the realistic *walk arounds*.

Phase transitions

- **Ising model, Landau theory**
- **Fluctuations and critical exponents**
- **Cumulants and mapping to the QCD systems**
- **Connections to the experimental programmes** (See Talk - 4 as well)

Critical phenomenon

Ising model

In naive terms Ising model is a description of a classical statistical system with degrees of freedom are the discrete *'spins'* with ensemble of configurations at a given temperature is determined by the $P \sim e^{\frac{-H}{T}}$.

Where the Hamiltonian is given by $H = -J\Sigma_{i,j}S_iS_j - h\Sigma_iS_i$

i, j : 3 dimensional unit vectors labelling each site on the lattice

The first term in the Hamiltonian would make it energetically advantageous for the spin to align.

The second term would influence the alignment of the spins. Depending up on the sign of the magnetic field h, the spin can align in an energetically advantageous way.

Let's define ***magnetisation per spin*****:** $M = \frac{1}{N}\Sigma_iS_i$**. N is the number of lattice**

Ising model

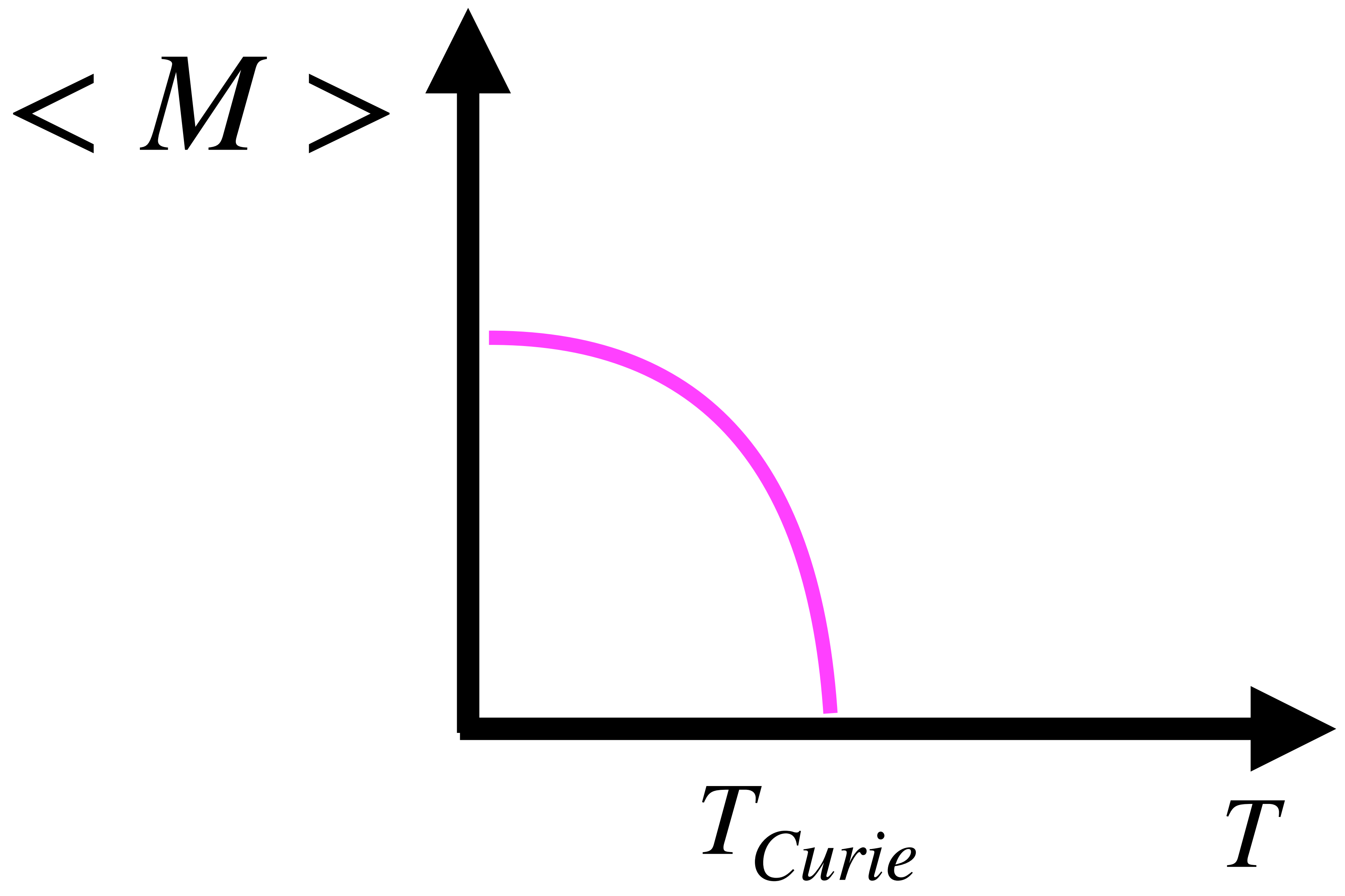
At temperature $T = 0$ *and* $h = 0$

The system doesn't care about the entropy. All spins are aligned which is the ordered state. But there are two such phases with identical energy (Up and down).

For a zero total magnetisation, then the system cannot be in a homogeneous state.

At temperature $T > 0$

Free energy replaces the energy at finite temperature. It takes in to account the entropy.



Landau theory

An effective analytical description in most of the cases for this model is provided by Landau theory though it has problems in making quantitative predictions.

The energetically favourable situation for a finite temperature system is $min\{F(M) - hM\}$

What Landau says is to consider the form of the free energy as follows:

$$F(M) = r(T)M^2 + \frac{u(T)}{4}M^4$$

Not an analytic function of M. It created issues in Landau's estimations!

Minimising this will give:

$$\frac{\partial F}{\partial M} = h$$

Which will result in the equation of state given by:

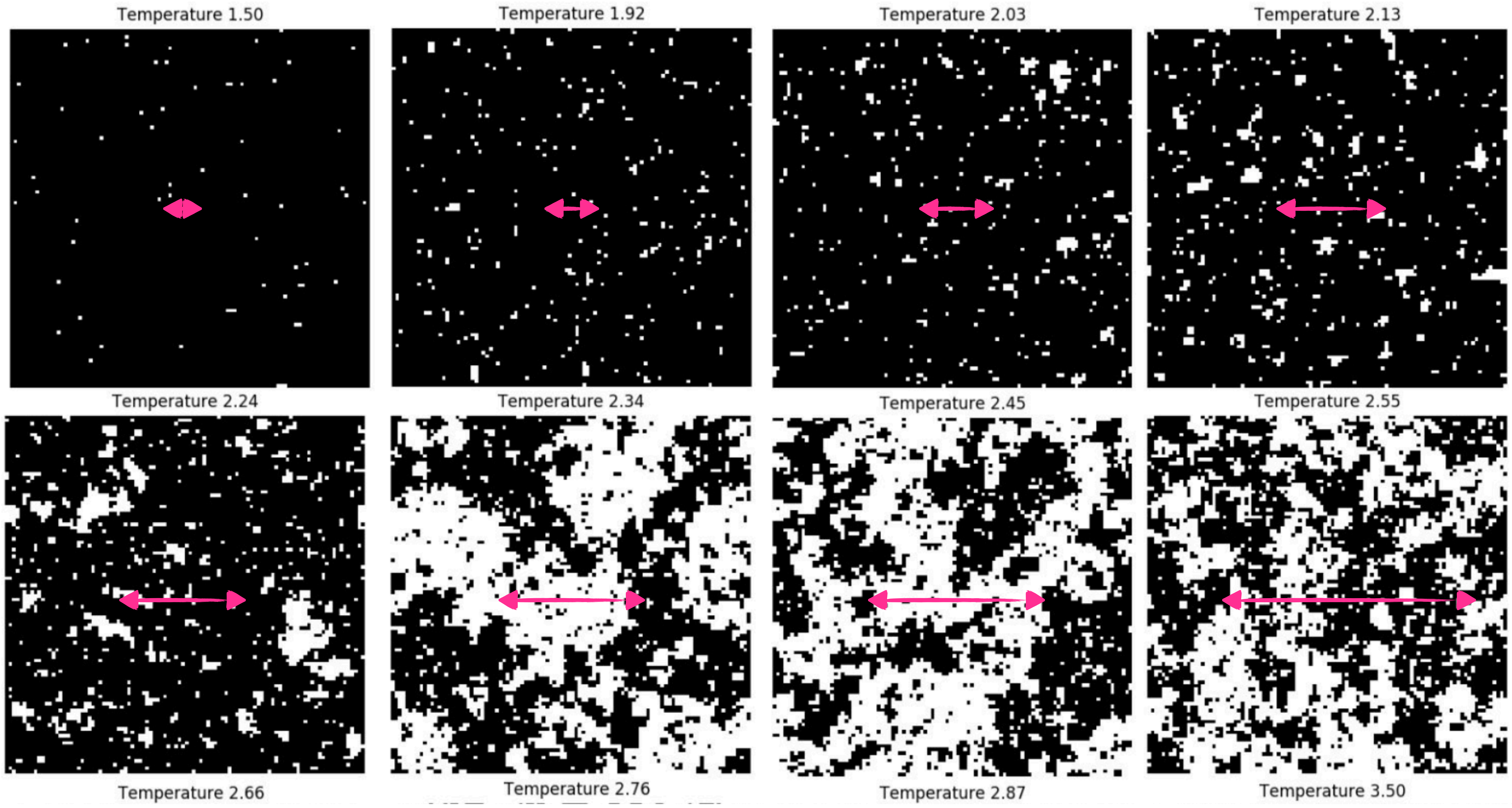
$$r(T)M + u(T)M^3 = h$$

Which is valid in in larger dimensions larger than 4.

The basic concept of correlation length

Condensed matter system vs heavy-ion collision systems: $6.02214076 \times 10^{23}$ v/s $10^2 - 10^4$

The size of the region over which the *fluctuations are no longer uncorrelated* **grows** as we approach towards the critical point. This is what is quantified in the measure of the **correlation length.**



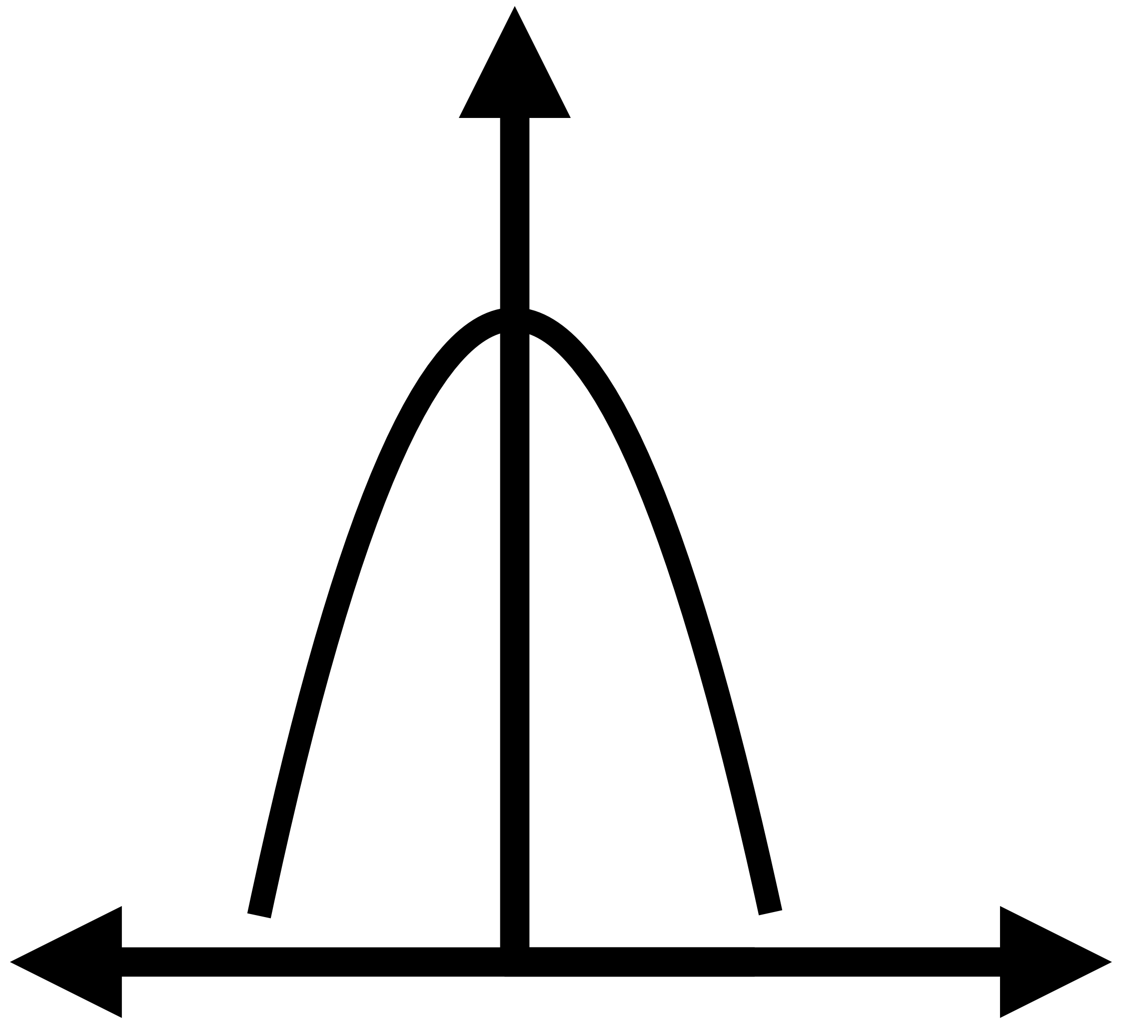
Picture Courtesy: University of Illinois Urbana-Champaign

Thermodynamic fluctuations

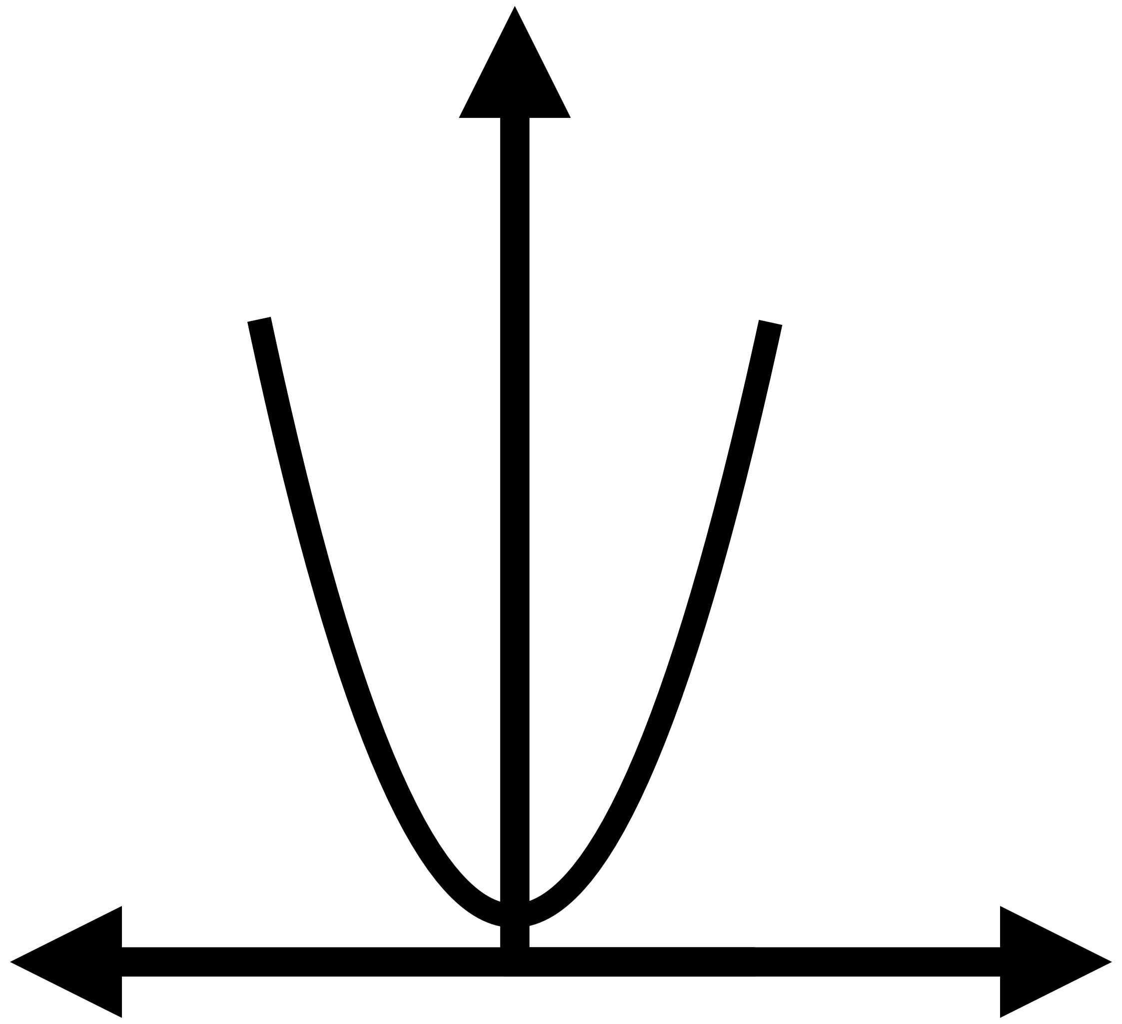
$P(M) \sim e^{-\frac{V}{T}(F-hM)}$

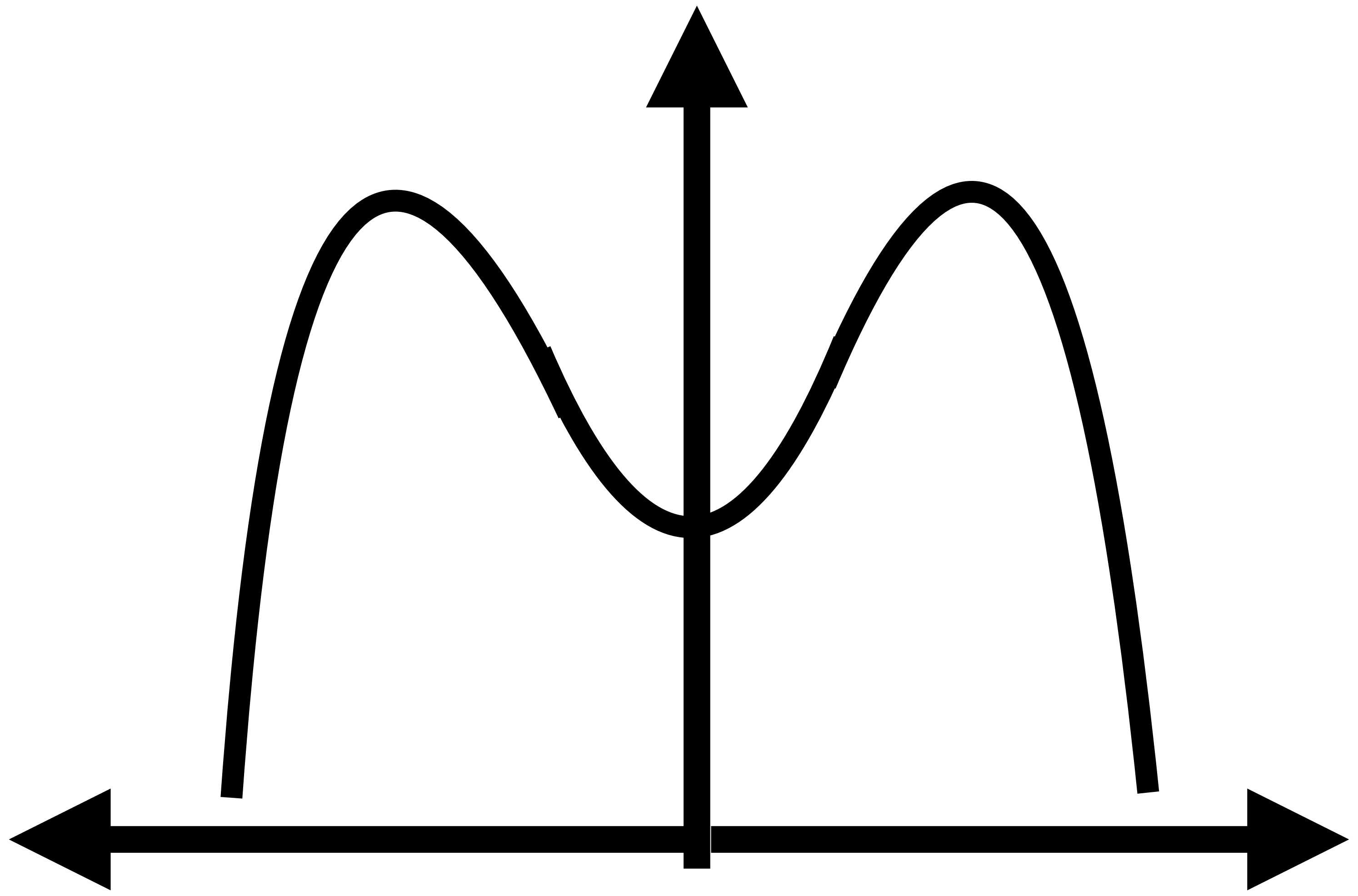
$\boxed{h=0}$

Free energy

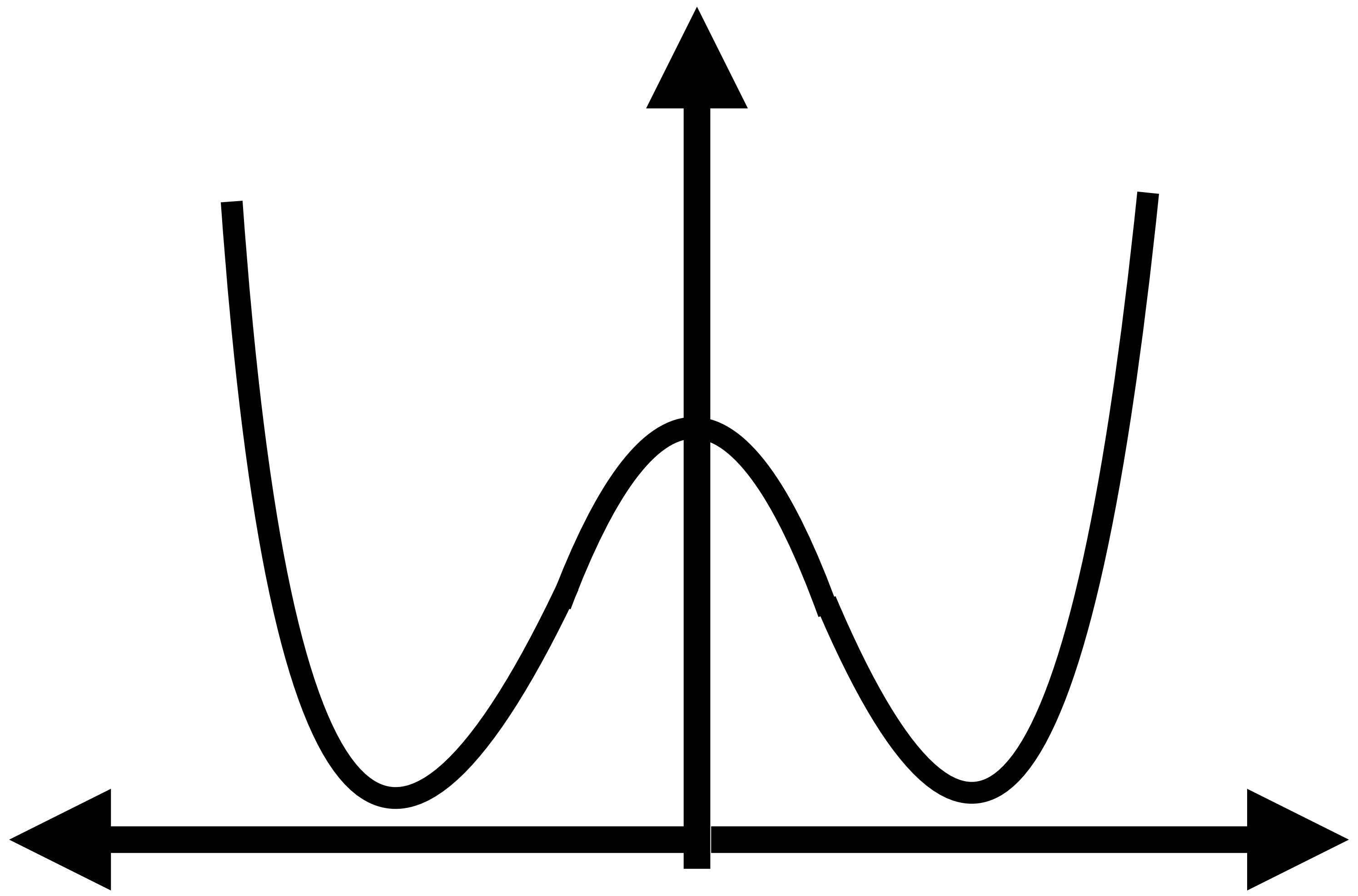


$T > T_c$





$T < T_c$



Landau theory : An analytic treatment of the fluctuations

The measure of fluctuation is given by $< M^2 > = \int dMP(M)M^2$

In the saddle point approximation:

$$< M^2 > = \frac{T}{V}\left(\frac{\partial^2 F}{\partial M^2}\right)^{-1}$$

$$\frac{\partial^2 F}{\partial M^2} = (r(T) + 3u(T)M^2)\Big|_{M=0} = r(T)$$

$$r(T) \to 0 \ at \ T = T_c$$

Consider the magnetisation per patch $M(x) = \frac{1}{V}\sum_{i \in v_b(x)} S_i$

$$b = V \to M(x) = M \to U \equiv F - hM$$

$$p[M(x)] = e^{\frac{-\Omega[M(x)]}{T}}$$

Landau theory : An analytic treatment

$b >> \xi :$ No correlation among the patches

$$\Omega[M(x)] = \int_x U(M(x))dx$$

$b \not\gg \xi :$

$$\Omega[M(x)] = \int_x U(M(x)) + \frac{z(M(x))}{2}(\nabla M(x))^2 + ..$$

- **Let us imagine that some functional for a given patch size b, we change the patch size.**
- **Then, we need to integrate out the fluctuations at those scales in-between the two sizes. which is a QFT calculation.**
- **The resulting contribution has to be sorted out in to each of the terms modifying the coefficients.**

Landau theory : Gaussian Approximation

$$< \delta M(x)\delta M(y) > = \int dMP(M)\delta M(x)\delta M(y)$$

Gaussian Approximation $$\Omega = \int_x \frac{r}{2}(\delta(M)^2 + \frac{z}{2}(\nabla \delta M^2)$$ $$\xi = \sqrt{\frac{z}{r}}$$

$$< \delta M(x)\delta M(y) > = \frac{T}{z}\frac{1}{4\pi(x-y)^{1.0}}exp(-\frac{(x-y)}{\xi}) \rightarrow < \delta M^2 > = \frac{1}{V}\frac{T}{z}\xi^2$$

- **When higher order loop terms are included, the exponents are not unity anymore.**
- **There will the infrared divergences which will induce corrections that needs to be resumed.**
- **This is what the Renormalisation group does.**

Critical exponents from the Landau theory v/s Ising model

$T \to T_c$	$\xi \to \infty$	$M \sim \|T - T_c\|^{-\nu}$	
$T < T_c$	$< M > \neq 0$	$M \sim \|T - T_c\|^{-\beta}$	
$T > T_c$	$h \neq 0$	$M \sim \chi h$	$\chi \sim \|T - T_c\|^{-\gamma}$
$T = T_c$		$M \sim h^{\frac{1}{\delta}}$	

Landau	3D Ising Actual
$\nu = \frac{1}{2}$	$\nu = \frac{2}{3}$
$\beta = \frac{1}{2}$	$\beta = \frac{1}{3}$
$\gamma = 1$	$\nu = \frac{4}{3}$
$\delta = 3$	$\delta = 5$

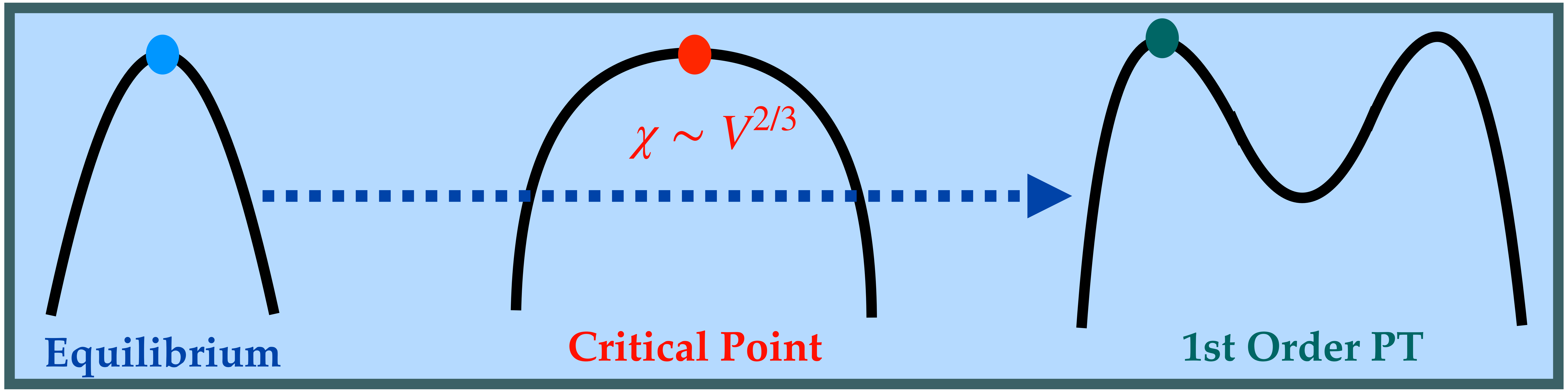
There is discrepancy between the actual and Landau values !

Thermodynamic fluctuations

The equilibrium state of a thermodynamic system is the one with maximum entropy. If we consider an order parameter σ, the probability distribution of the order parameter of a thermal state is give by:

$$P(\sigma) \sim e^{S(\sigma)}$$

At the critical point the $S(\sigma)$ should deviate from the Gaussian and flattens. The measure of the non-Gaussianity is a measure of the nearness of a critical point. Also at C.P: $\chi \equiv < \delta\sigma^2 > V \rightarrow \infty$



The central limit theorem seems to have violated near the critical point. However, its not the CLT that is violated, but the assumption that $\delta\sigma$ is an average of infinitely many uncorrelated contributions is the one which is violated near the critical point. $\chi \equiv < \delta\sigma^2 > \sim \xi^2/V$

From fluctuations to the cumulants

We introduce the variable σ

$$\sigma \equiv \delta M \sqrt{z}$$

In terms of which the Ω becomes

$$\Omega = \int_x \frac{1}{2}(\nabla \sigma)^2 + \frac{\xi^{-2}}{2}\sigma^2 + \frac{\lambda^3}{3}\sigma^3 + \frac{\lambda^4}{4}\sigma^4 + \ldots$$

We can derive that the higher order cumulants are proportional to the correlation lengths as follows:

$$< \sigma^3 > \sim \frac{T^2}{V^2} 2\lambda_3 \xi^6$$

$$< \sigma^4 > \sim \frac{T^3}{V^3} 6(2\lambda_3^2 \xi^2 - \lambda_4)\xi^8$$

Signatures of the QCD critical point

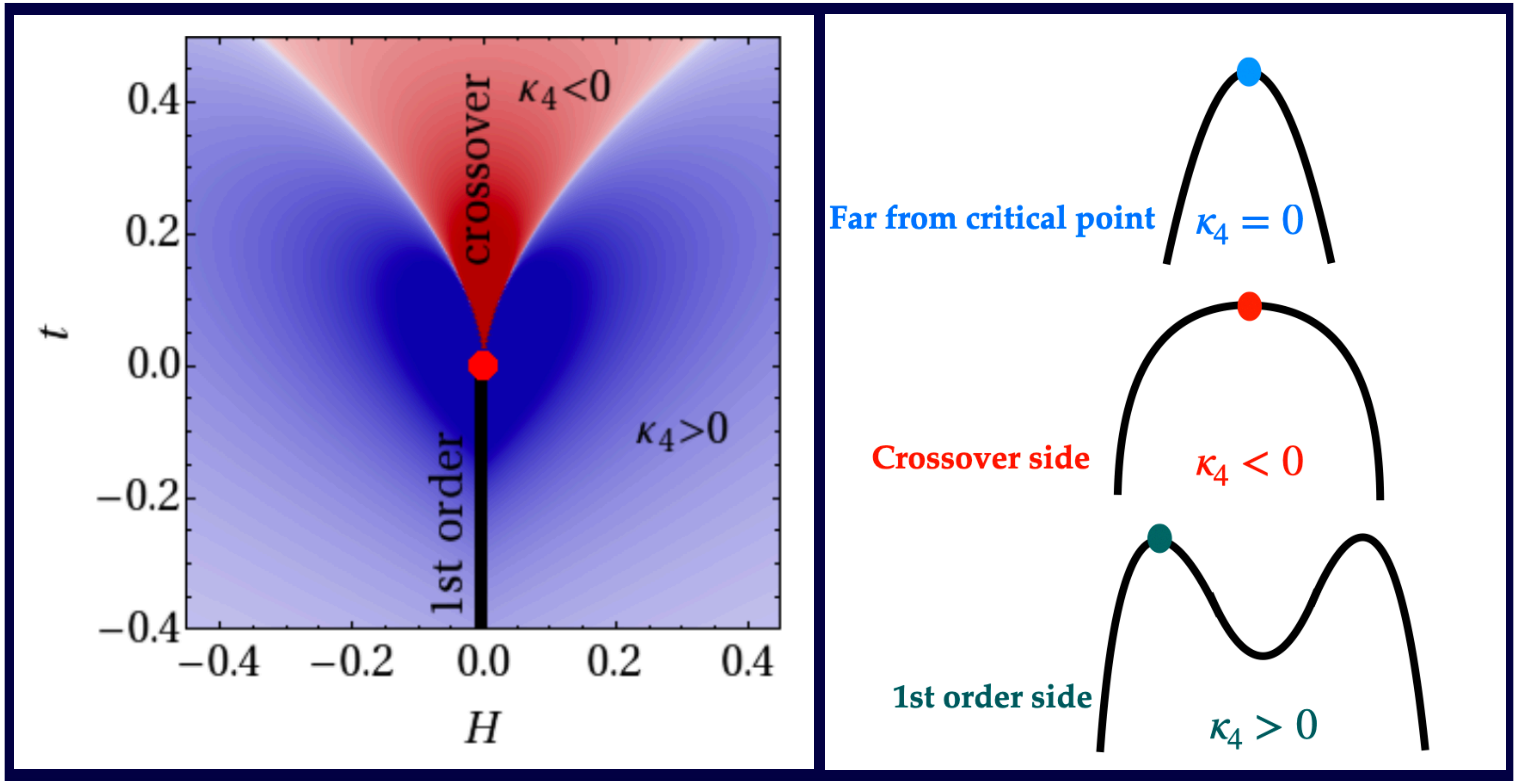
Higher order cumulants are found to be the tool for analysing the phase transitions. The Sign of the quartic cumulants depends on which side of the critical point we are in.

The behaviour of the cumulants as we scan the phase diagram is universal. Once we identify the universality class which the theory belongs to, it can be predicted without knowing the microscopic details of the system.

For $n > 2$, the sign of the cumulants κ_n depend strongly on the correlation length.

$$< \sigma^2 >_c \propto \xi^2$$
$$< \sigma^4 >_c \propto \xi^7$$

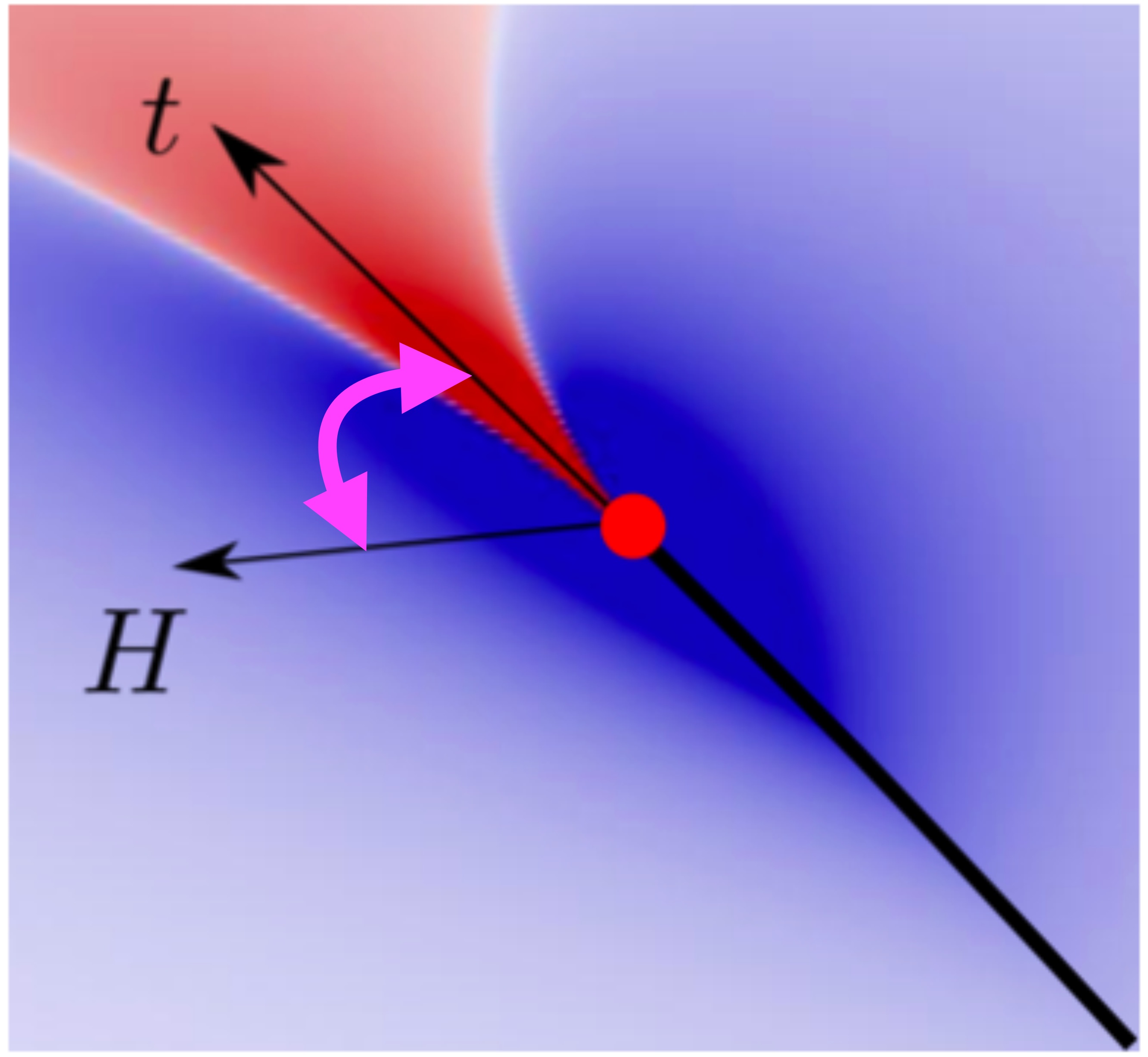
How do we map it from Ising model to QCD systems ?



Ising model phase diagram and critical behaviour

Phys. Rev. D 100, 056003 (2019)

Mapping of Ising model based estimations to QCD

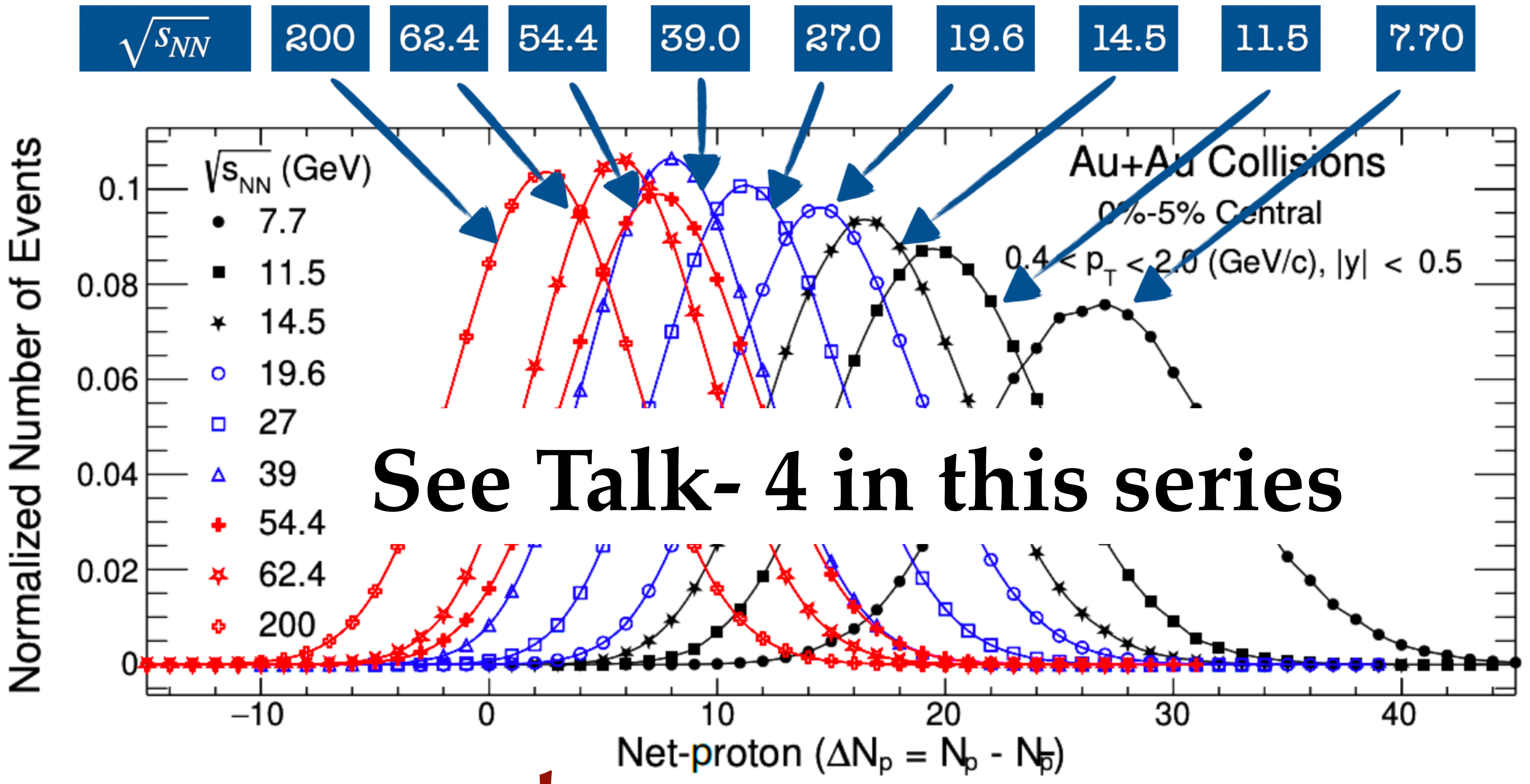


Map from Ising model phase diagram to QCD: $(t, h) \rightarrow (\mu - \mu_{CP}, T - T_{CP})$

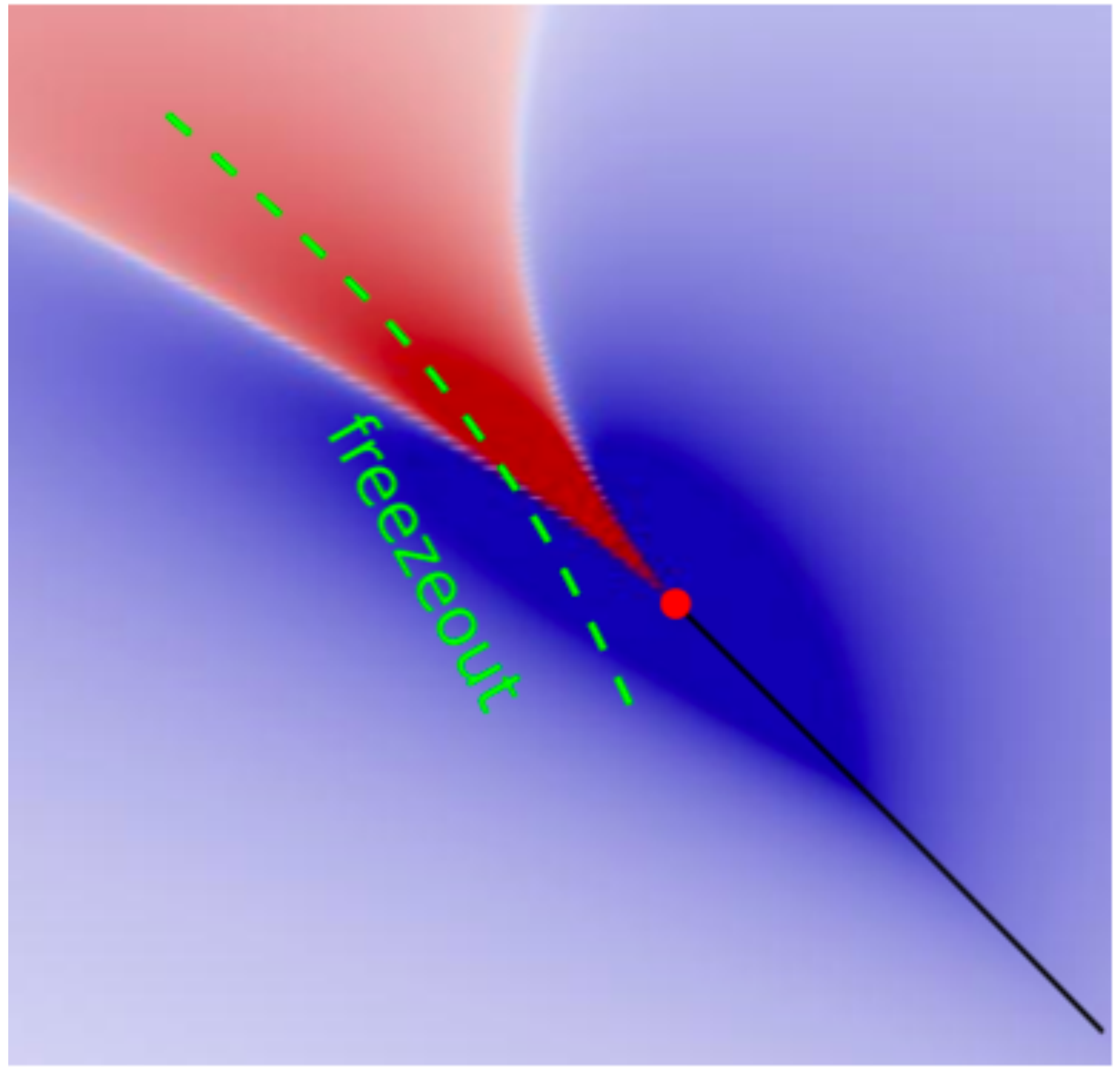
Map cumulants from multiplicity distributions: $\kappa_n(N) = \langle N \rangle + \mathcal{O}(\kappa_n(\sigma))$

What we know: The **t** falls in the same line as the phase transition

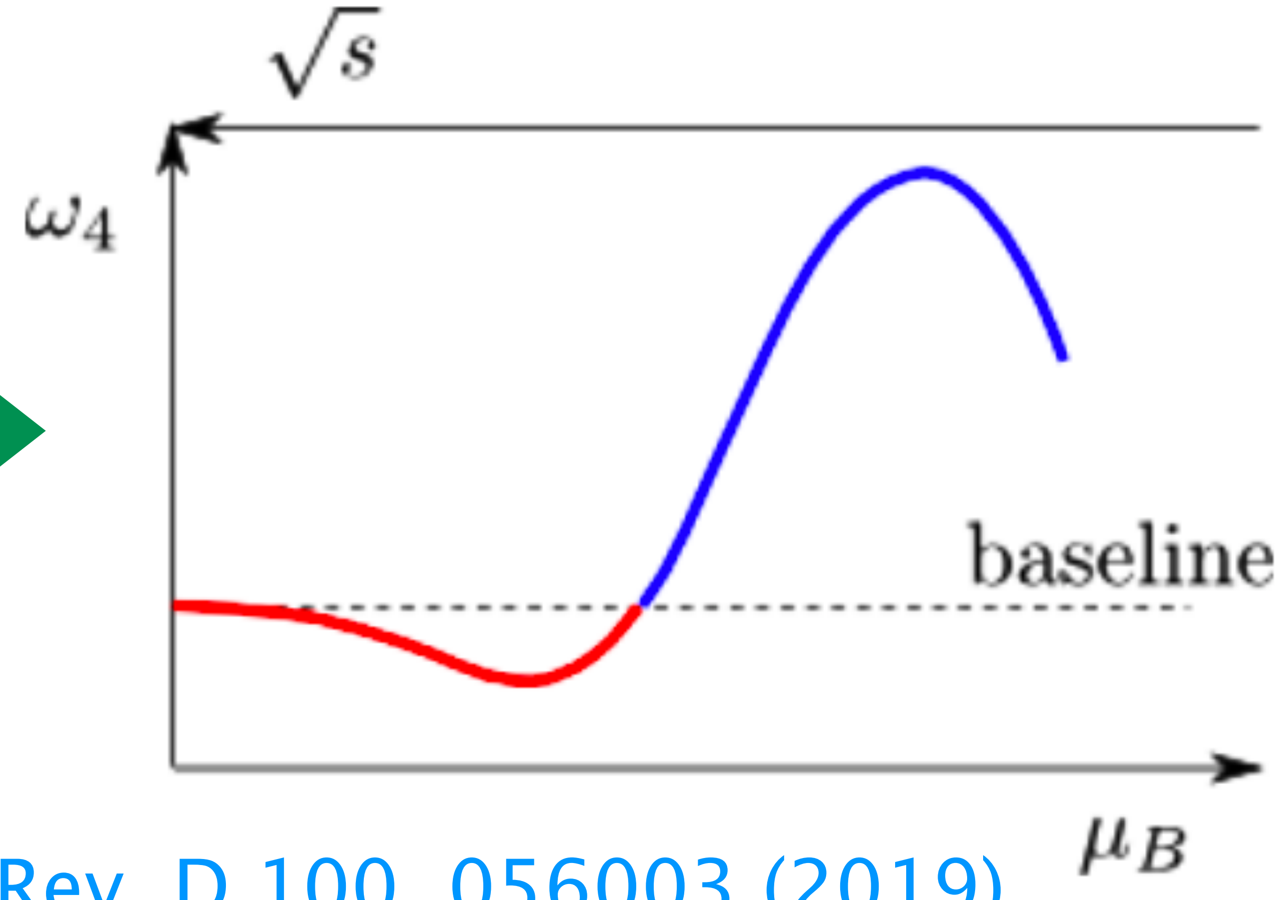
The angle between t and H are smaller for small quark masses.

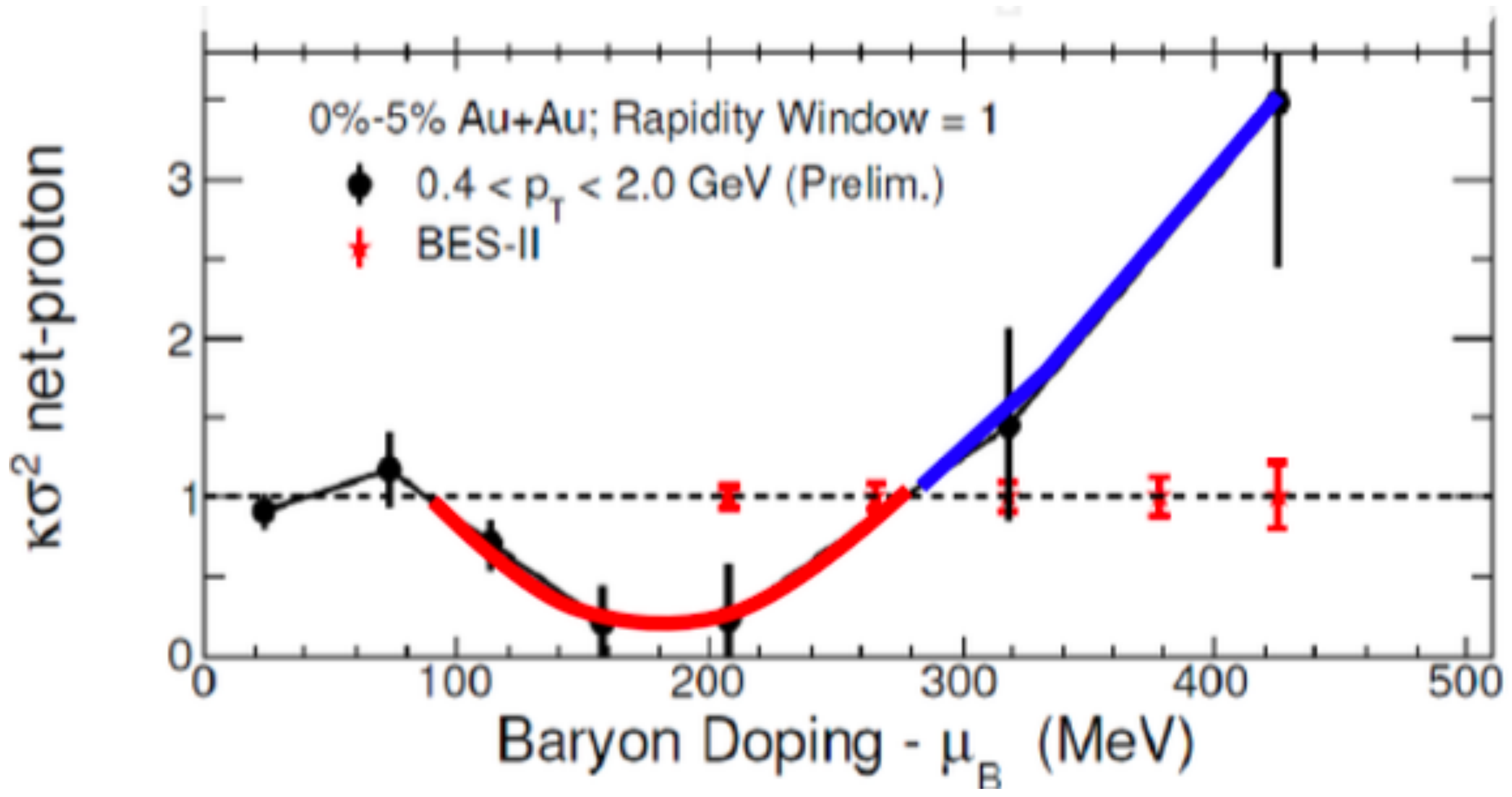


See Talk- 4 in this series



__For the speculation of a freezeout line, we expect:__





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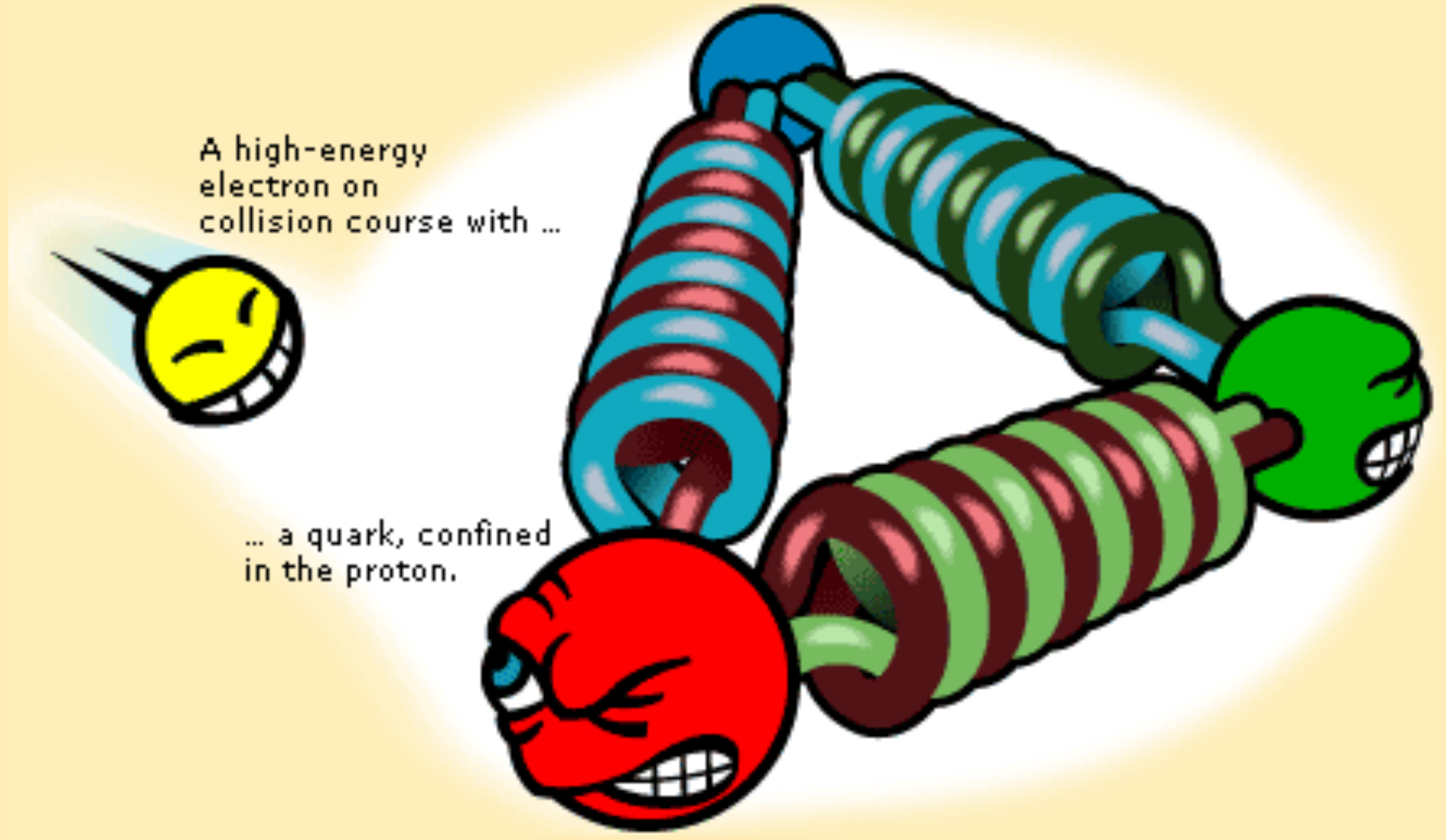

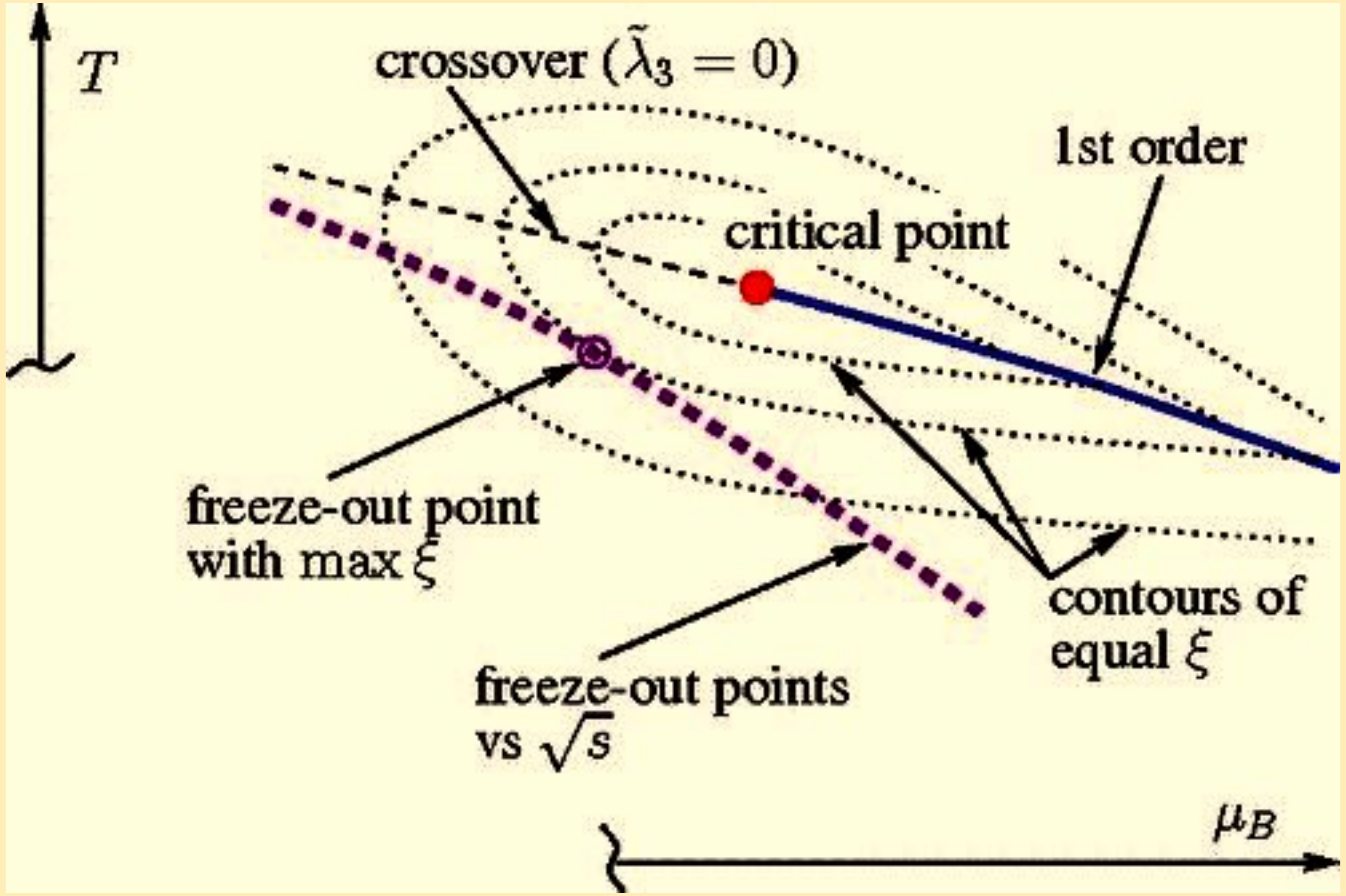

Thanks for your attention!

Picture Courtesy: nobelprize.com, Phys. Rev. Lett. 102, 032301